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Higher order $-\ell$ Alpha Difference Equations

Dominic Babu G[1], Rejin Rose D[2]
[1] Associate Professor [2]Research Scholar
Department of Mathematics, Annai Velankanni College,
Tholayavattam- 629157, Kanyakumari District.

Abstract

In this paper, we present some basic definitions and preliminary results $-\ell$ alpha difference operator and inverse. We derive the sum of infinite $-\ell$ alpha series and infinite $-\ell$ alpha multi-series formulae by equating summation and closed form of the generalized higher order $-\ell$ alpha difference equation.

Key words: Alpha Difference Equation, Alpha Difference Operator, Infinite Alpha Series, Infinite Alpha Multi-Series **AMS Classification:** 39A

1. Introduction

In 1984, Jerzy Popenda [1] introduced the difference operator $\Delta_\alpha$defined on $u(k)$ as as $\triangle_\alpha u\left(k\right) = u\left(k+l\right) - \alpha u(k)$. In 1989, Miller and Rose [2] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the inverse fractional difference operator ([3] [4]). Several formule on higher order partial sums on arithmetic, geometric progressions and products of n-consecutive terms of arithmetic progression have been derived in [5].

In 2011, M.Maria Susai Manuel, et.al, [7] extended the operator $\triangle_\alpha$ to generalized $\alpha-$difference operator as $\triangle_{\alpha(l)} v\left(k\right) = v\left(k+l\right) - \alpha v(k)$ for the real valued function $v(k)$. The generalized difference operator with n-shift values $\ell = (\ell_1, \ell_2, \ell_3, \cdots, \ell_n) \neq 0$ on a real valued function $v\left(k\right) : \mathbb{R}n \to \mathbb{R}$ is defined as

$$\Delta_{(\ell)} = v\left(k_1+\ell_1, k_2+\ell_2, \ldots\ldots, k_n+\ell_n\right) - v\left(k_1, k_2, \ldots.., k_2\right) \tag{1}$$

A linear generalized partial difference equation is of the form $\Delta_{(l)} v(k) = u(k)$, then the inverse of generalized partial difference equation

$$v\left(k\right) = \Delta^{-1} u\left(k\right) \tag{2}$$

Definition 1.1 Let $u(k)$ be a real valued function on $(-\infty, \infty)$ and $\ell \neq 0$. Then the

[1]dominicbabu202@gmail.com,[2]rejinrosejustes1980@gmail.com

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$-\ell$ alpha difference operator, denoted by $\underset{(-\ell)\alpha}{\Delta}$, on $u(k)$ is defined as

$$\underset{(-\ell)\alpha}{\Delta} v(k) = v(k-\ell) - \alpha v(k) \tag{3}$$

and the inverse of the $-\ell$ alpha difference operator, denoted by $\overset{-1}{\underset{(-\ell)\alpha}{\Delta}}$, on $u(k)$ is defined as if $\underset{(-\ell)\alpha}{\Delta} v(k) = u(k)$, then

$$v(k) = \Delta^{-1}_{(-l)\alpha} u(k) \tag{4}$$

Result 1.2 Let $k \in (-\infty,\infty)$ and $\alpha \neq 0$. Then we have

$$(i)\ \Delta^{-1}_{(-\ell)\alpha}\left(k^0\right) = \frac{1}{(k^{-\ell}-\alpha)} \tag{5}$$

$$(ii)\ \Delta^{-1}_{(-\ell)\alpha}\left(e^k\right) = \frac{e^k}{(e^{-\ell}-\alpha)} \tag{6}$$

Proof: The proof follows by replacing $u(k)$ by k^0 and e^k respectively in equation (3) and applying (4)

2. Solution of $-\ell$ alpha difference equation

Theorem 2.1 Let $l \neq 0$, $m \in N(1)$ and $u(k)$ be a real valued function on (∞,∞). Then we have

$$\overset{-1}{\underset{(-\ell)\alpha}{\Delta}} u(k-\ell) - \alpha^{m+1} \overset{-1}{\underset{(-\ell)\alpha}{\Delta}} u(k+m\ell) = \sum_{p=0}^{m} \alpha^p\, u(k+p\ell) \tag{7}$$

is a solution of $-\ell$ alpha difference equation $\underset{(-\ell)\alpha}{\Delta} v(k) = u(k)$ and hence

$$\alpha^s \overset{-1}{\underset{(-l)\alpha}{\Delta}} u(k+(s-1)l) - \alpha^{m+1} \overset{-1}{\underset{(-l)\alpha}{\Delta}} u(k+ml) = \sum_{p=s}^{m} \alpha^p\, u(k+pl), \text{for } s < m. \tag{8}$$

