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A Study On Entropy as Funtional Variant Of Mean Deviation

Syed Tahir Hussainy[1], Murugesan P[2]
[1]Associate professor, Department of Mathematics, Islamiah College (Autonomous) Vaniyambadi 635 752, Tirupattur District, Tamil Nadu India.
[2]Research scholar, Department of Mathematics, Islamiah College (Autonomous) Vaniyambadi 635 752, Tirupattur District, Tamil Nadu India.

Abstract

Under the given probability distribution and known expectation of a random variable, the functional variants of a particular type of weighted deviation have been maximized and many well known entropies in the existing literature have been obtained as functional variants of the mean deviation in Euclidean normed space.

Key words: Euclidean normed space, Functional variants, Mean deviation, Maximum, Lagrangian.

AMS classification:

1. Introduction

Let X be a discrete random variable taking finite number of possible values $x_1, x_2, \ldots, x_m$ with probabilities $p_1, p_2, \ldots, p_m \, (p_1 \geq 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} p_i = 1)$. Let $\mathbb{R}^m$ be the Euclidean m -dimensional real normed space. Let $\Delta_m = \{P = (p_1, p_2, \ldots, p_m), p_i \geq 0, \sum_{i=1}^{m} p_i = 1\} \subset \mathbb{R}^m$ be the set of complete finite discrete probability distributions.

Let $w_1, w_2, \ldots, w_m \; (w_i \geq 0, i = 1, 2, \ldots, m)$ be the weights quantifying the quality of $x_1, x_2, \ldots, x_m$ respectively, then $W = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$, If c is a real constant then vector $(c, c, \ldots, c) \in \mathbb{R}^m$ is denoted by the same letter c. If $Y = (y_1, y_2, \ldots, y_m)$ and $Z = (z_1, z_2, \ldots, z_m)$ are from $\mathbb{R}^m$ and f is real valued function defined on $\mathbb{R}$ then we denote $f(Y) = (f(y_1), f(y_2), \ldots, f(y_m)) \in \mathbb{R}^m$.

[1]*small1maths69@gmail.com,[2]murugesh5683@gmail.com

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Guiasu defined the deviation of Y from Z weighted by W by

$$D(Y:Z/W)=\sum_{i=1}^{m} w_i\left(y_i-z_i\right) \tag{1}$$

And f -variant of the deviation of Y and Z weighted by W by

$$D(f(Y):f(Z)/W)=\sum_{i=1}^{m} w_i\left(f\left(y_i\right)-f\left(z_i\right)\right) \tag{2}$$

The following results are direct consequences of (1)

$$D(1:P/P)=1-\sum_{i=1}^{m} P_i^2=1-\|P\|^2 \tag{3}$$

Where $\|\cdot\|$ is Euclidean norm in $\mathbb{R}^m$.

$$D(1:Q/F)=1-\sum_{i=1}^{m} p_i q_i=1=\langle p/q\rangle \tag{4}$$

Where $\langle\cdot/\cdot\rangle$ is inner product in $\mathbb{R}^m$.

$$D(P:Q/P)+D(Q:P/Q)=\sum_{i=1}^{m}\left(p_i-q_i\right)^2=\|P-Q\|^2 \tag{5}$$

(the Euclidean distance between P and Q).
When $W=P$, the corresponding weighted deviation becomes a mean deviation with respect to P.

In section 2 , the functional variants of (1,3) have been maximized.

In section 3, the particular cases of these functional variants have been discussed and different well known measures of entropy are obtained.

2. Maximizng Functional Variants

We maximize

$$D(f(1):f(P)/P)=\sum_{i=1}^{m}\left[f(1)-f\left(p_i\right)\right] p_i \tag{6}$$

[1*]small1maths69@gmail.com,[2]murugesh5683@gmail.com

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subject to the constraints

$$\sum_{i=1}^{m} p_i = 1 \text{ and } \sum_{i=1}^{m} x_i p_i = \mu \tag{7}$$

where μ is the expectation of the random variable X. The corresponding Lagrangian is

$$L = \sum_{i=1}^{m} \left(f(1) - f\left(p_i\right)\right) p_i - \nu \left(\sum_{i=1}^{m} p_i - 1\right) - \delta \left(\sum_{i=1}^{m} x_i p_i - \mu\right) \tag{8}$$

Differentiating with respect to $p_i (i = 1, 2, \ldots, m), \Upsilon$ and δ, the Lagrange's equation are:

$$f\left(p_i\right) + p_i f'\left(p_i\right) = f(1) - \nu - \delta x_i \tag{9}$$

$\sum_{i=1}^{m} p_i = 1$ and $\sum_{i=1}^{m} x_i p_i = \mu$
Taking $g\left(p_i\right) = \left[p_i f\left(p_i\right)\right]'$, (9) becomes

$$g\left(p_i\right) = K \tag{10}$$

Where $K = f(1) - \nu - \delta x_i$.
A unique value of p_i will exist if the function g is strictly monotonic on (0,1) and this happens only if f is twice differentiable on (0,1) and the derivative

