2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

ORIGINAL RESEARCH Open Access

Numerical Simulations for the Newton and Stokes Potentials using Approximate Approximations

Werner Varnhorn [1] and Saphir Mfogo [2]

Received: 12 February 2021/ Accepted: 07 March 2022/ Published online: 26 March 2022

Abstract

The method of approximate approximations is based on generating functions representing an approximate partition of the unity. In the present paper this method is used for the numerical solution of the Poisson equation and the Stokes system in $\mathbb{R}^n$ $(n = 2, 3)$. The corresponding approximate volume potentials will be computed explicitly in these cases, containing a one-dimensional integral, only. Numerical simulations show the efficiency of the method and confirm the expected approximation of essentially second order, depending on the smoothness of the data.

Key words: Approximate approximations, Poisson equation, Newton potential, Stokes system, Stokes potentials

AMS classification: 31A30, 35J05.

1 Introduction

In 1991, V. Maz'ya developed an approximation method called the method of approximate approximations [9]. Here a given function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is approximated by a linear combination f_h (step size $h > 0$) of radial smooth exponentially decreasing basis functions [7, 11, 12]. These basis functions, in contrast to splines, perform an approximate partition of the unity, only, hence do not converge as the step size h goes to zero [10, 13]. This lack of the convergence is not important since the approximation error can be kept below machine precision with help of certain parameters. An enormous advantage of the method is the smoothness and the simplicity of the basis functions, which can be generalized without problems to the multi-dimensional case due to their radial shape [11]. Another, and probably the most important advantage is the possibility to calculate exactly the values of almost all relevant operators in mathematical physics when applied to these basis functions [13].

[1] University of Kassel, Germany.
Email:varnhorn@mathematik.uni-kassel.de
[2] African Institute for Mathematical Sciences (AIMS) Cameroon.
Email:saphir.mfogo@aims-cameroon.org

Moreover, the method of approximate approximations can be used for the numerical solution of Cauchy problems for linear partial differential equations (PDEs) in $\mathbb{R}^n$, whenever a potential theory is available [4, 8, 15]. In these cases exact formulas for the approximate convolution type integrals (volume potentials) can be presented, if the right hand side f of the PDE is approximated by f_h. These formulas, instead of a multi-dimensional integration, contain a one-dimensional integral, only [1, 2]. Finally, the method of approximate approximations can be applied successfully for the numerical solution of boundary value problems (Boundary Point Method, see [6, 13, 14]), and, using modified Hermite polynomials, can be performed with an essentially general order of approximation [13]. These last two points are not covered here.

In the present paper the method of approximate approximations is introduced and carried out explicitly for two important Cauchy problems in $\mathbb{R}^n$, i.e. the Poisson equation $-\Delta v = f$ in $\mathbb{R}^n$ $(n = 2, 3)$ and the Stokes system $-\Delta u + \nabla p = f$ in $\mathbb{R}^2$, div $u = 0$ in $\mathbb{R}^2$.

After this introduction, in Chapter 2 we introduce the method of approximate approximations in one dimension, following the lines in [15]. With help of Fourier expansion we explain the notion of an approximate partition of the unity, using Gaussian bell curves as basis functions. This leads us to the approximation f_h $(h > 0)$ of a given function $f : \mathbb{R} \longrightarrow \mathbb{R}$, see (3). The important error estimate between f and f_h is established in Lemma 2.2. It has the form $\mathcal{O}(h^2) + \delta$ with $\delta << 1$ and is therefore of essentially second order. The chapter closes with the definition of the n-dimensional approximation, see (4).

In Chapter 3 we consider the Poisson equation $-\Delta v = f$ in $\mathbb{R}^n$ $(n = 2, 3)$. This PDE is of elliptic type and belongs to the most important PDEs of mathematical physics. Given $f \in C_0^1(\mathbb{R}^n)$, a solution of the Poisson equation is given by the volume potential Vf (see (6)), a convolution type integral containing the fundamental solution (7). Now replacing f by f_h $(h > 0)$, we obtain an approximate solution v_h in the form

$$v_h(x) = Vf_h(x) = \sum_{m \in \mathbb{Z}^n} S_{m,h}(x) f(hm),$$

where in Theorem 3.1 $(n = 2)$ and in Theorem 3.2 $(n = 3)$ the weights $S_{m,h}(x)$ are calculated exactly. As mentioned above, these weights contain a one-dimensional integral, only.

In Chapter 4, the numerical simulations confirm impressively the accuracy and the expected approximation of essentially second order, both for $n = 2$ and $n = 3$.

Chapter 5 deals with the Stokes system $-\Delta u + \nabla p = f$ in $\mathbb{R}^2$, div $u = 0$ in $\mathbb{R}^2$. This system is important in fluid dynamics and describes the steady motion of a viscous incompressible fluid in the plane, assuming small velocity gradients (creeping flow) such that the nonlinear convective term $(u \cdot \nabla)u$ in the Navier-Stokes equations can be neglected. Since a (hydrodynamical) potential theory for the Stokes system is available [3], we can proceed as follows: Using the given right hand side $f = (f_1, f_2)^T$ we define the vector function $F := (F_1, F_2, F_3)^T = (f_1, f_2, 0)^T$ and represent the Stokes system in the form $S^u_p = F$ in $\mathbb{R}^2$ with the formal Stokes operator S, compare (8). Then a solution $(u,p)^T$ of the Stokes system is given by the hydrodynamical volume potential VF (see (10)). Now replacing each component F_j (j=1,2,3) of the given function F by the approximation F_j^h, (see (13)), we obtain an approximate solution $\left(u^h, p^h\right)^T = \left(u_1^h, u_2^h, p^h\right)^T$ in the form

$$\left(u^h, p^h\right)^T (x) = \sum_{m \in \mathbb{Z}^2} A^{m,h}(x) F(hm),$$

where $A^{m,h}(x)$ denotes a 3 x 3 - matrix of weights. In Theorem 5.1 and in Theorem 5.2 the elements of this matrix are calculated exactly for the velocity field $u^h = \left(u_1^h, u_2^h\right)^T$ and the pressure function p^h, respectively.

Finally, in Chapter 6, the numerical simulations confirm the accuracy and the expected approximation of essentially second order also for the two-dimensional velocity vector.

2 An Approximate Partition of the Unity

In the present chapter we introduce the reader into the method of approximate approximations on the real line $\mathbb{R}$ and prove an important error estimate between a given function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and its approximation.

We consider the Gaussian probability density $\varphi_{\mu,\sigma}$ of the normal distribution with

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

expectation value $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, defined by

$$\mathbb{R} \ni x \longmapsto \varphi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mu - x)^2}{2\sigma^2}\right). \tag{1}$$

The function $\varphi_{\mu,\sigma}$ is always positive, it is C^∞ in $\mathbb{R}$, it takes its maximum value at $x = \mu$, and it has two turning points at $x = \mu \pm \sigma$. So the variance σ^2 somehow represents a measure of the thickness of the Gaussian bell $\varphi_{\mu,\sigma}$.

Since $\varphi_{\mu,\sigma}$ is a probability density[3] on $\mathbb{R}$, we find:

$$\int_{-\infty}^{+\infty} \varphi_{\mu,\sigma}(x)dx = 1.$$

Applying the rectangle rule, we can approximate the above integral by

$$\sum_{k\in\mathbb{Z}} \varphi_{\mu,\sigma}(k) \approx 1.$$

Now the sum on the left-hand side as a function of μ, i.e.

$$\mathbb{R} \ni \mu \mapsto \phi_\sigma(\mu) := \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu - k)^2}{2\sigma^2}\right),$$

forms the starting point of our consideration.

In a first step, we want to investigate the deviation of the function ϕ_σ from the constant 1. To do so, we use the Fourier series expansion of ϕ_σ. Since ϕ_σ is an even function with period p = 1, its Fourier series expansion is given by

$$\phi_\sigma(\mu) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(2m\pi\mu), \quad |\mu| < \frac{1}{2}.$$

Lemma 2.1 For the Fourier coefficients a_m in the above series, we have

$$a_m = 2\exp(-2\sigma^2 m^2 \pi^2), m \in \mathbb{N}_0.$$

[3] http://www.probabilityformula.org/probability-density-function.html

Proof: The Fourier coefficients of the above series are defined by:

$$a_m = 2 \int_{-1/2}^{1/2} \phi_\sigma(\mu) \cos(2m\pi\mu) d\mu \ , \ m \in \mathbb{N}_0.$$

For $m = 0$ we have:

$$a_0 = \frac{2}{\sqrt{2\pi\sigma^2}} \sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} \exp\left(-\frac{(\mu-k)^2}{2\sigma^2}\right) d\mu.$$

Substituting $t = \frac{\mu - k}{\sqrt{2}\sigma}$, hence $dt = \frac{d\mu}{\sqrt{2}\sigma}$, yields:

$$a_0 = \frac{2\sqrt{2}\sigma}{\sqrt{2\pi\sigma^2}} \sum_{k\in\mathbb{Z}} \int_{(-1/2-k)/\sqrt{2}\sigma}^{(1/2-k)/\sqrt{2}\sigma} \exp(-t^2) dt = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-t^2) dt = 2 \ ,$$

since[4]

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} exp(-t^2) = 1.$$

Analogously, for $m \in \mathbb{N}$, we obtain:

$$a_m = \frac{2}{\sqrt{\pi}} \sum_{k\in\mathbb{Z}} \int_{(-1/2-k)/\sqrt{2}\sigma}^{(1/2-k)/\sqrt{2}\sigma} \exp(-t^2) cos(2m\pi(\sqrt{2}\sigma t + k)) dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-t^2) \cos(2m\pi\sqrt{2}\sigma t) dt \ ,$$

[4] https://jakubmarian.com/integral-of-exp-x2-from-minus-infinity-to-infinity/

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

using the addition formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. Thus it follows[5]

$$a_m = 2\exp(-2\sigma^2 m^2\pi^2)$$

also for all $m \in \mathbb{N}$, and the lemma is proved.

Using the above lemma we can compute the deviation of the function ϕ_σ from the constant 1. Since $a_0 = 2$, we find:

$$\phi_\sigma(\mu) = 1 + \sum_{m=1}^{\infty} 2\exp(-2\sigma^2 m^2\pi^2)\cos(2m\pi\mu),$$

and thus the estimate

$$|\phi_\sigma(\mu) - 1| \leq 2\sum_{m=1}^{\infty} \exp(-2\sigma^2 m^2\pi^2).$$

Let us illustrate the right-hand side of this inequality for the values $\sigma = \frac{1}{2}$, $\sigma = 1$ and $\sigma = 2$.