Proof: By definition (1), we have $\Delta_{(-\ell)\alpha} v(k) = v(k+\ell) - \alpha v(k)$

$$v(k-\ell) = u(k) + \alpha v(k) \tag{9}$$

Replacing k by $k+\ell$ in equation (9) we get,

$$v(k) = u(k+l) + \alpha v(k+l) \tag{10}$$

[1]dominicbabu202@gmail.com,[2]rejinrosejustes1980@gmail.com

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Replacing k by $k+2\ell, k+3\ell,\ k+4\ell\ldots\ldots\ k+(m-1)\ell$ in equation (9) continuously and substituting the resultant expression in (9), we get

$$v\,(k-\ell) = u\,(k) + \alpha\ u\,(k+\ell) + \alpha^2\ u\,(k+2\ell) + \alpha^3\ u\,(k+3\ell) + \alpha^4\ u\,(k+4\ell)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots + \alpha^m\ u(k+ml) + \alpha^{m+1}\ v(k+ml)$$

$v\,(k-l) - \alpha^{m+1}\ v(k+ml) = \ \sum_{p=0}^{m} \alpha^p\ u(k+pl)$

$$\Delta^{-1}_{(-\ell)\alpha}\ u\,(k-\ell) - \alpha^{m+1}\ \Delta^{-1}_{(-\ell)\alpha}\ u\,(k+m\ell) = \sum_{p=0}^{m} \alpha^p\ u(k+p\ell) \tag{11}$$

Replacing m by $s-1$ in equation (11) where $s<m$, we arrive

$$\overset{-1}{\underset{(-\ell)\alpha}{\Delta}} u\,(k-\ell) - \alpha^s \overset{-1}{\underset{(-\ell)\alpha}{\Delta}} u\,(k+(s-1)\ell) = \sum_{p=0}^{s-1} c\alpha^p u\,(k+p\ell) \tag{12}$$

Subtracting (12) from (11), we get

$\alpha^s \Delta^{-1}_{(-l)\alpha}\ u\,(k+(s-1)l) - \alpha^{m+1}\ \Delta^{-1}_{(-l)\alpha}\ u\,(k+ml) = \sum_{p=s}^{m} \alpha^p\ u(k+pl)$, for $s<m$.

Corollary 2.2 Let $\ell \neq 0$, $m \in\ N(1)$ and $u(k)$ be a real valued function on $(-\infty,\infty)$. Then we have

$$\overset{-1}{\underset{(-\ell)\alpha}{\Delta}} \left(e^{(k-\ell)}\right) - \alpha^{m+1} \overset{-1}{\underset{(-\ell)\alpha}{\Delta}} \left(e^{(k+m\ell)}\right) = \sum_{p=0}^{m} \alpha^p \left(e^{(k+p\ell)}\right)$$

is a solution of $-\ell$ alpha difference equation $\Delta_{(-\ell)\alpha}\ v\,(k) = u\,(k)$ and hence

$$\alpha^s \overset{-1}{\underset{(-\ell)\alpha}{\Delta}} \left(e^{(k+(s-1)\ell)}\right)\ - \alpha^{m+1} \overset{-1}{\underset{(-\ell)\alpha}{\Delta}}\ \left(e^{(k+m\ell)}\right) = \sum_{p=s}^{m} \alpha^p\ \left(e^{(k+p\ell)}\right),\ \text{for } s<m \tag{13}$$

Proof: The proof follows by $u(k) = e^k$ in (8).