$$g'\left(p_i\right) = 2f'\left(p_i\right) + p_i f''\left(p_i\right) \tag{11}$$

Is either strictly positive or strictly negative on $(0, 1)$. Thus equation (10) gives $p_i = g^{-1}(K)$ which is the required solution. Below we discuss the concavity of the functional variant $D(f(1) : f(P)/P)$: We have

$$D(f(1) : f(P)/P) = \sum_{i=1}^{n} \psi\left(p_i\right) \tag{12}$$

Where

$$\psi\left(p_i\right) = \left(f(1) - f\left(p_i\right)\right) p_i \tag{13}$$

Now

$$\psi''\left(p_i\right) = -g'\left(p_i\right) \tag{14}$$

Obviously $\psi\left(p_i\right)$ is a concave function of p_i if $g'\left(p_i\right) > 0$ on (0,1).
Thus equation (12) implies that $D\left(f(1) : f(P/P)\right.$ is concave function of $p_i; i = 1, 2, \ldots, m$.

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Hence any solution of (6) found by the method of differential calculus yields an absolute maximum rather than relative maximum.

Particular Cases

I) Taking

$$f(t) = \log t, t \in (0,1) \tag{15}$$

Equation (6) becomes

$$D(f(1) : f(P)/P) = -\sum_{i=1}^{m} \log p_i$$

Which is Shannon's entropy.
Guiasu called it logarithmic variant of $1 - \|P\|^2$.
II) Taking

$$F(t) = \frac{1+\lambda t}{\lambda t} \log\left(\frac{1+\lambda t}{1+\lambda}\right), t \in (0,1), \lambda > 0 \tag{16}$$

Equation (6) becomes

$$D(f(1) : f(P)/P) = -\frac{1}{\lambda}\sum_{i=1}^{m} (1+\lambda p_i) \log\left(\frac{1+\lambda p_j}{1+\lambda}\right)$$

Which is Ferrari's entropy.
III) Taking

$$f(t) = \frac{t^{r-1}-1}{r-1}, r \neq 1, r > 0 \tag{17}$$

Equation (6) becomes

$$D(f(1) : f(P)/P) = \frac{\sum_{i=1}^{m} p_i^r - 1}{1-r}, \text{ which is}$$

Ha varda and Charvat's entropy.
IV) Taking

$$f(t) = \log t - \left(\frac{1+at}{at}\right) \log(1+at), a \neq 0, a \geq -1 \tag{18}$$

[1*]small1maths69@gmail.com,[2]murugesh5683@gmail.com

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Equation (6) becomes

$$D(f(1):f(P)/P) = -\sum_{i=1}^{m} p_i \log p_i + \frac{1}{a}\left[\sum_{i=1}^{m}(1+ap_i)\log(1+ap_i) \right.$$
$$\left. -(1+a)\log(1+a)p_i\right]$$

Which is Kapur's entropy of first type.
V) Taking

$$f(t) = \log t - \frac{1}{b^2 t}[(1+bt)\log(1+bt) - (1+b)\log(1+b)], b > 0 \tag{19}$$

equation (6) becomes

$$D(f(1):f(P)/P) = -\sum_{i=1}^{m} p_i \log p_i + \frac{1}{b^2}\left[\sum_{i=1}^{m}(1+bp_i)\log(1+bp_i) \right.$$
$$\left. -(1+b)\log(1+b)\right]$$

Which is Kapur's entropy of second type.
VI) Taking

$$F(t) = \frac{1}{\alpha - \beta}\left[t^{\alpha-1} - t^{\beta-1}\right], 0 < \alpha < 1, \beta > 1 \text{ or } \alpha > 1, 0 < \beta < 1, \tag{20}$$

Equation (6) becomes

$$D(f(1):f(P)/P) = \frac{1}{\beta - \alpha}\left[\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta}\right]$$

Which is Sharma and Taneja's entropy.
It can be easily verified that in all the cases discussed above, $g'(t) > 0$ and $f''(t)$ exists for $t \in (0,1)$ and for various values of the parameters specified. Next, we derive analytical expressions for p_i corresponding to the various functions discussed above:
I) The solution of equation (6) corresponding to (15) is

$$p_i = \exp(-1 - \nu - \delta x_i) \tag{21}$$

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Where $\sum_{i=1}^{m} p_i = 1$ and $\sum_{i=1}^{m} p_i x_i = \mu$, which is (7).
Using (7),(21) gives

$$p_i = \frac{\exp(-\delta x_1)}{\sum_{i=1}^{m} \exp(-\delta x_i)} \tag{22}$$

Where δ is the solution of

$$F(\delta) = \sum_{i=1}^{m} \frac{x_i \exp(-\delta x_i)}{\sum_{i=1}^{m} \exp(-\delta x_i)} - \mu \tag{23}$$

We can assume without any loss of generality that random value $x_1, x_2, \ldots, x_m$ are arranged in ascending order of magnitude, that is,