Because of the strong exponential decay, we find:

$$\sum_{m=1}^{\infty} 2\exp(-2\sigma^2 m^2\pi^2) \approx \begin{cases} 10^{-2}, & \sigma = \dfrac{1}{2} \\ 10^{-9}, & \sigma = 1 \\ 10^{-34}, & \sigma = 2 \end{cases}.$$

Analogously, for the derivatives:

$$\phi'_\sigma(\mu) = -4\pi\sum_{m=1}^{\infty} m\exp(-2\sigma^2 m^2\pi^2)\sin(2m\pi\mu),$$

$$\phi''_\sigma(\mu) = -8\pi^2\sum_{m=1}^{\infty} m^2\exp(-2\sigma^2 m^2\pi^2)\cos(2m\pi\mu),$$

[5] http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

we obtain:

$$|\phi_\sigma(\mu)'| \approx \begin{cases} 10^{-1}, & \sigma = \frac{1}{2} \\ 10^{-6}, & \sigma = 1 \\ 10^{-34}, & \sigma = 2 \end{cases} \quad \text{and} \quad |\phi''_\sigma(\mu)| \approx \begin{cases} 10^{-1}, & \sigma = \frac{1}{2} \\ 10^{-7}, & \sigma = 1 \\ 10^{-33}, & \sigma = 2 \end{cases} .$$

Now, let us fix $\sigma := 1$ and consider the function

$$\phi(\mu) := \phi_1(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \exp(-\frac{(\mu-k)^2}{2}) \tag{2}$$

as an approximate partition of the unity ($\phi(\mu) \approx 1$), in contrast to, for example,

$$\psi(\mu) := \sum_{k \in \mathbb{Z}} \psi_k(\mu)$$

with the piecewise linear spline

$$\psi_k(\mu) = \begin{cases} \mu + 1 - k, & k - 1 \leq \mu \leq k \\ 0, & |\mu - k| \geq 1 \\ -\mu + 1 + k & k \leq \mu \leq k + 1 \end{cases} ,$$

which defines an exact partition of the unity ($\psi(\mu) = 1$). Figure 1 and Figure 2 show an illustration of these functions.

In the following, we use this function to approximate a given function $f : \mathbb{R} \longrightarrow \mathbb{R}$. To do so, chose $h > 0$ and define:

$$f_h(x) := \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x - kh}{h}\right)^2\right) f(kh). \tag{3}$$

Since we are using an approximate partition of the unity, only, we cannot expect convergence of the resulting sequence if h tends to zero. Anyway, let us study the error:

$$\varepsilon_h(x) := f_h(x) - f(x), \qquad \text{as } h \longrightarrow 0 .$$

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

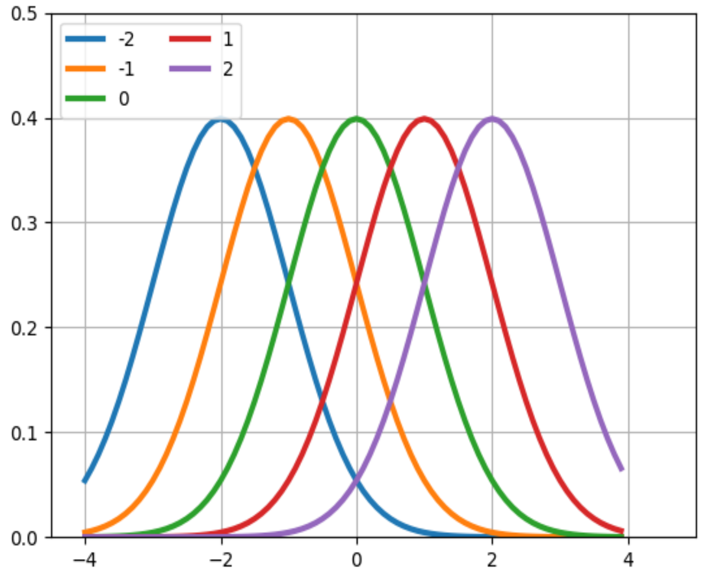

Figure 1: Gaussian kernel: Approximate partition of the unity

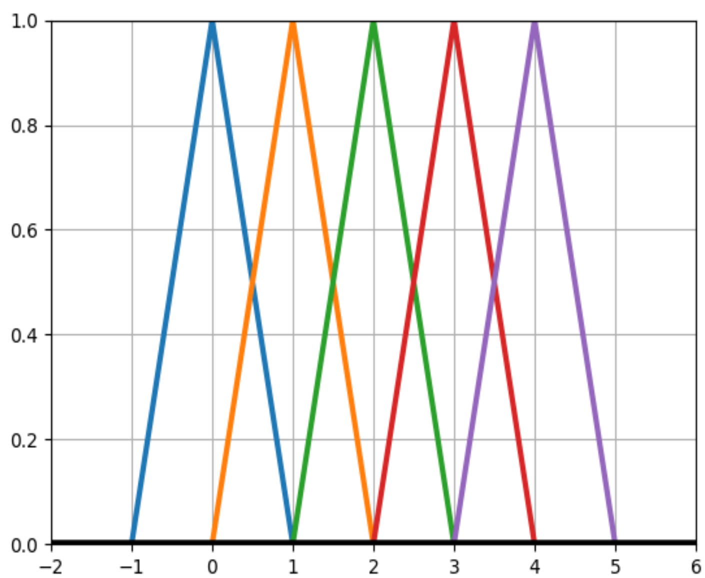

Figure 2: Hat function: Exact partition of the unity

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

In the following, the space $C_b^m(\mathbb{R})$ contains functions having bounded continuous derivatives on $\mathbb{R}$ up to the order $m \in \mathbb{N}$.

Lemma 2.2 Let $f \in C_b^2(\mathbb{R})$, $h > 0$, and f_h defined by (3). Then the error $\varepsilon_h(x)$ satisfies in $x \in \mathbb{R}$ the following estimate:

$$
\begin{aligned}
|\varepsilon_h(x)| \; &\leq \frac{h^2}{2}||f''||_\infty \left(\left|\phi\left(\frac{x}{h}\right)\right| + \left|\phi''\left(\frac{x}{h}\right)\right|\right) + h\,|f'(x)|\left|\phi'\left(\frac{x}{h}\right)\right| + |f(x)|\left|\phi\left(\frac{x}{h}\right) - 1\right| \\
&= \mathcal{O}(h^2) + \delta.
\end{aligned}
$$

Here, ϕ is the function defined by (2), $||f||_\infty := \operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)|$ is the norm in $L^\infty(\mathbb{R})$, and δ is some very small positive constant.

Proof: We use the decomposition:

$$
\begin{aligned}
\varepsilon_h(x) \; &= f_h(x) - f(x) \\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) f(kh) - f(x) \\
&-\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) f(x) \\
&+\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) f(x) \\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) (f(kh) - f(x)) + f(x) \\
&\times \left(\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) - 1\right) \\
\varepsilon_h(x) \; &= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) (f(kh) - f(x)) + f(x)\left(\phi\left(\frac{x}{h}\right) - 1\right) \\
&=: S_1(x) + S_2(x).
\end{aligned}
$$

We have:

$$S_1(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right)\left(f(kh)-f(x)\right),$$

and the Taylor expansion of $f(kh)$ at the point x yields:

$$f(kh)-f(x) = (kh-x)\,f'(x) + \frac{(kh-x)^2}{2} f''(\zeta_h),$$

with some $\zeta_h \in \mathbb{R}$ between x and kh. It follows:

$$\begin{aligned} S_1(x) &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right)\left((kh-x)\,f'(x) + \frac{(kh-x)^2}{2} f''(\zeta_h)\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right)(kh-x)\,f'(x) \\ &+\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right)\frac{(kh-x)^2}{2} f''(\zeta_h) =: s_1(x)+s_2(x). \end{aligned}$$

Since

$$\phi'\left(\frac{x}{h}\right) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right)\frac{kh-x}{h},$$

for the first summand, we find:

$$s_1(x) = hf'(x)\phi'\left(\frac{x}{h}\right).$$

From

$$\phi(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2}\right),$$

$$\phi'(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2}\right)(k-\mu),$$

$$\phi''(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2}\right)\left((k-\mu)^2-1\right),$$

we quote:

$$\phi''\left(\frac{x}{h}\right) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-kh}{h}\right)^2\right) \frac{(kh-x)^2}{h^2} - \phi\left(\frac{x}{h}\right),$$

and applying this to the second summand, it follows

$$s_2(x) = \frac{h^2}{2} f''(\zeta_h)\left(\phi''\left(\frac{x}{h}\right) + \phi\left(\frac{x}{h}\right)\right).$$

Thus we obtain:

$$\begin{aligned} \varepsilon_h(x) &= S_1(x) + S_2(x) = s_1(x) + s_2(x) + S_2(x) \\ &= hf'(x)\phi'\left(\frac{x}{h}\right) + \frac{h^2}{2} f''(\zeta_h)\left(\phi''\left(\frac{x}{h}\right) + \phi\left(\frac{x}{h}\right)\right) + f(x)\left(\phi\left(\frac{x}{h}\right) - 1\right). \end{aligned}$$

Applying the triangle inequality, it follows

$$|\varepsilon_h(x)| \leq \frac{h^2}{2}||f''||_\infty \left(\left|\phi\left(\frac{x}{h}\right)\right| + \left|\phi''\left(\frac{x}{h}\right)\right|\right) + h\,|f'(x)|\left|\phi'\left(\frac{x}{h}\right)\right| + |f(x)|\left|\phi\left(\frac{x}{h}\right) - 1\right|.$$

The estimate of Lemma 2.2 shows that we are using an approximation essentially of second order, since in practice, only the term

$$\frac{h^2}{2}||f''||_\infty \left|\phi\left(\frac{x}{h}\right)\right|$$

has to be taken into account, all other factors are neglectably small. Therefore it is clear that, the expression approximate approximation seems to be reasonable.

The method carries over immediately to the n-dimensional case, where a given function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ can be approximated by:

$$f_h(x) := \frac{1}{\sqrt{(2\pi)^n}} \sum_{k\in\mathbb{Z}^n} \exp\left(-\frac{1}{2}\left|\frac{x-kh}{h}\right|^2\right) f(kh). \tag{4}$$

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

All the above statements hold true in this case, too.