Example 2.3 Here $s=4, m=7, \alpha=3, k=4, \ell=5$ in equation (13)

$$\alpha^4 \overset{-1}{\underset{(-\ell)\alpha}{\Delta}} \left(e^{(k+3\ell)}\right) - \alpha^8 \overset{-1}{\underset{(-\ell)\alpha}{\Delta}} \left(e^{(k+7\ell)}\right) = \sum_{p=4}^{7} \alpha^p \left(e^{(k+p\ell)}\right) \tag{14}$$

By result (1.2), we have

$$\overset{-1}{\underset{(-\ell)\alpha}{\Delta}} \left(e^{k+3\ell}\right) = \frac{e^{k+3\ell}}{(e^{-\ell}-\alpha)} \tag{15}$$

[1]dominicbabu202@gmail.com,[2]rejinrosejustes1980@gmail.com

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Using (15) in (14) we get,

$$\frac{\alpha^4\left(e^{(k+3l)}\right)}{(e^{-l}-\alpha)} - \frac{\alpha^8\left(e^{(k+7l)}\right)}{(e^{-l}-\alpha)} = \sum_{p=4}^{7}\alpha^p\ \left(e^{(k+pl)}\right) \tag{16}$$

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Theorem 2.4 Let $\ell \neq 0$, $m \in\ N(1)$ and $k \in -\infty, \infty$ and $\alpha \neq 0$. Then we have

$$\sum_{i=1}^{t-1}\alpha_{i+1}^{m+1}\sum_{(r)_{1\to i}}^{m}\prod_{p=1}^{i}\alpha_p{}^{r_p}\ \Delta^{-1}_{(-\ell,\alpha)_{i+1\to t}}\ u(\sum_{p=i+2}^{t}(k-\ell_p)+\sum_{p=1}^{i}(r_p\ell_p)+m\ell_{i+1})$$

$$+\sum_{(r)_{1\to t}}^{m}\prod_{p=1}^{t}\alpha_p{}^{r_p}\ u(k+\sum_{i=1}^{t}(r_i\ell_i))$$

$$=\underset{(-\ell,\alpha)_{1\to t}}{\overset{-1}{\Delta}}\ u\sum_{i=1}^{t}(k-\ell_i)-\alpha_1^{m+1}\underset{(-\ell,\alpha)_{1\to t}}{\overset{-1}{\Delta}}\ u(\sum_{p=2}^{t}(k-\ell_p+m\ell_i) \tag{17}$$

Proof: From theorem (2.1) we have,

$$\Delta^{-1}_{(-l)\alpha}\ u\,(k-l)-\alpha^{m+1}\ \Delta^{-1}_{(-l)\alpha}\ u\,(k+ml)=\sum_{r=0}^{m}\alpha^r\ u(k+rl)$$

Replacing l by l_1 and α by α_1 in previous equation

$$u\,(k)+\alpha_1\ u\,(k+l_1)+\alpha_1^2\ u\,(k+2l_1)+\alpha_1^3\ u\,(k+3l_1)+\dots\dots\dots\dots+\alpha_1{}^m\ u\,(k+ml_1)$$

$$=\Delta^{-1}_{(-l_1)\alpha_1}\ \ u\,(k-l_1)-\alpha_1{}^{m+1}\ \Delta^{-1}_{(-l_1)\alpha_1}\ \ u\,(k+ml_1)\ \ (1.19)$$

Replacing $u\,(k)$ by $\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k-l_2)$, $u\,(k+l_1)$ by $\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+l_1-l_2)$,

$u\,(k+2l_1)$by $\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+2l_1-l_2),\dots\ u\,(k+ml_1)$ by $\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+ml_1-l_2)$ in (1.19)

$$\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k-l_2)+\alpha_1\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+l_1-l_2)+\alpha_1^2\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+2l_1-l_2)$$

$$+\alpha_1^3\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+3l_1-l_2)+\cdots+\alpha_1^m\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+ml_1-l_2)$$

$$=\Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ u\,(k-l_1-l_2)-\alpha_1{}^{m+1}\ \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+ml_1-l_2) \quad (1.20)$$

$$\sum_{r=0}^{m}\alpha_1^r\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+rl_1-l_2)$$

[1]dominicbabu202@gmail.com,[2]rejinrosejustes1980@gmail.com

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$$= \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ u\,(k-l_1-l_2) - {\alpha_1}^{m+1}\ \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ u\,(k+ml_1-l_2)$$

Changing the subscripts 1 to 2 in (1.19)

$$u\,(k) + \alpha_2\ u\,(k+l_2) + \alpha_2^2\ u\,(k+2l_2) + \alpha_2^3\ u\,(k+3l_2) + \cdots + {\alpha_2}^m\ u\,(k+ml_2)$$

$$= \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k-l_2) - {\alpha_2}^{m+1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+ml_2) \tag{1.21}$$

