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Generalized Discrete Finite Fourier Series with Two Parameter

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Abstract

In this paper, we obtain a Generalized Discrete Finite Fourier Series by using generalized difference operator with two parameters. Suitable examples are given to illustrate the results.

Key words: Difference Equation, Generalized Difference Operator, ℓ-Difference Operator, Fourier Series, Polynomial.

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1 Introduction

The Fourier Series is the most widely used series expansion in mathematics modeling of engineering systems. It serves as the basis for the Fourier integral, the Laplace transform, the solution of autonomous linear differential equations, frequency response methods and many engineering applications. There are many good treatments on the subject; too many to mention in a comprehensive manner. However, the treatment by Tolstov[9] is classical. Jerri[5] provides an excellent overview on convergence of the Fourier series and discusses Gibbs-like phenomena in continuous and discrete wavelet representations. Sidi[7] reviews the state of the art of extrapolation methods giving applied scientists and engineers a practical guide to accelerating convergence in difficult computational problems. Also, accelerated convergence by means of periodic bridge functions was developed by Anguelov[1].

A Fourier Series is a series of sines and cosines of an angle and its multiples of the form $\frac{a_0}{2} + a_1 cosk + a_2 cos2k + a_3 cos3k + ... + a_\omega cos\omega k + ... + b_1 sink + b_2 sin2k + b_3 sin3k ... + b_\omega sin\omega k ... = \frac{a_0}{2} + \sum_{\omega=1}^{\infty} a_\omega cos\omega k + \sum_{\omega=1}^{\infty} b_\omega sin\omega k.$ is called the

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Fourier series.Where a_1,a_2,...,a_ω,...b_1,b_2,...b_ω,... are constants.

A periodic function $f(x)$ can be expanded in a Fourier Series. The series consists of the following: (i) A constant term a_0. (ii) A component at the fundamental frequency determined by the values of a_1,b_1. (iii) Components of the harmonics (multiples of the fundamental frequency) determined by a_2,a_3,...b_2,b_3,... and a_0,a_1,a_2,...b_1,b_2,... are known as Fourier coefficients of Fourier constants. And the Fourier coefficients are defined by,
$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, $a_\omega = \frac{1}{\pi}\int_0^{2\pi} f(x)cos\omega x dx$ and $b_\omega = \frac{1}{\pi}\int_0^{2\pi} f(x)sin\omega x dx$.

2 Discrete Fourier series

The discrete Fourier series may be studied merely from a mathematical point of view. But, it is also of interest to know where it originated or where it is applied. As far as the author is aware, it originated in telecommunication engineering in connection with the digitalization of data (signals) in order to transmit them. Formerly, signals were transmitted in the form of waves that were a superposition of continuous periodic time functions[8]. Nowadays, the signals are digitalized that is to say, they are sampled at discrete points and are then transmitted. In this context, the fast Fourier transform (FFT) allows one to transmit the digitalized (discrete) signals much faster than the pertinent analog (continuous) signals. Thus, there is a very important application of the discrete Fourier series.

Definition 2.1 Let $S_\ell = \{\phi_0, \phi_1, \phi_2, ...\phi_\eta\}$ be an orthonormal on $I = [a, a + 2\pi]$, $\ell = \frac{\pi}{\eta}$, $u(k)$ is bounded function on I and $c_\omega = (u, \phi_\omega)_\ell$, say finite Fourier coefficients. Then the FFS of $u(k)$ related to S_ℓ is defined as

$$u(k) = \sum_{\omega=0}^{\eta} c_\omega \phi_\omega(k), \quad k \in \{a,\, a+\ell, a+2\ell, \cdots, a+(\eta-1)\ell\}. \tag{1}$$

3 Generalized finite Fourier series

In this section, we present some basic definitions and results.