3 Application to the Poisson Equation

To use the approximation method of Chapter 2 for the numerical solution of the Poisson equation

$$-\Delta v = f \quad \text{in} \quad \mathbb{R}^n \quad (n = 2,3), \tag{5}$$

we proceed as follows: It is well-known that a solution of (5) is given by the volume potential

$$Vf(x) := \int_{\mathbb{R}^n} e(x-y) f(y) dy \quad (n = 2,3). \tag{6}$$

Here,

$$e(x) := \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x|}, & n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x|}, & n = 3 \end{cases} \tag{7}$$

denotes the fundamental solution of the Laplacian $-\Delta$ in $\mathbb{R}^n (n = 2,3)$. To approximate Vf, we replace f by f_h defined in (4). This leads to an approximate solution v_h of (5) in the form:

$$\begin{aligned} v_h(x) := Vf_h \; &= \int_{\mathbb{R}^n} e(x-y) f_h(y) dy \\ &= \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} e(x-y) \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\left|\frac{y}{h} - m\right|^2\right) f(hm) \\ &=: \sum_{m \in \mathbb{Z}^n} S_{m,h}(x) f(hm), \end{aligned}$$

with

$$S_{m,h}(x) = \begin{cases} \dfrac{1}{4\pi^2} \displaystyle\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) dy, & n=2, \\ \dfrac{1}{4\pi\sqrt{(2\pi)^3}} \displaystyle\int_{\mathbb{R}^3} \frac{1}{|x-y|} \exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) dy, & n=3. \end{cases}$$

The weights $S_{m,h}(x)$ can be determined analytically. First we consider the case $n = 2$:

Theorem 3.1 For $n = 2$ we have:

$$S_{m,h}(x) = -\frac{h^2}{4\pi}\left\{\ln\left(2h^2\right) - C + \operatorname{exint}\left(\frac{1}{2}\left|\frac{1}{h}x - m\right|^2\right)\right\}.$$

Here $C = 0.577215\ldots$ is Euler's constant, and the exponential integral exint is defined by:

$$\operatorname{exint}(x) := \int_0^x \frac{1-\exp(-t)}{t} dt.$$

Proof: We set $\zeta := x/h - m$ and $z := y/h - m$, i.e. $x - y = h(\zeta - z)$. With $dy = h^2 dz$ it follows

$$\begin{aligned} S_{m,h}(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \ln\frac{1}{|h(\zeta - z)|} \exp\left(-\frac{|z|^2}{2}\right) h^2 dz \\ &= -\frac{h^2}{4\pi^2}\int_{\mathbb{R}^2} \ln(h) \exp\left(-\frac{|z|^2}{2}\right) dz - \frac{h^2}{4\pi^2}\int_{\mathbb{R}^2} \ln|\zeta - z| \exp\left(-\frac{|z|^2}{2}\right) dz \\ &=: -S_1(x) - S_2(x). \end{aligned}$$

For the first summand $S_1(x)$ we use two-dimensional polar coordinates (r, φ) and obtain:

$$S_1(x) = \frac{h^2}{4\pi^2}\int_{\mathbb{R}^2} \ln(h)\exp\left(-\frac{|z|^2}{2}\right)dz$$

$$= \frac{h^2}{4\pi^2}\ln(h)\int_0^{2\pi}\int_0^{\infty}\exp\left(-\frac{r^2}{2}\right)rdrd\varphi$$

$$= \frac{h^2}{2\pi}\ln(h)\int_0^{\infty}\exp\left(-\frac{r^2}{2}\right)rdr,$$

and substituting $s = \frac{1}{2}r^2$, hence $ds = rdr$, we find:

$$S_1(x) = \frac{h^2}{2\pi}\ln(h)\int_0^{\infty}\exp(-s)\,ds = \frac{h^2}{2\pi}\ln(h).$$

For the second summand $S_2(x)$, it follows

$$S_2(x) = \frac{2\pi h^2}{4\pi^2}\int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln|\zeta - z|\left[-\exp\left(-\frac{|z|^2}{2}\right)\right]dz$$

$$= \frac{h^2}{2\pi}\int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln|\zeta - z|\left[-\exp\left(-\frac{|z|^2}{2}\right)\right]dz =: \frac{h^2}{2\pi}S_3(\zeta),$$

with the function $S_3(\zeta)$ given by:

$$S_3(\zeta) = \int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln|\zeta - z|\left[-\exp\left(-\frac{|z|^2}{2}\right)\right]dz.$$

The function S_3 is a volume potential for the Laplacian $-\Delta$ in $\mathbb{R}^2$, and we obtain

$$-\Delta S_3(\zeta) = -\exp\left(-\frac{|\zeta|^2}{2}\right) \quad \text{for all } \zeta \in \mathbb{R}^2.$$

Since the right hand side of this Poisson equation depends only on $|\zeta|$, we expect the same for the solution S_3. To prove this, we consider $S_3(\zeta)$ at the point $\zeta =$

$(|\zeta|\cos\psi, |\zeta|\sin\psi)$, obtaining

$$\begin{aligned} S_3(\zeta) &= \int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln|\zeta - z|\left[-\exp\left(-\frac{|z|^2}{2}\right)\right]dz \\ &= \int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln\sqrt{|\zeta|^2+|z|^2-2|\zeta||z|\cos(\measuredangle(\zeta,z))}\left[-\exp\left(-\frac{|z|^2}{2}\right)\right]dz \\ &= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\ln\sqrt{|\zeta|^2+r^2-2|\zeta|r\cos(\psi-\varphi)}\exp\left(-\frac{r^2}{2}\right)rd\varphi dr, \end{aligned}$$

where $\measuredangle(\zeta, z)$ denotes the angle between $\zeta, z \in \mathbb{R}^2$. Now substituting $\theta := \varphi - \psi$, we find (note $\cos\theta = \cos(-\theta)$)

$$S_3(\zeta) = \frac{1}{2\pi}\int_0^\infty\int_{-\psi}^{2\pi-\psi}\ln\sqrt{|\zeta|^2+r^2-2|\zeta|r\cos(\theta)}\exp\left(-\frac{r^2}{2}\right)rd\theta dr,$$

hence, due to the 2π-periodicity of the cosine function,

$$S_3(\zeta) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\ln\sqrt{|\zeta|^2+r^2-2|\zeta|r\cos(\theta)}\exp\left(-\frac{r^2}{2}\right)rd\theta dr.$$

This shows that S_3 depends only on $\rho := |\zeta|$ and not on the angle ψ. Since the two-dimensional Laplace operator in polar coordinates is defined by:

$$\Delta S_3(\rho,\psi) = \frac{\partial^2 S_3}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial S_3}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 S_3}{\partial\psi^2},$$

from $\Delta S_3(\zeta) = \exp\left(\frac{-|\zeta|^2}{2}\right)$ we obtain $(\zeta = \rho\cos\psi, \rho\sin\psi)$

$$S_3''(\rho) + \frac{1}{\rho}S_3'(\rho) = \frac{1}{\rho}\left(\rho S_3'(\rho)\right)' = \exp\left(-\frac{1}{2}\rho^2\right).$$

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

Setting $\rho = t$, integration yields

$$\int_0^r (tS_3'(t))'\,dt = \int_0^r t\ \exp\left(\frac{-t^2}{2}\right) dt,$$

hence,

$$rS_3'(r) = \int_0^r t\ \exp\left(\frac{-t^2}{2}\right) dt,$$

and, substituting $s = \frac{t^2}{2},\ ds = tdt$, for the right hand side we obtain:

$$\int_0^r t\ \exp\left(\frac{-t^2}{2}\right) dt = \int_0^{\frac{r^2}{2}} \exp(-s)\,ds = 1 - \exp\left(\frac{-r^2}{2}\right).$$

Thus it follows

$$rS_3'(r) = 1 - \exp\left(\frac{-r^2}{2}\right),$$

and another integration yields

$$\int_0^r S_3'(t)dt = \int_0^r \frac{1}{t}\left(1 - \exp\left(\frac{-t^2}{2}\right)\right) dt,$$

hence,

$$S_3(r) - S_3(0) = \int_0^r \frac{t}{2t^2/2}\left(1 - \exp\left(\frac{-t^2}{2}\right)\right) dt,$$

which implies, setting $s = \frac{1}{2}t^2,\ ds = tdt$,

$$S_3(r) = S_3(0) + \frac{1}{2}\int_0^{\frac{r^2}{2}} \left(\frac{1 - \exp(-s)}{s}\right) ds = S_3(0) + \frac{1}{2}\ \text{exint}\left(\frac{r^2}{2}\right).$$

To calculate

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

$$S_3(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |z| \exp\left(-\frac{|z|^2}{2}\right) dz,$$

we use polar coordinates and find:

$$\begin{aligned} S_3(0) &= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \ln(r) \exp\left(-\frac{r^2}{2}\right) r d\varphi dr \\ &= \frac{1}{2}\left[\int_0^\infty \ln \frac{r^2}{2} \exp\left(-\frac{r^2}{2}\right) rdr + \ln 2 \int_0^\infty \exp\left(-\frac{r^2}{2}\right) rdr\right] \\ &= \frac{1}{2}\left[\int_0^\infty \ln s \ \exp\left(-s\right) ds + \ln 2 \int_0^\infty \exp\left(-s\right) ds\right]. \end{aligned}$$

Using SAGE (System for Algebra and Geometry Experimentation), we find:

$$\int_0^\infty \ln s \ \exp\left(-s\right) ds = -0.577215... = -C,$$

and thus,

$$S_3(0) = \frac{1}{2}\left(-C + \ln 2\right).$$

It follows

$$S_3(\zeta) \quad = S_3(0) + \frac{1}{2} \text{ exint}\left(\frac{|\zeta|^2}{2}\right) = \frac{1}{2}\left[-C + \ln 2 + \text{ exint}\left(\frac{|\zeta|^2}{2}\right)\right].$$