Replacing k by $k+rl_1$ and multiplying $\alpha_1^r, r = 1, 2, 3, \ldots m$ in (1.21)

$$\alpha_1^r[u\,(k+rl_1) + \alpha_2\ u\,(k+rl_1+l_2) + \alpha_2^2\ u\,(k+rl_1+2l_2) + \cdots + {\alpha_2}^m\ u\,(k+rl_1+ml_2)]$$

$$= \alpha_1^r\left[\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+rl_1-l_2) - {\alpha_2}^{m+1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+rl_1+ml_2)\right] \tag{1.22}$$

Adding equation (1.21) and (1.22) for $r = 1, 2, 3, \ldots m$ and then applying (1.20), we arrive

$$u\,(k) + \alpha_1^r u\,(k+rl_1) + \alpha_2\ u\,(k+l_2) + \alpha_1^r\alpha_2\ u\,(k+rl_1+l_2) + \alpha_2^2\ u\,(k+2l_2)$$
$$+\alpha_1^r\alpha_2^2\ u\,(k+rl_1+2l_2) + \ldots + {\alpha_2}^m\ u\,(k+ml_2) + \alpha_1^r{\alpha_2}^m\ u\,(k+rl_1+ml_2)$$
$$= \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k-l_2) + \alpha_1^r\Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+rl_1-l_2) - {\alpha_2}^{m+1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+ml_2)$$
$$-\alpha_1^r{\alpha_2}^{m+1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ u\,(k+rl_1+ml_2)$$

$$\sum_{r_1=0}^{m}\sum_{r_2=0}^{m}\alpha_1^{r_1}\alpha_2^{r_2}\ u\,(k+r_1l_1+r_2l_2) = \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ u\,(k-l_1-l_2) - {\alpha_1}^{m+1}$$

$$\Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ u\,(k+ml_1-l_2) - {\alpha_2}^{m+1}\sum_{r_1=0}^{m}\alpha_1^{r_1}\Delta^{-1}_{(l_2)\alpha_2}\ u\,(k+r_1l_1+ml_2) \tag{1.23}$$

Changing the subscripts 1 to 2 and 2 to 3 in (1.23)

$$\sum_{r_2=0}^{m}\sum_{r_3=0}^{m}\alpha_2^{r_2}\alpha_3^{r_3}\ u\,(k+r_2l_2+r_3l_3) = \Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\,(k-l_2-l_3) - {\alpha_2}^{m+1}$$

$$\Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\,(k+ml_2-l_3) - {\alpha_3}^{m+1}\sum_{r_2=0}^{m}\alpha_2^{r_2}\Delta^{-1}_{(-l_3)\alpha_3}\ u\,(k+r_2l_2+ml_3) \tag{1.24}$$

Replacing k by $k+rl_1$ and multiplying $\alpha_1^r, r = 1, 2, 3, \ldots m$ in (1.24) and then adding

Corresponding expressions, we arrive

$$\sum_{r_1=0}^{m}\sum_{r_2=0}^{m}\sum_{r_3=0}^{m}{\alpha_1}_1^{r_1}{\alpha_2}^{r_2}{\alpha_3}^{r_3}u\,(k+r_1l_1+r_2l_2+r_3l_3) = \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\,(k-l_1-l_2-l_3)$$

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$$-{\alpha_1}^{m+1}\ \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\left(k+ml_1-l_2-l_3\right)$$

$$-{\alpha_2}^{m+1}\sum_{r_1=0}^{m}\alpha_1^{r_1}\Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\left(k+r_1l_1+ml_2-l_3\right)$$

$$-{\alpha_3}^{m+1}\sum_{r_1=0}^{m}\sum_{r_2=0}^{m}\alpha_1^{r_1}\alpha_2^{r_2}\Delta^{-1}_{(-l_3)\alpha_3}\ u\left(k+r_1l_1+r_2l_2+ml_3\right)$$

Continuing the above process and rearranging the terms, we get proof.