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_m \tag{24}$$

From (23) and (24), we have $F(-\infty) = x_m - \mu, F(0) = \bar{x} - \mu, F(\infty) = x_1 - \mu$.
Also

$$\begin{aligned} F'(\delta) &= -\left[\frac{\sum_{i=1}^{m} x_i^2 \exp(-\delta x_i)}{\sum_{i=1}^{m} \exp(-\delta x_i)} - \left\{\frac{\sum_{i=1}^{m} x_i \exp(-\delta x_i)}{\sum_{i=1}^{m} \exp(-\delta x_i)}\right\}^2\right] \\ &= -\sigma^2(X) \text{ where } \sigma^2(\cdot) \text{ denotes the variance of } X. \end{aligned}$$

Thus $F'(\delta) \leq 0$. Now $F'(\delta) = 0$ iff $x_1 = x_2 = x_3 = \cdots = x_m$, which is not feasible. Thus $F'(\delta) < 0$ and $F(\delta)$ is strictly decreasing function of δ decreases from $(x_m - \mu)$ to $(x_1 - \mu)$ as δ goes from $-\infty$ to ∞.
Now we discuss the different cases:
a) $F(\delta) = 0$ has no real solution if $\mu < x_1$ or $\mu > x_m$.
b) $F(\delta) = 0$ has positive real solution if $x_1 < \mu < \bar{x}$.
c) $F(\delta) = 0$ has negative real solution if $\bar{x} < \mu < x_m$.
d) $F(\delta) = 0$ is satisfied by $\delta = 0$ if $\bar{x} = \mu$.

Thus we conclude that if $x_1 < \mu < x_m$ then $F(\delta) = 0$ has a unique real solution. If $\mu = x_1$ or x_m, we get degererate probability distribution $(1, 0, 0, \ldots, 0)$ or $(0, 0, \ldots 0, 1)$. If $\mu = \bar{x}$, we get the uniform distribution $\left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$. If $\bar{x} < \mu < x_m$ then $\delta > 0$ and probability p_i decreases with i and on the other hand if $\bar{x} < \mu < x_m$ then $\delta < 0$ and the probability p_i increases with i.
II) The solution of equation (6) corresponding to (16) is

$$1 + \lambda p_i = (m + \lambda) \frac{\exp(-\delta x_i)}{\sum_{i=1}^{m} \exp(-\delta x_i)} \tag{25}$$

1*small1maths69@gmail.com,2murugesh5683@gmail.com

Where δ is the solution of

$$G(\delta)=\frac{\sum_{i=1}^{m} x_i \exp\left(-\delta x_i\right)}{\sum_{i=1}^{m} \exp\left(-\delta x_i\right)}-\frac{m\bar{x}+\lambda\mu}{m+\lambda}=0$$

Proceeding as above, we see that $G'(\delta)<0$ and $G(\delta)$ is strictly monotonic decreasing function of δ which decreases from $\left(x_m-\frac{m\bar{x}+\lambda\mu}{m+\lambda}\right)$ to $\left(x_1-\frac{m\bar{z}+\lambda\mu}{m+\lambda}\right)$ As δ goes from $-\infty$ to ∞.

The following cases arise:

a) $G(\delta)=0$ has no real solution if $\frac{m\bar{x}+\lambda\mu}{m+\lambda}<x_1$ or $\frac{m\bar{x}+\lambda\mu}{m+\lambda}>x_m$

b) $G(\delta)=0$ has positive real solution if $x_1<\frac{m\bar{x}+\lambda\mu}{m+\lambda}<\bar{x}$.

c) $G(\delta)=0$ has negative real solution if $\bar{x}<\frac{m\bar{x}+\lambda\mu}{m+\lambda}<x_m$

d) $G(\delta)=0$ is satisfied by $\delta=0$ if $\bar{x}=\frac{m\bar{x}+\lambda\mu}{m+\lambda}$.

Thus for $x_1<\frac{m\bar{x}+\lambda\mu}{m+\lambda}<x_m, G(\delta)=0$ has a unique real solution.

III) The solution of (6) corresponding to (17) and (18) are

$$P_i=\left[\frac{1-\left(\nu+\delta x_i\right)(r-1)}{r}\right]^{\frac{1}{r-1}} \tag{26}$$

And

$$\frac{1}{p_i}=\exp\left[\frac{1+a}{a}\log(1+a)+\nu+\delta x_i\right]-a; a\neq 0, a>-1 \tag{27}$$

Where ν,δ are obtained by introducing (26) and (27) into the constraints of (6).

No analytical solutions of (6) corresponding to (19) and (20) exist, though g^{-1} exists.

The solutions p_i from (26) and (27) may have negative values. In order to avoid that we have to add $p_i\geq 0(i=1,2,\ldots,m)$ to the constraints. Then by applying the Kuhn-Tucker conditions we obtain the expressions (26) and (27) respectively but only for those values of index i belonging to a subset $J\subset\{1,2,\ldots,m\}$ in which case the sum $\sum$ is taken with respect to $i\in J$ while for $i\notin J$, we have $p_i=0$.

3. Conclusion

In this paper, we discussed functional variants and concavity of functional variants. Also we derived lagrange's equation, in particular case we derived logarithmic variant also we derived entropy of first type and second type.

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[1*]small1maths69@gmail.com,[2]murugesh5683@gmail.com

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[1*]small1maths69@gmail.com,[2]murugesh5683@gmail.com