Thus, we obtain:

$$\begin{aligned} S_{m,h}(x) &= -S_1(x) - S_2(x) = -S_1(x) - \frac{h^2}{2\pi} S_3(\zeta) \\ &= -\frac{h^2}{2\pi}\ln(h) - \frac{h^2}{2\pi}\left(\frac{1}{2}\left[-C + \ln 2 + \text{ exint}\left(\frac{1}{2}|\zeta|^2\right)\right]\right) \\ &= -\frac{h^2}{4\pi}\ln(h^2) - \frac{h^2}{4\pi}\ln 2 - \frac{h^2}{4\pi}\left[-C + \text{ exint}\left(\frac{1}{2}|\zeta|^2\right)\right] \\ &= -\frac{h^2}{4\pi}\left[\ln\left(2h^2\right) - C + \text{ exint}\left(\frac{1}{2}|\zeta|^2\right)\right]. \end{aligned}$$

Finally, using $\zeta = \frac{1}{h}x - m$, the theorem is proved :

$$S_{m,h}(x) = -\frac{h^2}{4\pi}\left\{\ln\left(2h^2\right) - C + \text{ exint}\left(\frac{1}{2}\left|\frac{1}{h}x - m\right|^2\right)\right\}.$$

Theorem 3.2 For $n = 3$, with $\zeta := \frac{1}{h}x - m \neq 0$ we have

$$S_{m,h}(x) = \frac{h^2}{\sqrt{(2\pi)^3}}\frac{1}{|\zeta|}\int_0^{|\zeta|} \exp\left(\frac{-t^2}{2}\right) dt,$$

and for $x = mh$ it holds

$$S_{m,h}(mh) = \frac{h^2}{\sqrt{(2\pi)^3}}.$$

Proof: Let $\zeta := x/h - m$ and $z := y/h - m$. With $dz = \frac{1}{h^3}dy$, hence $dy = h^3 dz$, we obtain:

$$S_{m,h}(x) = \frac{1}{4\pi\sqrt{(2\pi)^3}}\int_{\mathbb{R}^3} \frac{1}{|x-y|} \exp\left(-\frac{1}{2}\left|\frac{y}{h} - m\right|^2\right) dy =: \frac{1}{4\pi\sqrt{(2\pi)^3}} S(x)$$

with

$$S(x) = \frac{1}{h}\int_{\mathbb{R}^3} \frac{1}{|\zeta - z|} \exp\left(-\frac{1}{2}|z|^2\right) h^3 dz = S(\zeta)$$

$$= h^2 \int_{\mathbb{R}^3} \frac{1}{|\zeta - z|} \exp\left(-\frac{1}{2}|z|^2\right) dz =: S(\zeta).$$

Obviously, $\zeta \longmapsto S(\zeta) = S(|\zeta|)$ depends only on $|\zeta|$, hence we assume $\zeta = (0, 0, |\zeta|) \neq 0$ and obtain:

$$S(\zeta) = h^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(-\frac{1}{2}r^2\right) r^2 \sin\theta \frac{1}{\sqrt{|\zeta|^2 + r^2 - 2|\zeta| r\cos\theta}} dr d\theta d\varphi$$

$$= 2\pi h^2 \int_0^{\infty} \exp\left(-\frac{1}{2}r^2\right) r \int_0^{\pi} \frac{r\sin\theta}{\sqrt{|\zeta|^2 + r^2 - 2|\zeta| r\cos\theta}} d\theta \; dr$$

$$=: 2\pi h^2 \int_0^{\infty} r \exp\left(-\frac{1}{2}r^2\right) I(r) \; dr,$$

with

$$I(r) = \int_0^{\pi} \frac{r\sin\theta}{\sqrt{|\zeta|^2 + r^2 - 2|\zeta| r\cos\theta}} d\theta$$

$$= \int_0^{\pi} \frac{1}{|\zeta|} \frac{r|\zeta|\sin\theta}{\sqrt{|\zeta|^2 + r^2 - 2|\zeta| r\cos\theta}} d\theta$$

$$= \frac{1}{|\zeta|}\left[\sqrt{|\zeta|^2 + r^2 - 2|\zeta| r\cos\theta}\right]_{\theta=0}^{\theta=\pi} = \frac{1}{|\zeta|}\left(|\zeta| + r - \big||\zeta| - r\big|\right).$$

It follows

$$
\begin{aligned}
S(\zeta) &= 2\pi h^2 \int_0^{\infty} r \exp\left(-\frac{1}{2}r^2\right) I(r)\, dr, \\
S(\zeta) &= 2\pi h^2 \frac{1}{|\zeta|} \int_0^{|\zeta|} r \exp\left(-\frac{1}{2}r^2\right) \left(|\zeta| + r - (|\zeta| - r)\right)\, dr \\
&\quad + 2\pi h^2 \frac{1}{|\zeta|} \int_{|\zeta|}^{\infty} r \exp\left(-\frac{1}{2}r^2\right) \left(|\zeta| + r - (r - |\zeta|)\right)\, dr \\
&= 4\pi h^2 \frac{1}{|\zeta|} \int_0^{|\zeta|} r^2 \exp\left(-\frac{1}{2}r^2\right)\, dr + 4\pi h^2 \int_{|\zeta|}^{\infty} r \exp\left(-\frac{1}{2}r^2\right) dr \\
&=: 4\pi h^2 \left(I_1(\zeta) + I_2(\zeta)\right).
\end{aligned}
$$

For

$$
I_1(\zeta) = \frac{1}{|\zeta|} \int_0^{|\zeta|} r^2 \exp\left(-\frac{1}{2}r^2\right)\, dr,
$$

applying integration by part with $u' = r \exp\left(-\frac{1}{2}r^2\right)$ and $v = r$, we obtain

$$
\begin{aligned}
I_1(\zeta) &= \frac{1}{|\zeta|} \left(\left[-r \exp\left(-\frac{1}{2}r^2\right) \right]_0^{|\zeta|} + \int_0^{|\zeta|} \exp\left(-\frac{1}{2}r^2\right)\, dr \right) \\
&= -\exp\left(-\frac{1}{2}|\zeta|^2\right) + \frac{1}{|\zeta|} \int_0^{|\zeta|} \exp\left(-\frac{1}{2}r^2\right)\, dr,
\end{aligned}
$$

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

and

$$I_2(\zeta) = \int_{|\zeta|}^{\infty} r \exp\left(-\frac{1}{2}r^2\right) dr = \left[-\exp\left(-\frac{1}{2}r^2\right)\right]_{|\zeta|}^{\infty} = \exp\left(-\frac{1}{2}|\zeta|^2\right).$$

Therefore it follows

$$S(\zeta) \quad = 4\pi h^2 \left(I_1(\zeta) + I_2(\zeta)\right) = 4\pi h^2 \frac{1}{|\zeta|} \int_0^{|\zeta|} \exp\left(-\frac{r^2}{2}\right) dr,$$

hence,

$$S_{m,h}(x) \quad = \frac{1}{4\pi\sqrt{(2\pi)^3}} S(\zeta) = \frac{h^2}{\sqrt{(2\pi)^3}} \frac{1}{|\zeta|} \int_0^{|\zeta|} \exp\left(-\frac{t^2}{2}\right) dt,$$

if $|\zeta| \neq 0$, finally, using

$$\frac{1}{|\zeta|} \int_0^{|\zeta|} \exp\left(-\frac{t^2}{2}\right) dt \longrightarrow 1 \quad \text{as} \quad |\zeta| \longrightarrow 0,$$

the theorem is proved.

4 Numerical Simulation for the Poisson Equation

In the following, we present some numerical simulations using the above formulas for the 2-d and 3-d Poisson equation. First, let us consider the case $n = 2$.

In this case, for $2 \leq \beta \in \mathbb{N}$, we consider the test function:

$$v(x_1, x_2) = \begin{cases} 16^\beta \left(\frac{1}{4} - x_1^2\right)^\beta \left(\frac{1}{4} - x_2^2\right)^\beta & \text{in} \quad \overline{Q} \\ \\ 0 & \text{in} \quad \mathbb{R}^2 \backslash \overline{Q}, \end{cases}$$

where Q denotes the open 2-d unit square

$$Q := \left\{ (x_1, x_2) \in \mathbb{R}^2 | \quad |x_1| < \frac{1}{2}, \quad |x_2| < \frac{1}{2} \right\}.$$

The above function $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a solution of the 2-d Poisson equation $-\Delta v = f$ in $\mathbb{R}^2$ with $f(x) := f(x_1, x_2) := -\Delta v(x_1, x_2)$. Of course it holds $f = 0$ in $\mathbb{R}^2 \backslash \overline{Q}$ and, defining $h_i := \frac{1}{4} - x_i^2$ for $i = 1, 2$, an easy calculation shows

$$f(x) \quad = 2\beta 16^\beta \left(h_1 h_2\right)^{\beta-2} . \left[h_1 h_2 \left(h_1 + h_2\right) - 2\left(\beta - 1\right)\left(\left(x_2 h_1\right)^2 + \left(x_1 h_2\right)^2\right)\right] \quad \text{in} \quad \overline{Q}.$$

The first table below shows the approximation error $\varepsilon_h := \max |v(x) - v_h(x)|$, the maximum taken in 225 points in the unit square Q, for different step sizes and different values of β. Here the exponential integral function exint(x) in Theorem 3 and v_h have been evaluated with help of SageMath (www.sagemath.org).