Theorem: 2.5. Let α be a non zero real, $l \neq 0$ *and* $k \in (-\infty,\infty)$. Then we have

$$\sum_{r=0}^{m}\binom{r+t-1}{t-1}\alpha^r u\left(k+rl\right)+\alpha^{m+1}\sum_{r=1}^{t-1}\frac{(m+r)^{(r)}}{r!}\Delta^{-(t-r)}_{(l)\alpha}u\left(k+(t-r)\,l+(m+1)\,l\right)$$

$$=\Delta^{-1}_{(l)\alpha}u\left(k+tl\right)-\alpha^{m+1}\Delta^{-t}_{(l)\alpha}u\left(k+tl+(m+1)\,l\right) \qquad (1.25)$$

Proof: From theorem (2.1.) we have

$$u\left(k\right)+\alpha\ u\left(k+l\right)+\alpha^2\ u\left(k+2l\right)+\alpha^3\ u\left(k+3l\right)+\alpha^4\ u\left(k+4l\right)+........+\alpha^m\ u(k+ml)$$

$$=\Delta^{-1}_{(-l)\alpha}\ u\left(k-l\right)-\alpha^{m+1}\ \Delta^{-1}_{(-l)\alpha}\ u\left(k+ml\right) \qquad (1.27)$$

Replacing k by $k+l$ and multiplying by α in equation (1.27)

$$\alpha u\left(k+l\right)+\alpha^2\ u\left(k+2l\right)+\alpha^3\ u\left(k+3l\right)+\alpha^4\ u\left(k+4l\right)+...........+\alpha^{m+1}\ u(k+(m+1)l)$$

$$=\alpha\Delta^{-1}_{(-l)\alpha}\ u\left(k\right)-\alpha^{m+2}\ \Delta^{-1}_{(-l)\alpha}\ u\left(k+(m+1)l\right) \qquad (1.28)$$

Replacing k by $k+2l$ and multiplying by α^2 in equation (1.27)

$$\alpha^2u\left(k+2l\right)+\alpha^3\ u\left(k+3l\right)+\alpha^4\ u\left(k+4l\right)+\alpha^5\ u\left(k+5l\right)+.........+\alpha^{m+2}\ u(k+(m+2)l)$$

$$=\alpha^2\Delta^{-1}_{(-l)\alpha}\ u\left(k+l\right)-\alpha^{m+3}\ \Delta^{-1}_{(-l)\alpha}\ u\left(k+(m+2)l\right) \qquad (1.29)$$

Replacing k by $k+3l$ and multiplying by α^3 in equation (1.27)

$$\alpha^3u\left(k+3l\right)+\alpha^4\ u\left(k+4l\right)+\alpha^5\ u\left(k+5l\right)+\alpha^6\ u\left(k+6l\right)+.........+\alpha^{m+3}\ u(k+(m+3)l)$$

$$=\alpha^3\Delta^{-1}_{(-l)\alpha}\ u\left(k+2l\right)-\alpha^{m+4}\ \Delta^{-1}_{(-l)\alpha}\ u\left(k+(m+3)l\right) \qquad (1.30)$$

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Again replacing k by $k+4l, k+5l, k+6l, \ldots$, and multiplying by $\alpha^4, \alpha^5, \alpha^6, \ldots$ in equation (1.27) repeatedly and then adding all resultant expressions, we have

$$\sum_{r=0}^{m} \binom{r+1}{1} \alpha^r u(k+rl) = \Delta^{-2}_{(-l)\alpha} u(k+2l) - \alpha^{m+1} [\Delta^{-2}_{(-l)\alpha} u(k+(m-1)l)$$

$$+ (m+1) \Delta^{-1}_{(-l)\alpha} u(k+ml)] \tag{1.31}$$

Applying the above process mentioned above to (1.31), we find that

$$\sum_{r=0}^{m} \binom{r+2}{2} \alpha^r u\Big(k+rl) = \Delta^{-3}_{(-l)\alpha} u(k+3l) - \alpha^{m+1} [\Delta^{-3}_{(-l)\alpha} u(k+(m-2)l)\Big)$$

$$+(m+1)\Delta^{-2}_{(-l)\alpha} u(k+(m-1)l) + \frac{(m+2)^{(2)}}{2} \Delta^{-1}_{(-l)\alpha} u(k+ml)] \tag{1.32}$$

Proceeding like this, we get proof of this theorem.

