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A Study on some Stochastic models in Time series

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Abstract

A first order autoregressive model is introduced which is an extension of the EAR(1) process of Gaver and Lewis (1980) if the marginals are exponentially distributed. Discrete version of the model is introduced and studied. Some applications of the models are also mentioned.

Key words: Characteristic function, Stochastic models, Markovian, Laplace transformation, Bernoulli random variable.

1. Introduction

Class - L distributions and geometrically infinitely divisible distributions have important role in time series modelling. A distribution with characteristic function $\phi(\lambda)$ is in class $-L$ if for each $\rho, 0<\rho<1, \phi(\lambda)/\phi(\rho n\lambda)$ is a characteristic function. A random variable Y is said to be geometrically infinitely divisible if for every $p \in (0,1), Y=\sum_{j=1}^{N_p} X_j^{(p)}$, where N_p is a geometric random variable such that$P\left\{N_p=k\right\}=p(1-p)^{k-1}, k=1,2\ldots$ and $\left\{X_j^{(p)}, j=1,2,\ldots\right\}$ are independent and identically distributed random variables and $Y, N_p, X_j^{(p)}$are independent.

Jayakumar and Pillai(1992) characterized semi α -Laplace distribution which was introduced by Pillai(1985). A distribution with characteristic function $f(\lambda)$ is called semi α-Laplace if $f(\lambda)=\frac{1}{1+\psi(\lambda)}$, where $\psi(\lambda)$ satisfies $\psi(\lambda)=a\psi(b\lambda), 0<b<1$ and a is the unique solution of $ab^{\alpha}=1$ for some $0<\alpha\leq 2$. For distributions with positive support, if $0<\alpha<1, f(\lambda)$ is the characteristic funtion of semi MittagLeffler distribution. Gaver and Lewis(1980) proved that only class L

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distributions can be marginals of stationary first order autoregressive equation $X_n = \rho X_{n-1} + \epsilon_n, 0 < \rho < 1$ and $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables. Here we introduce a model which is a generalization of this and is presented in section 2. The model is also solved by assuming exponential marginals under stationarity. The discrete version of this model is given in section 3. In section 4 some applications of the models are given.

2. The Autoregressive model

Let $\{X_n, n \geq 1\}$ be a discrete time stochastic process on $(0, \infty)$ defined by

$$X_n = \begin{cases} \rho X_{n-1} & \text{with probability} \quad p^2 \\ \epsilon_n & \text{with probability} \quad 2p(1-p) \\ \rho X_{n-1} + \epsilon_n & \text{with probability} \quad (1-p)^2 \end{cases} \tag{1}$$

where $0 < \rho \leq 1, 0 \leq p < 1$ and $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables such that ϵ_n is independent of X_{n-1}. Clearly if a solution exists, then the process is Markovian. Denoting the Laplace transform of the random variable X by $\phi_X(\lambda)$, from (1) we arrive at the following.

$$\phi_{X_n}(\lambda) = p^2 \phi_{X_{n-1}}(\rho\lambda) + 2p(1-p)\phi_{\epsilon_n}(\lambda) + (1-p)^2 \phi_{X_{n-1}}(\rho\lambda)\phi_{\epsilon_n}(\lambda) \tag{2}$$

If we assume that the process is stationary, we get

$$\phi_X(\lambda) = p^2 \phi_X(\rho\lambda) + 2p(1-p)\phi_\epsilon(\lambda) + (1-p)^2 \phi_X(\rho\lambda)\phi_\epsilon(\lambda) \tag{3}$$

Now we have the following theorem.