$h \quad \backslash \quad \beta$	2	3	4	5
0.1	5.0572493e-01	1.4148691e-01	2.4969445e-01	0.2976052e-01
0.05	2.1495065e-01	4.2375107e-02	7.4950321e-02	9.2231701e-02
0.025	9.6771751e-02	1.1084570e-02	1.9672619e-02	2.4488300e-02
0.0125	4.5629427e-02	2.8027145e-03	4.9793479e-03	6.2175874e-03
0.00625	2.2120503e-02	7.0266619e-04	1.2487062e-03	1.5604673e-03
0.003125	1.0886379e-02	1.7579099e-04	3.12419092e-04	3.9049785e-04
0.0015625	5.3997014e-03	4.3955530e-05	7.8119942e-05	9.7648301e-05

Table 4.1: Maximal error ε_h

From this table, an order $\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$ of approximation can be easily calculated:

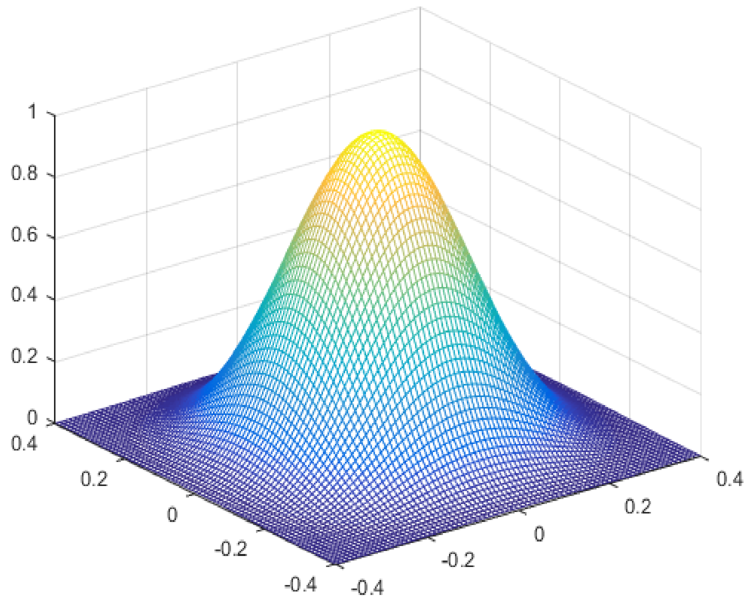

Figure 3: Test function v $(\beta = 5)$

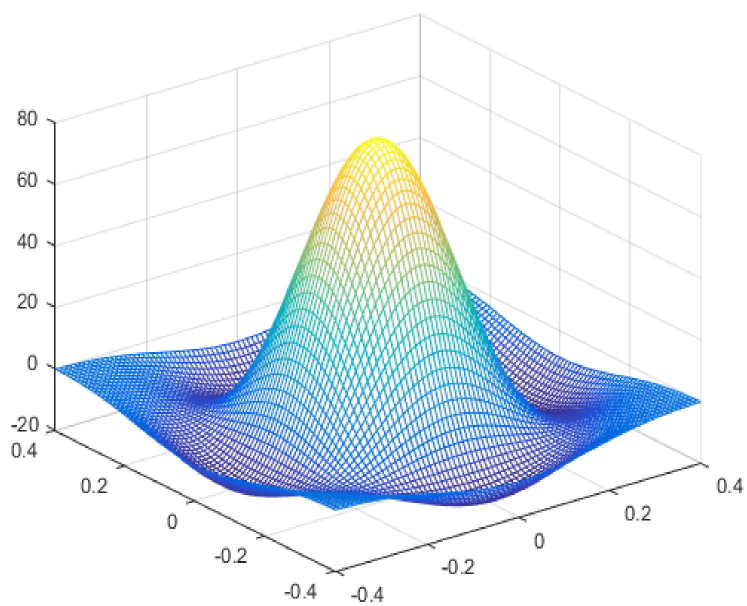

Figure 4: $f := -\Delta v$ $(\beta = 5)$

$h \quad \backslash \quad \beta$	2	3	4	5
0.1	-	-	-	-
0.05	1.2343474	1.7393797	1.7361572	1.6900654
0.025	1.1513476	1.9346642	1.9297456	1.9131700
0.0125	1.0846214	1.9836561	1.9821602	1.9776658
0.00625	1.04458036	1.9959134	1.9955227	1.9943766
0.003125	1.0228600	1.9989783	1.9988795	1.9985916
0.0015625	1.0115726	1.9997445	1.9997198	1.9996477

Table 4.2: Order of approximation α_h

Since in the case $\beta = 2$ the function $f \notin C^0(\mathbb{R}^2)$ jumps on the boundary ∂Q of the unit square Q, here, only a first order approximation is shown. For $3 \leq \beta \in \mathbb{N}$, we have $f \in C_0^{\beta-3}(\mathbb{R}^n)$ and obtain a second order approximation. This behavior corresponds to the statement of Lemma 2 if $5 \leq \beta$, since in this case we have $f \in C_b^2(\mathbb{R}^n)$. Due to the regularizing effect of the convolution type integral, however, we obtain this behavior even for $3 \leq \beta$.

Secondly, let us consider the case $n = 3$. In this case, for $2 \leq \beta \in \mathbb{N}$, we consider the test function:

$$v(x_1, x_2, x_3) = \begin{cases} 64^\beta \left(\frac{1}{4} - x_1^2\right)^\beta \left(\frac{1}{4} - x_2^2\right)^\beta \left(\frac{1}{4} - x_3^2\right)^\beta & \text{in} \quad \overline{P} \\ \\ 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{P}, \end{cases}$$

where P denotes the open 3-d unit cube

$$P := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 | \quad |x_1| < \frac{1}{2}, \quad |x_2| < \frac{1}{2}, \quad |x_3| < \frac{1}{2} \right\}.$$

The above function $v : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a solution of the 3-d Poisson equation $-\Delta v = f$

in $\mathbb{R}^3$ with $f(x) := f(x_1, x_2, x_3) := -\Delta v(x_1, x_2, x_3)$. Of course it holds $f = 0$ in $\mathbb{R}^3 \backslash \overline{P}$ and, defining $h_i := \frac{1}{4} - x_i^2$ for $i = 1, 2, 3$, an easy calculation shows

$$f(x) = 2\beta 64^\beta \left(h_1 h_2 h_3\right)^{\beta-2} \left[h_1 h_2 h_3 \left\{h_1 + h_2\right\} - 2\left(\beta - 1\right) \left\{x_3^2 \left(h_1 h_2\right)^2 + x_2^2 \left(h_1 h_3\right)^2 + x_1^2 \left(h_2 h_3\right)^2\right\}\right]$$

in $\overline{P}$. The Tables 4.3 and 4.4 below show similar results as in the case $n = 2$. Here the extensive summation in a 3-d unit cube on a simple laptop leads to a restriction of the smallness of the step size.

$h \quad \backslash \quad \beta$	2	3	4	5
0.1	1.1975408e-01	2.7438660e-01	4.1138694e-01	4.8621201e-01
0.05	1.2114626e-01	8.1703728e-02	1.2556494e-01	1.5404790e-01
0.025	7.8346660e-02	2.1384350e-02	3.3175341e-02	4.1261496e-02
0.0125	4.3784184e-02	5.4084498e-03	8.4122752e-03	1.0501918e-02

Table 4.3 : Maximum error ε_h

From this table, an order $\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$ of approximation can be easily calculated:

$h \quad \backslash \quad \beta$	2	3	4	5
0.1	-	-	-	-
0.05	1.2985123	1.7477362	1.7120623	1.6582064
0.025	1.2066054	1.9338465	1.9202504	1.90051104
0.0125	1.1201218	1.9832682	1.9795433	1.9741432

Table 4.4 : Order of approximation α_h

This shows that also in 3-d case the approximation yields the expected order 2.

5 Application to the Stokes System

We will use the approximation method of Chapter 2 to construct a numerical solution of the Stokes equations in $\mathbb{R}^2$. To do so, we define the formal Stokes operator S by

$$\begin{pmatrix} u \\ p \end{pmatrix} \mapsto S_p^u := \begin{pmatrix} -\Delta u + \nabla p \\ \text{div } u \end{pmatrix} := \begin{pmatrix} -\sum_{i=1}^{2} \frac{\partial^2 u_1}{\partial x_i^2} + \frac{\partial p}{\partial x_1} \\ -\sum_{i=1}^{2} \frac{\partial^2 u_2}{\partial x_i^2} + \frac{\partial p}{\partial x_2} \\ \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} \end{pmatrix}. \tag{8}$$

This leads us to the Stokes Cauchy problem in $\mathbb{R}^2$: For a given vector function $f := (f_1, f_2)^T$ construct a vector function $u := (u_1, u_2)^T$ and some scalar function p with

$$\begin{cases} -\Delta u + \nabla p = f \\ \text{div } u = 0 \end{cases} \text{in } \mathbb{R}^2, \tag{9}$$

where $u := (u_1, u_2)^T$ is the unknown velocity field and p an unknown pressure function of a viscous incompressible fluid flow with some given external force density $f := (f_1, f_2)^T$.

It is well known that under suitable assumptions on f, a solution of (9) is given by the hydrodynamical volume potential

$$\begin{pmatrix} u \\ p \end{pmatrix}(x) = VF(x) := \int_{\mathbb{R}^2} E(x-y)F(y)dy, \quad \text{with} \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} := \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix}. \tag{10}$$

Here, $E(x) = E_{ij}(x)$ denotes the 3 x 3 fundamental matrix of the Stokes system, defined by:

$$E_{ij}(x) = \frac{1}{4\pi}\left(\delta_{ij}\ln\frac{1}{|x|} + \frac{x_i x_j}{|x|^2}\right), \quad i,j = 1,2, \tag{11}$$

$$E_{3j}(x) = E_{j3}(x) = \frac{x_j}{2\pi|x|^2}, \quad j = 1,2, \quad E_{33}(x) = \delta(x), \tag{12}$$

where δ_{ij} is the Kronecker symbol and δ the Dirac distribution in $\mathbb{R}^2$. In (10), EF means matrix-vector multiplication.

To approximate the volume potential VF, we replace each component F_j $(j = 1,2,3)$ of the given function F by the approximation (4), i.e. by:

$$F_j^h(y) := \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^2}\exp\left(-\frac{1}{2}\left|\frac{y-mh}{h}\right|^2\right)F_j(hm). \tag{13}$$

This yields an approximate solution $\left(u^h, p^h\right)^T = \left(u_1^h, u_2^h, p^h\right)^T$ of (9) in the form:

$$\begin{aligned}\left(u^h, p^h\right)^T(x) &= VF_h(x) := \int_{\mathbb{R}^2} E(x-y)F_h(y)dy \\ &= \sum_{m\in\mathbb{Z}^2}\int_{\mathbb{R}^2} E(x-y)\frac{1}{2\pi}\exp\left(-\frac{1}{2}\left|\frac{y-hm}{h}\right|^2\right)F(hm)dy \\ &=: \sum_{m\in\mathbb{Z}^2} A^{m,h}(x)F(hm)\end{aligned}$$

with the 3 x 3-matrix $A^{m,h}(x) = A_{ij}^{m,h}(x)$, defined by

$$A_{ij}^{m,h}(x) = \int_{\mathbb{R}^2} E_{ij}(x-y)\frac{1}{2\pi}\exp\left(-\frac{1}{2}\left|\frac{y-mh}{h}\right|^2\right)dy.$$