Corollary: 2.6.. Let $l \neq 0$, $m \in N(1)$ and $k \in (-\infty,\infty)$ and $\alpha \neq 0$. Then we have

$$\sum_{i=1}^{t-1} \alpha_{i+1}^{m+1} \sum_{(r)_{1\to i}}^{m} \prod_{p=1}^{i} \alpha_p{}^{r_p} \ \Delta^{-1}_{(-l,\alpha)_{i+1\to t}} \ e^{u(\sum_{p=i+2}^{t}(k-l_p)+\sum_{p=1}^{i}(r_p l_p)+ml_{i+1})}$$

$$+ \sum_{(r)_{1\to t}}^{m} \prod_{p=1}^{t} \alpha_p{}^{r_p} \ e^{u(k+\sum_{i=1}^{t}(r_i l_i))} \ = \ \Delta^{-1}_{(-l,\alpha)_{1\to t}} \ e^{u(\sum_{i=1}^{t}(k-l_i)} - \alpha_1^{m+1} \ \Delta^{-1}_{(-l,\alpha)_{1\to t}} e^{u(\sum_{p=2}^{t}(k-l_p+ml_i)}$$

Proof: The proof follows by taking $u(k) = e^k$ in (1.25).

Corollary: 2.7.. Let $l \neq 0$ *and* $k \in (-\infty,\infty)$. Then we have

$$\sum_{r=0}^{m} \binom{r+t-1}{t-1} u(k+rl) + \sum_{r=1}^{t-1} \frac{(m+r)^{(r)}}{r!} \Delta^{-(t-r)}_{(l)} u(k+(t-r)l-(m+1)l)$$

$$= \Delta^{-1}_{(l)} u(k+tl) - \Delta^{-t}_{(l)} u(k+tl+(m+1)l) \tag{1.33}$$

Proof: The proof follows by taking $\alpha = 1$ in (1.25).

Example: 2.8. Here $m=2, \alpha=3, k=4, l=2$ in Corollary: 2.6. we get,

$$e^4 + 6e^6 + 27e^8 = \frac{e^8}{(e^{-2}-3)^2} - 3^3 \left(\frac{e^6}{(e^{-2}-3)^2} - \frac{3e^8}{(e^{-2}-3)} \right)$$

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[1]dominicbabu202@gmail.com,[2]rejinrosejustes1980@gmail.com

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3. Summation formula of Alpha multi infinite series

Theorem: 3.1. Let $l, \alpha \neq 0$, and $u(k)$ be a real valued function on (-8,8). If

$$\lim_{q\to\infty} \frac{1}{\alpha^q} \Delta^{-1}_{(-l)\alpha}\, u\,(k-ql) \; = 0 \qquad (\ 1.34)$$

then $$\Delta^{-1}_{(-l)\alpha}\, u\,(k) = \frac{-1}{\alpha}\sum_{q=0}^{\infty}\frac{1}{\alpha^q}u\,(k-ql) \qquad (1.35)$$

Proof: By definition 1.1 we arrive

$$\Delta_{(-l)\alpha}\; v\,(k) = v\,(k-l) - \alpha v(k)$$

$$v\,(k) = \frac{1}{\alpha}\; v\,(k-l) - \frac{1}{\alpha}\; u\,(k) \qquad (1.36)$$

Replacing k by $k-l$ in equation (1.36) and then (1.36) becomes,

$$v\,(k-l) = \frac{1}{\alpha}\; v\,(k-2l) - \frac{1}{\alpha}\; u(k-l)$$

$$v\,(k) = \frac{1}{\alpha^2}\; v\,(k-2l) - \frac{1}{\alpha^2}\; u\,(k-l) - \frac{1}{\alpha}\; u\,(k) \qquad (1.37)$$

Replacing k by $k-2l$ in equation (1.36) and then (1.37) becomes,

$$v\,(k-2l) = \frac{1}{\alpha}\; v\,(k-3l) - \frac{1}{\alpha}\; u(k-2l)$$

$$v\,(k) = \frac{1}{\alpha^3}v\,(k-3l) - \frac{1}{\alpha^3}u\,(k-2l) - \frac{1}{\alpha^2}\; u\,(k-l) \qquad (1.38)$$

Again replacing k by $k-3l, k-4l, \ldots..$ in equation (1.36) repeatedly and putting the resultant expressions in (1.38), we arrive

$$v\,(k) = \frac{-1}{\alpha}\sum_{q=0}^{\infty}\frac{1}{\alpha^q}u\,(k-ql)$$

Which completes proof of this theorem.