Theorem 2.1 For $\rho = 1$ the model (1) exists under stationarity if and only if X is geometrically infinitely divisible.

proof Assume that the model (1) exists under stationarity. Then (3) with $\rho = 1$ has solution for all $p \in (0, 1)$. From (3) we get,

$$\phi_X(\lambda) = \frac{b\phi_\epsilon(\lambda)}{1 - (1-b)\phi_\epsilon(\lambda)} \tag{4}$$

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where $b = \frac{2p(1-p)}{(1-p^2)}$. Clearly $0 < b < 1$. Thus $X \stackrel{d}{=} \sum_{j=1}^{N_b} Y_j^{(b)}$, where N_b is a geometric random variable such that $P\{N_b = n\} = b(1-b)^{n-1}$. and $\left\{Y_j^{(b)}, j \geq 1\right\}$ are independent and identically distributed random variables such that $Y_1^{(b)} \stackrel{d}{=} \epsilon_n$. Hence X is geometrically infinitely divisible.

Conversely assume that X is geometrically infinitely divisible. Then for each $q \in (0,1)$, there exists a sequence of independent and identically distributed random variables $\left\{Y_j^{(q)}, j \geq 1\right\}$ such that $X \stackrel{d}{=} \sum_{j=1}^{N_q} Y_j^{(q)}$, where N_q is a geometric random variable such that $P\{N_q = n\} = q(1-q)^{n-1}, n \geq 1$. Hence we have

$$\phi_X(\lambda) = \frac{q\phi_Y(\lambda)}{1-(1-q)\phi_Y(\lambda)} \tag{5}$$

where $\phi_Y(\lambda)$ is the Laplace transform of $Y_1^{(q)}$. Let p be such that $\frac{2p(1-p)}{(1-p^2)=q}$ Then (4) will imply (3) with $\rho = 1$ and $\epsilon \stackrel{d}{=} Y_1^{(q)}$. Hence the proof of the theorem is complete.

Also in (3) if we assume that $\epsilon_n \stackrel{d}{=} X$, we get

$$\phi_X(\lambda) = \frac{c\phi_X(\rho\lambda)}{1-(1-c)\phi_X(a\lambda)} \text{ where } c = \frac{p^2}{1-2p(1-p)}$$

That is, $X \stackrel{d}{=} a\sum_{j=1}^{N_c} X_j^{(c)}$ where $\{X_j, j \geq 1\}$ is a sequence of independent and identically distributed random variables which are identically distributed as X and N_c is a geometric random variable with $P(N_c = n) = c(1-c)^{n-1}, n \geq 1$ Thus X is the geometric sum of its own type and by Jayakumar and Pillai(1993), we have the following theorem.

Theorem 2.2 Assume that the process in (1) is stationary. Then X has semi-Mittag - Leffler distribution with exponent $\alpha \in (0,1)$ if and only if $X_n \stackrel{d}{=} \epsilon_n$.

Remark 2.3 If the distribution of X_n has real support then Theorem 2.2 hold good, with X_n distributed as semi α -Laplace.

The bivariate Laplace transform of (X_n, X_{n-1}) is given by

$$\begin{aligned}\phi_{X_n,X_{n-1}}(\lambda_1,\lambda_2) =& E\{\exp[-\lambda_1 X_1 - \lambda_2 X_2]\} \\ =& p^2\phi_{X_{n-1}}(\rho\lambda_1+\lambda_2) + 2p(1-p)\phi_{\epsilon_n}(\lambda_1)\phi_{X_{n-1}}(\lambda_2) \\ &+ (1-p)^2\phi_{\epsilon_n}(\lambda_1)\phi_{X_{n-1}}(\rho\lambda_1+\lambda_2)\end{aligned}$$

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From this it is clear that the process is not time reversible.