In the next theorem, we compute the 2 x 2 left upper part of that matrix, which concerns the velocity field, and which is defined by

$$A_{ij}^{m,h}(x) = \frac{1}{8\pi^2} \quad \int_{\mathbb{R}^2} \left(\delta_{ij} \ln \frac{1}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right) \times \exp\left(-\tfrac{1}{2} \left| \frac{y - mh}{h} \right|^2 \right) dy, \quad i,j = 1,2.$$

Theorem 5.1 For $i, j = 1, 2$ and $x \neq hm$ it holds

$$A_{ij}^{m,h}(x) = \frac{1}{8\pi} h^2 \delta_{ij} \left[-\ln(2h^2) + C - \operatorname{exint}\left(\frac{1}{2} \left| \frac{x}{h} - m \right|^2 \right) + \frac{1 - \exp\left(-\frac{1}{2} \left| \frac{x}{h} - m \right|^2 \right)}{\frac{1}{2} \left| \frac{x}{h} - m \right|^2} \right]$$

$$+ \frac{1}{8\pi} h^2 \left[\frac{\left(\frac{x_i}{h} - m_i \right) \left(\frac{x_j}{h} - m_j \right)}{\frac{1}{2} \left| \frac{x}{h} - m \right|^2} - \frac{\left(\frac{x_i}{h} - m_i \right) \left(\frac{x_j}{h} - m_j \right)}{\frac{1}{4} \left| \frac{x}{h} - m \right|^4} \left[1 - \exp\left(-\frac{1}{2} \left| \frac{x}{h} - m \right|^2 \right) \right] \right],$$

and for $x = hm$, it holds

$$A_{ij}^{m,h}(hm) \quad = \delta_{ij} \frac{1}{8\pi} h^2 \left(C - \ln\left(2h^2\right) + 1 \right).$$

Here, $C = 0.577215\ldots$ is Euler's constant, and the exponential integral exint is defined as in Theorem 3.

Proof: Let us consider the function

$$H(x) = \frac{1}{2} |x|^2 \left(\ln|x| - \frac{1}{2} \right), \quad x \in \mathbb{R}^2.$$

An easy calculation shows

$$\frac{\partial^2 H(x)}{\partial x_i \partial x_j} = -\delta_{ij} \ln \frac{1}{|x|} + \frac{x_i x_j}{|x|^2}, \quad 0 \neq x \in \mathbb{R}^2.$$

Thus, setting $\alpha := -\frac{1}{2}\left|\frac{y-mh}{h}\right|^2$ for abbreviation, we obtain:

$$\begin{aligned} A_{ij}^{m,h}(x) &= \frac{1}{8\pi^2}\left[\int_{\mathbb{R}^2}\left(-\delta_{ij}\ln\frac{1}{|x-y|}+\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right)\exp(\alpha)\right. \\ &\left. +2\delta_{ij}\ln\frac{1}{|x-y|}\exp(\alpha)\,dy\right] \\ &= \frac{1}{8\pi^2}\left[\int_{\mathbb{R}^2}\frac{\partial^2 H(x-y)}{\partial x_i\partial x_j}\exp(\alpha)\,dy+2\delta_{ij}\int_{\mathbb{R}^2}\ln\frac{1}{|x-y|}\exp(\alpha)\,dy\right] \\ &= \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\frac{\partial^2 H(x-y)}{\partial x_i\partial x_j}\exp(\alpha)\,dy+\delta_{ij}\int_{\mathbb{R}^2}\frac{1}{4\pi^2}\ln\frac{1}{|x-y|}\exp(\alpha)\,dy \\ &= \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\frac{\partial^2 H(x-y)}{\partial x_i\partial x_j}\exp\left(-\frac{1}{2}\left|\frac{y-mh}{h}\right|^2\right)dy+\delta_{ij}S_{m,h}(x) \qquad \text{(Theorem 3)} \\ &=: \frac{1}{16\pi^2}\frac{\partial^2}{\partial x_i\partial x_j}B(x)+\delta_{ij}S_{m,h}(x) \end{aligned}$$

with

$$B(x) := \int_{\mathbb{R}^2}|x-y|^2\left(\ln|x-y|-\frac{1}{2}\right)\exp\left(-\frac{1}{2}\left|\frac{y-mh}{h}\right|^2\right)dy.$$

In the following, we compute the integral $B(x)$. To do so, let $\zeta := \dfrac{x}{h}-m$,

$z := \dfrac{y}{h} - m$, $x - y = h(\zeta - z)$ and $dy = h^2 dz$. Then we find:

$$\begin{aligned} B(x) &= \int_{\mathbb{R}^2} |h(\zeta - z)|^2 \left(\ln |h(\zeta - z)| - \frac{1}{2} \right) \exp\left(-\frac{1}{2} |z|^2 \right) h^2 dz \\ &= h^4 \int_{\mathbb{R}^2} \left[|\zeta - z|^2 \ln |h(\zeta - z)| \exp\left(-\frac{1}{2} |z|^2 \right) - \frac{1}{2} |(\zeta - z)|^2 \exp\left(-\frac{1}{2} |z|^2 \right) \right] dz \\ &= h^4 \int_{\mathbb{R}^2} |\zeta - z|^2 \ln |\zeta - z| \exp\left(-\frac{1}{2} |z|^2 \right) dz + \left(\ln h - \frac{1}{2} \right) h^4 \int_{\mathbb{R}^2} |\zeta - z|^2 \exp\left(-\frac{1}{2} |z|^2 \right) dz \\ &=: h^4 F(x) + \left(\ln h - \frac{1}{2} \right) h^4 G(x), \end{aligned}$$

with

$$F(x) := \int_{\mathbb{R}^2} |\zeta - z|^2 \ln |\zeta - z| \exp\left(-\frac{1}{2} |z|^2 \right) dz, \ \ G(x) := \int_{\mathbb{R}^2} |\zeta - z|^2 \exp\left(-\frac{1}{2} |z|^2 \right) dz.$$

Using two-dimensional polar coordinates (r, φ) we obtain:

$$\begin{aligned} G(x) &= \int_0^\infty \int_0^{2\pi} \left(|\zeta|^2 + r^2 - 2|\zeta| r \cos\varphi \right) \exp\left(-\frac{1}{2} r^2 \right) r d\varphi dr \\ &= \int_0^\infty \left(r \exp\left(-\frac{1}{2} r^2 \right) \left[|\zeta|^2 \varphi + r^2 \varphi - 2|\zeta| r \sin\varphi \right]_0^{2\pi} \right) dr \\ &= \int_0^\infty r \exp\left(-\frac{1}{2} r^2 \right) \left(2\pi |\zeta|^2 + 2\pi r^2 \right) dr \\ &= 2\pi |\zeta|^2 + 4\pi = 2\pi \left(|\zeta|^2 + 2 \right) \end{aligned}$$

and

$$F(x) = \int_0^\infty \int_0^{2\pi} \left(|\zeta|^2 + r^2 - 2|\zeta| r\cos\varphi\right) \ln\left(|\zeta|^2 + r^2 - 2|\zeta| r\cos\varphi\right)^{1/2} \exp\left(-\frac{1}{2}r^2\right) r d\varphi dr$$

$$= \frac{1}{2}\int_0^\infty \Bigg[\left(|\zeta|^2 + r^2\right) \int_0^{2\pi} \ln\left(|\zeta|^2 + r^2 - 2|\zeta| r\cos\varphi\right) d\varphi$$

$$- 2|\zeta| r \int_0^{2\pi} \cos\varphi \ln\left(|\zeta|^2 + r^2 - 2|\zeta| r\cos\varphi\right) d\varphi \Bigg] r\exp\left(-\tfrac{1}{2}r^2\right) dr$$

$$=: \frac{1}{2}\int_0^\infty \left[\left(|\zeta|^2 + r^2\right) I_1(\zeta, r) - 2|\zeta| r I_2(\zeta, r)\right] r\exp\left(-\frac{1}{2}r^2\right) dr,$$

with

$$I_1(\zeta, r) = \int_0^{2\pi} \ln\left(|\zeta|^2 + r^2 - 2|\zeta| r\cos\varphi\right) d\varphi, \quad I_2(\zeta, r) = \int_0^{2\pi} \cos\varphi \ln\left(|\zeta|^2 + r^2 - 2|\zeta| r\cos\varphi\right) d\varphi.$$

Using

$$\int_0^{2\pi} \ln\left(a \pm b\cos\varphi\right) d\varphi = 2\pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \quad \text{if} \quad a \geq b, \quad \text{we find}$$

$$I_1(\zeta, r) = 2\pi \ln \frac{|\zeta|^2 + r^2 + \left||\zeta|^2 - r^2\right|}{2} = \begin{cases} 4\pi \ln|\zeta|, & r < |\zeta| \\ 4\pi \ln r, & |\zeta| \leq r. \end{cases}$$

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

Applying ([5], p.589, 4.397 Nr.6), we finally obtain

$$I_2(\zeta, r) = \begin{cases} -\dfrac{2\pi r}{|\zeta|}, & r < |\zeta| \\ -\dfrac{2\pi|\zeta|}{r}, & |\zeta| \leq r. \end{cases}$$

Thus it follows:

$$\begin{aligned} F(x) &= \frac{1}{2}\int\limits_0^{|\zeta|} \left[\left(|\zeta|^2 + r^2\right) 4\pi \ln|\zeta| + 2|\zeta| r \frac{2\pi r}{|\zeta|} \right] r \exp\left(-\frac{1}{2}r^2\right) dr \\ &+ \frac{1}{2}\int\limits_{|\zeta|}^{\infty} \left[\left(|\zeta|^2 + r^2\right) 4\pi \ln r + 2|\zeta| r \frac{2\pi|\zeta|}{r} \right] r \exp\left(-\frac{1}{2}r^2\right) dr \\ &= 2\pi \left[|\zeta|^2 \ln|\zeta| \int\limits_0^{|\zeta|} r \exp\left(-\frac{1}{2}r^2\right) dr + (\ln|\zeta| + 1) \int\limits_0^{|\zeta|} r^3 \exp\left(-\frac{1}{2}r^2\right) dr \right. \\ &\left. + |\zeta|^2 \int\limits_{|\zeta|}^{\infty} (\ln r + 1)\, r \exp\left(-\frac{1}{2}r^2\right) dr + \int\limits_{|\zeta|}^{\infty} r^3 \ln r \exp\left(-\frac{1}{2}r^2\right) dr \right] \\ &= 2\pi \left[|\zeta|^2 \ln|\zeta| \left(1 - \exp\left(-\frac{1}{2}|\zeta|^2\right)\right) + (\ln|\zeta| + 1)\left(2 - (|\zeta|^2 + 2)\exp\left(-\frac{1}{2}|\zeta|^2\right)\right) \right. \\ &+ |\zeta|^2 \left\{ -\frac{1}{2}\mathrm{Ei}\left(-\frac{1}{2}|\zeta|^2\right) + \exp\left(-\frac{1}{2}|\zeta|^2\right)(\ln|\zeta| + 1) \right\} \\ &\left. + \left\{ -\mathrm{Ei}\left(-\frac{1}{2}|\zeta|^2\right) + \exp\left(-\frac{1}{2}|\zeta|^2\right)\left((|\zeta|^2 + 2)\ln|\zeta| + 1\right) \right\} \right] \end{aligned}$$

with

$$\mathrm{Ei}(z) := -\int_{-z}^{\infty} \frac{\exp(-t)}{t} dt = \mathrm{exint}(z) - C - \ln z.$$