Corollary: 3.2. Let $l, \alpha \neq 0$, and $u(k)$ be a real valued function on (-8,8). If

$$\lim_{q\to\infty}\; \frac{1}{\alpha^q}\Delta^{-1}_{(-l)\alpha}\; e^{(k-ql)} \quad = 0 \quad \text{then} \quad \Delta^{-1}_{(-l)\alpha}\, u\,(k) = \frac{-1}{\alpha}\sum_{q=0}^{\infty}\frac{1}{\alpha^q}e^{(k-ql)} \qquad (1.39)$$

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Proof: The proof follows by $u(k) = e^k$ in (1.35)

Theorem: 3.3. Let $k \in (-\infty,\infty)$ and $l_i, \alpha_i \neq 0$. If

$\lim_{s_i\to\infty} \frac{1}{{\alpha_i}^{s_i}} \Delta^{-1}_{(-l,\alpha)_{1\to i}}\ u\left(k - s_i l_i\right) \ = 0, for\ i = 1, 2, \ldots, t$ then we have

$$\sum_{(s)_{1\to t}}^{\infty} \prod_{p=1}^{t} {\alpha_p}^{-s_p}\ u\left(\sum_{p=1}^{t} \left(k - s_p l_p\right)\right) = (-1)^t \prod_{p=1}^{t} \alpha_p\ \ \Delta^{-1}_{(-l,\alpha)_{1\to t}} u\left(k\right) \quad (1.40)$$

Proof: From theorem (3.1.) we have

$$v\left(k\right) = \frac{-1}{\alpha} \sum_{s=0}^{\infty} \frac{1}{\alpha^s} u\left(k - sl\right) \quad (1.41)$$

Replacing l, α, s by l_2, α_2, s_2 in (1.41)

$$v\left(k\right) = \frac{-1}{\alpha_2} \sum_{s_2=0}^{\infty} \frac{1}{{\alpha_2}^{s_2}} u\left(k - s_2 l_2\right)$$

$$-\alpha_2 \Delta^{-1}_{(-l_2)\alpha_2}\ u\left(k\right) = \ u\left(k\right) + \frac{1}{\alpha_2} u\left(k - l_2\right) + \frac{1}{{\alpha_2}^2}\ u\left(k - 2l_2\right) + \frac{1}{{\alpha_2}^2}\ u\left(k - 3l_2\right) + \cdots + \infty \quad (1.42)$$

Replacing k by $k - s_1 l_1$, and dividing by ${\alpha_1}^{s_1}$, $s_1 = 1, 2, \ldots.\infty$ in (1.42)

$$\frac{-\alpha_2}{{\alpha_1}^{s_1}} \Delta^{-1}_{(-l_2)\alpha_2}\ u\left(k - s_1 l_1\right) = \frac{1}{{\alpha_1}^{s_1}} [\ u\left(k + s_1 l_1\right) + \frac{1}{\alpha_2} u\left(k - s_1 l_1 - l_2\right) + \frac{1}{{\alpha_2}^2}\ u\left(k - s_1 l_1 - 2l_2\right)$$

$$+ \frac{1}{{\alpha_2}^2}\ u\left(k - s_1 l_1 - 3l_2\right) + \cdots + \infty]$$

Summing the above equation with (1.42), we arrive

$$\sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \frac{u\left(k - s_1 l_1 - s_2 l_2\right)}{{\alpha_1}^{s_1}\ {\alpha_2}^{s_2}} = \left(-\alpha_2\right)\ \sum_{s_1=0}^{\infty} \Delta^{-1}_{(-l_2)\alpha_2}\ \frac{u\left(k - s_1 l_1\right)}{{\alpha_1}^{s_1}} \quad (1.43)$$

Applying (1.35) in (1.43), we obtain

$$\sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \frac{u\left(k - s_1 l_1 - s_2 l_2\right)}{{\alpha_1}^{s_1}\ {\alpha_2}^{s_2}} = \left(-\alpha_1\right)\left(-\alpha_2\right) \Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ u\left(k\right) \quad (1.44)$$