Now we shall prove that under stationarity if the marginals are exponentially distributed, there exists an innovation sequence such that the model (1) is properly defined. From (3) we get

$$\phi_\epsilon(\lambda) = \frac{\phi_X(\lambda) - p^2\phi_X(\rho\lambda)}{2p(1-p) + (1-p)^2\phi_X(\rho\lambda)} \tag{6}$$

If X is exponential with mean s, we have $\phi_X(\lambda) = \frac{1}{1+s\lambda}$. Therefore,

$$\begin{aligned}\phi_\epsilon(\lambda) &= \frac{\frac{1}{1+s\lambda} - \frac{p^2}{1+\rho s\lambda}}{2p(1-p) + (1-p)^2\frac{1}{1+\rho s\lambda}} \\ &= \frac{\delta}{1+s\lambda} + \frac{1-\delta}{1+ds\lambda}\end{aligned}$$

where $\delta = \frac{1-\rho}{(1+p)[1-p(1+2\rho)]}$ and $d = \frac{2p\rho}{1-p}$. Thus the innovations are mixtures of exponentials. Now we have the following.

$$\begin{array}{ll} EX_{n-1} = s & \operatorname{var}(X_{n-1}) = s^2 \\ EX_n = s\left(1+4\rho p^2\right) & \operatorname{Var}(X_n) = s^2 + \frac{8s^2p^3\rho}{1-\rho^2}\left[p+2\rho-p\rho+2p^3\rho\right] \end{array}$$

$$\begin{aligned} E(X_nX_{n-1}) &= s^2(1+\rho-2p\rho+6p^2\rho) \\ \operatorname{cov}(X_n, X_{n-1}) &= s^2\rho\left(1-2p+2p^2\right) \\ \operatorname{corr}(X_n, X_{n-1}) &= \frac{\rho\left(1-2p+2p^2\right)}{\left[1+\frac{8p^3\rho}{1-p^2}\left(p+2\rho-p\rho+2p^3\rho\right)\right]^{1/2}} \end{aligned}$$

Clearly the correlation lies between 0 and 1.

Remark 2.4 If $0 < p < \frac{-\rho+\left(3a^2+a\right)^{1/2}}{a+1}$ then we get $0 < \delta < 1$

Remark 2.5 If $p = 0$, we get EAR(1) model of Gaver and Lewis (1980).

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3. The Integer Valued First-Order Autoregressive model

Here wendefine and study the discrete version of the model in section 2. Let $\{X_n, n \geq 1\}$ be a non-negative integer valued stochastic process defined by

$$X_n = \begin{cases} \rho * X_{n-1} & \text{with probability} \quad p^2 \\ \epsilon_n & \text{with probability} \quad 2p(1-p) \\ \rho * X_{n-1} + \epsilon_n & \text{with probability} \quad (1-p)^2 \end{cases} \tag{7}$$

where $0 < \rho \leq 1, 0 \leq p < 1$ and $\{\epsilon_n\}$ is a sequence of independent and identically distributed innovations taking non-negative integer values such that ϵ_n is independent of X_{n-1} and $\rho * X_{n-1} = \sum_{j=1}^{X_{n-1}} N_j$ where N_j 's are independent and identically distributed Bernoulli random variables with
$P(N_j = 1) = \rho = 1 - P(N_j = 0), 0 < \rho < 1$

Now we will show that in the stationary case the model (7) is properly defined by assuming geometric marginals. The probability generating function of (7) is given by the following.

$$G(s) = p^2 G(1-\rho+\rho s) + 2p(1-p)G_\epsilon(s) + (1-p)^2 G(1-\rho+\rho s)G_\epsilon(s)$$
$$G_\epsilon(s) = \frac{G(s) - p^2 G(1-\rho+\rho s)}{2p(1-p) + (1-p)^2 G(1-\rho+\rho s)}$$

If $P\{X_n = n\} = b(1-b)^n, n = 0, 1, \ldots$ then $G(s) = \frac{b}{1-(1-b)s}$. Hence after some calculations, we get

$$G_\epsilon(s) = \delta \frac{b}{1-(1-b)s} + (1-\delta) \frac{\gamma}{1-(1-\gamma)s}$$

where δ is as in section 2 and $\gamma = b/\left(b + \frac{2(1-b)\rho p}{1-p}\right)$

4. Conclusion

In this paper we discussed the autoregressive model, the integer valued first order autoregressive model and proved some theorems.

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