Here the first integral is trivial, and the other three integrals are evaluated using WolframAlpha [6]. Collecting all terms yields

$$F(x) = 2\pi \left[2 - \exp\left(-\frac{1}{2}|\zeta|^2\right) + \left(\frac{1}{2}|\zeta|^2 + 1\right) . \left\{\ln 2 - C + \mathrm{exint}\left(\frac{1}{2}|\zeta|^2\right)\right\}\right],$$

and it follows

$$\begin{aligned} B(x) &= h^4 F(x) + \left(\ln h - \frac{1}{2}\right) h^4 G(x) \\ &= 2\pi h^4 \left\{2 - \exp\left(-\frac{1}{2}|\zeta|^2\right) + \left(\frac{1}{2}|\zeta|^2 + 1\right)\left[\ln\left(2h^2\right) - C - 1 + \mathrm{exint}\left(\frac{1}{2}|\zeta|^2\right)\right]\right\}. \end{aligned}$$

It remains to calculate the second order derivatives of the function B. Assuming $\zeta \neq 0$, we find:

$$\frac{\partial}{\partial x_i} B(x) = 2\pi h^3 \left[\zeta_i \left(\ln\left(2h^2\right) - C + \mathrm{exint}\left(\frac{1}{2}|\zeta|^2\right)\right) + \zeta_i \frac{1 - \exp\left(-\frac{1}{2}|\zeta|^2\right)}{\frac{1}{2}|\zeta|^2}\right],$$

and thus,

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} B(x) &= 2\pi h^3 \left[\delta_{ij}\left(\ln\left(2h^2\right) - C + \mathrm{exint}\left(\frac{1}{2}|\zeta|^2\right)\right)\right. \\ &\left. + \delta_{ij} \frac{1 - \exp\left(-\frac{1}{2}|\zeta|^2\right)}{\frac{1}{2}|\zeta|^2} + \frac{\zeta_i \zeta_j}{\frac{1}{2}|\zeta|^2} - \frac{\zeta_i \zeta_j}{\frac{1}{4}|\zeta|^4}\left(1 - \exp\left(-\frac{1}{2}|\zeta|^2\right)\right)\right]. \end{aligned}$$

[6] https://www.wolframalpha.com/

Finally, applying Theorem 3 and using $\zeta = x/h - m$, for $\zeta \neq 0$ we obtain:

$$A_{ij}^{m,h}(x) = \frac{1}{16\pi^2}\frac{\partial^2}{\partial x_i \partial x_j}B(x) + \delta_{ij}S_{m,h}(x)$$

$$= \frac{1}{8\pi}h^2\delta_{ij}\left[-\ln(2h^2) + C - \text{exint}\left(-\frac{1}{2}\left|\frac{x}{h} - m\right|^2\right) + \frac{1 - \exp\left(-\frac{1}{2}\left|\frac{x}{h} - m\right|^2\right)}{\frac{1}{2}\left|\frac{x}{h} - m\right|^2}\right]$$

$$+ \frac{1}{8\pi}h^2\left[\frac{\left(\frac{x_i}{h} - m_i\right)\left(\frac{x_j}{h} - m_j\right)}{\frac{1}{2}\left|\frac{x}{h} - m\right|^2} - \frac{\left(\frac{x_i}{h} - m_i\right)\left(\frac{x_j}{h} - m_j\right)}{\frac{1}{4}\left|\frac{x}{h} - m\right|^4}\left[1 - \exp\left(-\frac{1}{2}\left|\frac{x}{h} - m\right|^2\right)\right]\right],$$

and for $\zeta = 0$, we have:

$$A_{ij}^{m,h}(x) = A_{ij}^{m,h}(hm) = \frac{1}{8\pi}h^2\delta_{ij}\left(C - \ln\left(2h^2\right) + 1\right).$$

Analogously, in the next theorem, we present the missing components of the 3 x 3-matrix $A^{m,h}(x)$ for the pressure function. The proof can be carried out analogously.

Theorem 5.2 For i=3, j=1,2, and $x \neq hm$, it holds

$$A_{3j}^{m,h}(x) = 2\pi h\left(\frac{x_j}{h} - m\right)\frac{1 - \exp\left(-\frac{1}{2}\left|\frac{x}{h} - m\right|^2\right)}{\left|\frac{x}{h} - m\right|^2},$$

and for $x = hm$

$$A_{3j}^{m,h}(hm) = 0.$$

Proof: We have

$$A_{3j}^{m,h}(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{x_i - y_i}{|x - y|^2}\exp\left(-\frac{1}{2}\left|\frac{y - mh}{h}\right|^2\right)dy.$$

Since, $\dfrac{\partial}{\partial x_j}\ln|x| = \dfrac{x_j}{|x|^2}$ if $x \neq 0$ it follows $A_{3j}^{m,h}(x) = -\dfrac{\partial}{\partial x_j}S_{m,h}(x)$, and since,

$$S_{m,h}(x) = -\frac{h^2}{4\pi}\left\{\ln\left(2h^2\right) - C + \text{exint}\left(\frac{1}{2}\left|\frac{1}{h}x - m\right|^2\right)\right\}$$

we obtain for $x \neq hm$

$$A_{3j}^{m,h}(x) = -\frac{\partial}{\partial x_j}S_{m,h}(x) = \pi h\left(\frac{x_j}{h} - m\right)\frac{1 - \exp\left(-\frac{1}{2}\left|\frac{x}{h} - m\right|^2\right)}{\frac{1}{2}\left|\frac{x}{h} - m\right|^2},$$

and for $x = hm$

$$A_{3j}^{m,h}(hm) = 0.$$

This proves the theorem.

6 Numerical Simulation for the Stokes System

In the following we present some numerical simulations using the formula of Chapter 5 for the 2-d Stokes system. In this case, for $3 \leq \beta \in \mathbb{N}$ we consider the test functions $u = (u_1, u_2)^T$ and p defined by:

$$u_1(x_1, x_2) := 4x_2\left(\frac{16}{3}\right)^{2\beta-1}\left(\frac{1}{4} - x_1^2\right)^{\beta}\left(\frac{1}{4} - x_2^2\right)^{\beta-1} \qquad \text{in} \quad \overline{Q},$$

$$u_2(x_1, x_2) := -4x_1\left(\frac{16}{3}\right)^{2\beta-1}\left(\frac{1}{4} - x_1^2\right)^{\beta-1}\left(\frac{1}{4} - x_2^2\right)^{\beta} \qquad \text{in} \quad \overline{Q},$$

$$p(x_1, x_2)^T := 16^{\beta-1}\left(\frac{1}{4} - x_1^2\right)^{\beta-1}\left(\frac{1}{4} - x_2^2\right)^{\beta-1} \qquad \text{in} \quad \overline{Q},$$

Here, Q denotes the open 2-d unit square as in Chapter 4, and we set $u_1 := 0,\ u_2 := 0,\ p := 0$ in $\mathbb{R}^2\backslash\overline{Q}$. The Figures 5, 6 bellow shows an illustration of each component of the function u.

An easy calculation shows that u is divergence-free in $\mathbb{R}^2$. Hence the functions u_1, u_2, p represent a solution of the 2-d Stokes system $-\Delta u + \nabla p = f$ in $\mathbb{R}^2$, div $u = 0$

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

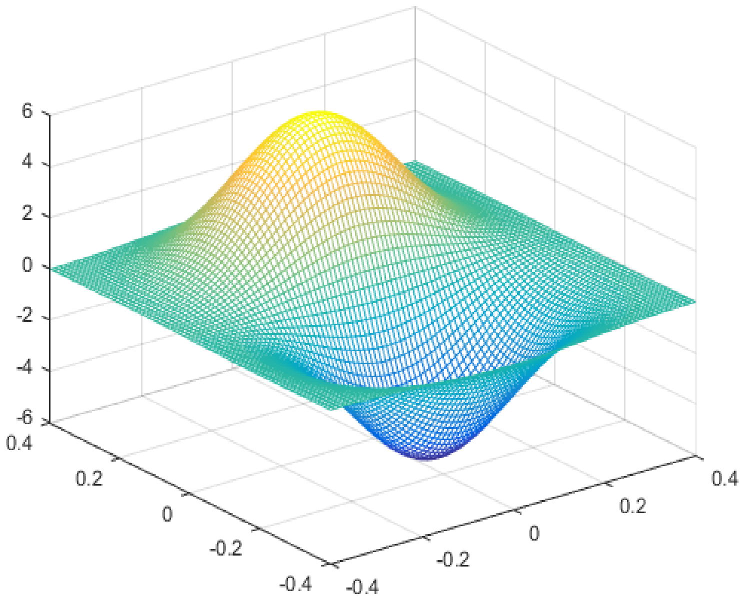

Figure 5: u_1 for $\beta = 5$

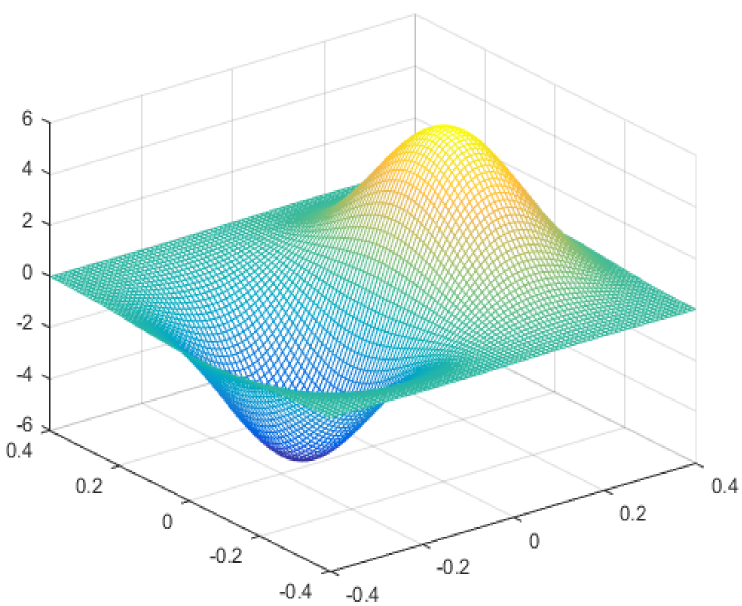

Figure 6: u_2 for $\beta = 5$

in $\mathbb{R}^2$, with $f := -\Delta u + \nabla p$ in $\mathbb{R}^2$. Setting $h_i := \frac{1}{4} - x_i^2$ for $i = 1, 2$ and defining $j \in \{1, 2\}$ by $j := 1$ if $i = 2$ and $j := 2$ if $i = 1$, a lengthy calculation yields for $i = 1, 2$:

$$f_i(x) = f_i(x_1, x_2) = (-1)^i 8 \left(\frac{16}{3}\right)^{2\beta-1} x_j h_i^{\beta-2} h_j^{\beta-3} \left[(\beta-1) h_i^2 \left\{2(\beta-2) x_j^2 - 3h_j\right\}\right.$$

$$\left. + \beta h_j^2 \left\{2(\beta-1) x_i^2 - h_i\right\}\right] - 16^{\beta-1} 2(\beta-1) x_i h_i^{\beta-2} 2h_j^{\beta-1}.$$

The Figures 7, 8 bellow shows an illustration of each component of the function f.

The table 6.1 below shows the approximation error $\varepsilon_h := \max |u_i(x) - u_i^h(x)|$ (the results are identical for i = 1, 2), the maximum taken in 225 points in the unit square Q, for different step sizes and different values of β. The functions u_i^h have been evaluated with help of SageMath (www.sagemath.org).

$h \quad \backslash \quad \beta$	3	4	5	6
0.1	1.0011340e-00	1.0850504e-00	2.3241189e-00	4.2213356e-00
0.05	2.7613415e-01	3.4703009e-01	7.6838811e-01	1.4439411e-00
0.025	7.6513041e-02	9.2533098e-02	2.0751514e-01	3.9527866e-01
0.0125	1.997655e-02	2.3512980e-02	5.2915240e-02	1.0117698e-01
0.00625	4.6772811e-02	5.9022680e-03	1.3294823e-02	2.5444871e-02
0.003125	1.1713465e-02	1.4770762e-03	3.3278477e-03	6.3706901e-03
0.0015625	2.9296106e-03	3.6936276e-04	8.3222152e-04	1.5932702e-03

Table 6.1: Maximum error ε_h

From this table, an order $\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$ of approximation can be easily calculated:

2456-8686, 6(1), 2022: 159-199
https://doi.org/10.26524/cm128

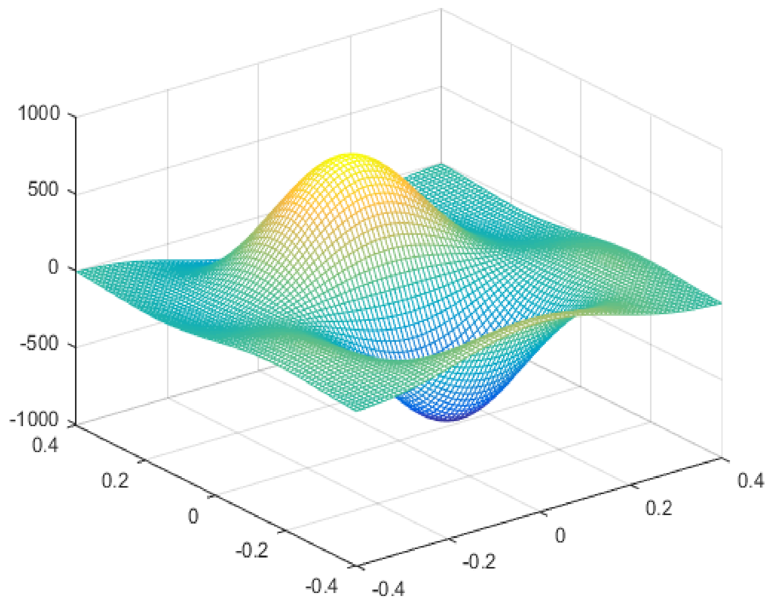

Figure 7: f_1 for $\beta = 5$

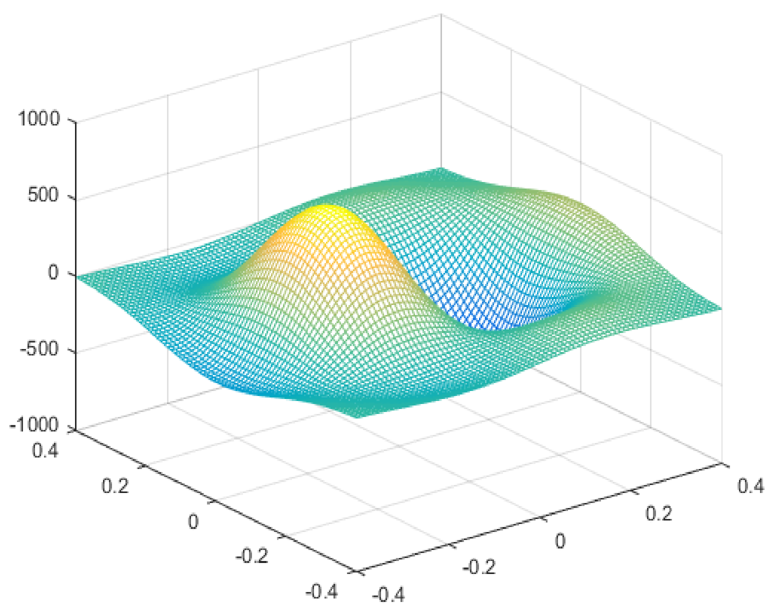

Figure 8: f_2 for $\beta = 5$

$h \quad \backslash \quad \beta$	3	4	5	6
0.05	1.8581938	1.6446293	1.5967767	1.5476876
0.025	1.8515917	1.9070193	1.8886185	1.8690699
0.0125	1.9373982	1.9765120	1.9714614	1.9659889
0.00625	1.9944283	1.9941160	1.9928187	1.9914343
0.003125	1.9975098	1.9985251	1.9982032	1.9978533
0.0015625	1.9993790	1.9996339	1.9995499	1.9994587

Table 6.2: Order of approximation α_h

7 Conclusion

The present paper deals with the numerical solution of two important problems in mathematical physics, i.e. the Poisson equation in 2 and 3 space dimensions and the Stokes system in 2 space dimensions. Using methods of potential theory based on the weakly singular fundamental solutions of the Poisson and the Stokes equations, the solution of these Cauchy problems can be represented by two- and three-dimensional convolution type integrals (volume potentials).

These volume potentials are calculated numerically, and in all cases, the simulations impressively confirm an approximation of essentially second order, which we prove for the method of approximate approximations used in this paper.

This method is based on an approximate partition of the unity with Gaussian bell curves as generating functions to approximate the given right hand sides of the partial differential equations considered here. The corresponding volume potentials with the approximated right hand sides can be calculated exactly, up to a one-dimensional integral.

The result is a non-convergent approximate approximation delivering an error estimate between exact and numerical solution in the form $\mathcal{O}(h^2) + \delta$ with a very small positive δ below machine precision. In all simulations this δ is not seen since the standard deviation $\sigma = 1$ of the Gaussian bell is large enough.

References

[1] F. Müller and W. Varnhorn, An Approximation Method using Approximate Approximations, Applicable Analysis, 85(6-7), 669–680 (2006).

[2] F. Müller and W. Varnhorn. Error estimates for approximate approximations with Gaussian kernels on compact intervals. Journal of Approximation Theory, 145(2), 171–181 (2007).

[3] F. K. G. Odquist. Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten. Math. Z., 32:, 329–375 (1930).

[4] F. Müller and W. Varnhorn, On Approximate Approximations of Potentials, PAMM, Proc. Appl. Math. Mech., 4, 710–711 (2004).

[5] I. Gradshteyn and I. Ryzhik, Table of Integrals, Series, and Products (in Russian), Handbook of Mathematical Formulas and Integrals, 4(6):589 (1963).

[6] S. König and W. Varnhorn, Approximate Approximations and a Boundary Point Method for the Linearized Stokes System, Int. J. Pure Appl. Math., 37(3), 441–451 (2007).

[7] T. Ivanov, V. Maz'ya, and G. Schmidt, Boundary Layer Approximate Approximations and Cubature of Potentials in Domains. Advances in Computational Mathematics, 10(3-4), 311–342 (1999).

[8] T. Samrowski and W. Varnhorn. Approximate Approximations for the Poisson and the Stokes equations. Appl. Math. Sci., Ruse, 2(17-20), 929–946 (2008).

[9] V. Maz'ya, A New Approximation Method and its Applications to the Calculation of Volume Potentials, Boundary point method, 3:18 (1991).

[10] V. Maz'ya and G. Schmidt, "Approximate Approximations" and the Cubature of Potentials. Weierstrass-Inst. für Angewandte Analysis und Stochastik (1995).

[11] V. Maz'ya and G. Schmidt, On Approximate Approximations using Gaussian kernels, IMA Journal of Numerical Analysis, 16(1), 13–29 (1996).

[12] V. Maz'ya and G. Schmidt, Construction of Basis Functions for High Order Approximate Approximations, Mathematical Aspects of Boundary Element Methods, 414:191 (1999).

[13] V. Maz'ya and G. Schmidt, Approximate Approximations., volume 141, Providence, RI: American Mathematical Society (AMS) (2007).

[14] V. Maz'ya, G. Schmidt, and W. Wendland, On the Computation of Multidimensional Single Layer Harmonic Potentials via Approximate Approximations, Calcolo, 40(1), 33–53 (2003).

[15] W. Varnhorn. Approximate Approximations and the Poisson Equation in the Whole Space. International Journal of Pure and Applied Mathematics, 49(4), 539–545 (2008).