Changing the subscripts 1 to 2 and 2 to 3 in (1.44)

$$\sum_{s_2=0}^{\infty} \sum_{s_3=0}^{\infty} \frac{u\left(k - s_2 l_2 - s_3 l_3\right)}{{\alpha_2}^{s_2}\ {\alpha_3}^{s_3}} = \left(-\alpha_2\right)\left(-\alpha_3\right) \Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\left(k\right) \quad (1.45)$$

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Replacing k by $k - s_1 l_1$, and dividing by ${\alpha_1}^{s_1}$, $s_1 = 1, 2, \ldots \infty$ in (1.45) and then corresponding expressions

$$\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_3=0}^{\infty}\frac{u\left(k - s_1 l_1 - s_2 l_2 - s_3 l_3\right)}{{\alpha_1}^{s_1}\ {\alpha_2}^{s_2}\ {\alpha_3}^{s_3}}$$

$$= \left(-\alpha_1\right)\left(-\alpha_2\right)\left(-\alpha_3\right)\Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \Delta^{-1}_{(-l_3)\alpha_3}\ u\left(k\right) \tag{1.46}$$

Proceeding like this, we arrive

$$\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\cdots\sum_{s_t=0}^{\infty}\frac{u\left(k - s_1 l_1 - s_2 l_2 - \cdots - s_t l_t\right)}{{\alpha_1}^{s_1}\ {\alpha_2}^{s_2}\ \ldots {\alpha_t}^{s_t}}$$

$$= \left(-\alpha_1\right)\left(-\alpha_2\right)\ldots\left(-\alpha_t\right)\Delta^{-1}_{(-l_1)\alpha_1}\ \Delta^{-1}_{(-l_2)\alpha_2}\ \ldots\Delta^{-1}_{(-l_t)\alpha_t}\ u\left(k\right) \tag{1.47}$$

Which yields the theorem 3.3.

Corollary: 3.4. For any real valued function $u\left(k\right)$ on $(-\infty,\infty)$. If $\alpha, l_i \neq 0$

$\lim_{s_i\to\infty}\frac{1}{\alpha^{s_i}}\ \Delta^{-1}_{(-l,\alpha)_{1\to t}}\ u\left(k - s_i l_i\right) = 0, for\ i = 1, 2, \ldots, t$ then we have

$$\sum_{(s)_{1\to t}}^{\infty}\prod_{p=1}^{t}\alpha^{-s_p}\ u(\textstyle\sum_{p=1}^{t}\left(k - s_p l_p\right)) = \left(-\alpha\right)^t\ \Delta^{-1}_{(-l,\alpha)_{1\to t}} u\left(k\right) \tag{1.48}$$

Proof: Put $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha$ in the theorem 3.3.

Corollary: 3.5. For any real valued function $u\left(k\right)$ on $(-\infty,\infty)$. If $l_i \neq 0$

$\lim_{s_i\to\infty}\Delta^{-1}_{(-l)_{1\to i}}\ u\left(k - s_i l_i\right) = 0, for\ i = 1, 2, \ldots, t$ then we have

$$\sum_{(s)_{1\to t}}^{\infty} u(\sum_{p=1}^{t}\left(k - s_p l_p\right)) = \left(-1\right)^t\ \Delta^{-1}_{(-l)_{1\to t}} u\left(k\right) \tag{1.49}$$

Proof: Put $\alpha = 1$ in the equation (1.48)

Corollary: 3.6. For any real valued function $u\left(k\right)$ on $(-\infty,\infty)$. If

$\lim_{s_i\to\infty}\Delta^{-1}_{-l}\ u\left(k - s_i l\right) = 0, for\ i = 1, 2, \ldots, t$ then we have

$$\sum_{(s)_{1\to t}}^{\infty} u(\sum_{p=1}^{t}\left(k - s_p l\right)) = \left(-1\right)^t\ \Delta^{-1}_{-l} u\left(k\right) \tag{1.50}$$

Proof: Put $l_i = l$ in equation (1.49).

Conclusion: We have obtained alpha summation formula, alpha multi-series formulae by equating summation and closed form of higher order alpha difference equation which will be used to our further research.

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[1]dominicbabu202@gmail.com,[2]rejinrosejustes1980@gmail.com