Definition 3.1 [6] Let $u(k),\ k \in [0, \infty)$, be a real or complex valued function and

$\ell > 0$ be a fixed shift value. Then, the ℓ-difference operator Δ_ℓ on $u(k)$ is defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), \tag{2}$$

and its infinite $\ell-$ difference sum is defined by

$$\Delta_\ell^{-1} u(k) = \sum_{r=0}^{\infty} u(k+r\ell). \tag{3}$$

Lemma 3.2 [2] Let p be real $\ell > 0$, $k \in (\ell, \infty)$ and $p\ell \neq \beta 2\pi$. Then we have

$$\Delta_\ell^{-1} \sin pk = \frac{(\sin p(k-\ell) - \sin pk)}{2(1-cosp\ell)} + c_j, \tag{4}$$

and

$$\Delta_\ell^{-1} cospk = \frac{(cosp(k-\ell) - cospk)}{2(1-cosp\ell)} + c_j. \tag{5}$$

In particular

$$\Delta_\ell^{-1} \sin pk|_0^{2\pi} = 0 = \Delta_\ell^{-1} \cos pk|_0^{2\pi}. \tag{6}$$

Definition 3.3 [4] The generalized finite Fourier series is defined by

$$u(k) = \frac{a_0}{2} + \sum_{\omega=1}^{\eta-1}(a_\omega cos\omega k + b_\omega sin\omega k) + \frac{a_\eta}{2} cos\eta k, \quad k \in \{a + r\ell\}_{r=0}^{2\eta-1}, \tag{7}$$

where $a_\omega = \frac{\ell}{\pi}\Delta_\ell^{-1} u(k) cos\omega k\Big|_a^{a+2\pi}$,$b_\omega = \frac{\ell}{\pi}\Delta_\ell^{-1} u(k) sin\omega k\Big|_a^{a+2\pi}$.

Lemma 3.4 If $k \in (-\infty, \infty)$ and $\ell_1, \ell_2 > 0$, then $\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1} 2^k = \frac{2^k}{(2^{\ell_1}-1)(2^{\ell_2}-1)}$.

Lemma 3.5 Let $\ell_1, \ell_2 > 0$ and $u(k)$, $w(k)$ are real valued bounded functions. Then $\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1} u(k)v(k) = \Delta_{\ell_2}^{-1}\Big[u(k)\Delta_{\ell_1}^{-1} v(k)\Big] - \Delta_{\ell_2}^{-1}\Big[\Delta_{\ell_1}^{-1}\Big(\Delta_{\ell_1}^{-1} v(k+\ell_1)\Delta_{\ell_1} u(k)\Big].$

Theorem 3.6 Let $k \in (-\infty, \infty)$ and $\ell_1, \ell_2 > 0$, then we have

$$\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]} \sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]} (k-r_2\ell_2-r_1\ell_1)2^{k-r_2\ell_2-r_1\ell_1}$$

$$= \Big(\frac{(k+\ell_2+\ell_1)2^{k+\ell_2+\ell_1}}{(2^{\ell_1}-1)(2^{\ell_2}-1)} - \frac{\ell_2 2^{\ell_2}2^{k+\ell_2+\ell_1}}{(2^{\ell_2}-1)^2(2^{\ell_1}-1)} - \frac{\ell_1 2^{\ell_1}2^{k+\ell_2+\ell_1}}{(2^{\ell_1}-1)^2(2^{\ell_2}-1)}\Big)$$

$$-\Big(\frac{\ell_2+\hat{\ell}_1(k)2^{\ell_2+\hat{\ell}_1(k)}}{(2^{\ell_1}-1)(2^{\ell_2}-1)} - \frac{\ell_2 2^{\ell_2}2^{\ell_2+\hat{\ell}_1(k)}}{(2^{\ell_2}-1)^2(2^{\ell_1}-1)} - \frac{\ell_1 2^{\ell_1}2^{\ell_2+\hat{\ell}_1(k)}}{(2^{\ell_1}-1)^2(2^{\ell_2}-1)}\Big)$$

$$-\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]} \Big(\frac{\hat{\ell}_2(k-r_1\ell_1+\ell_2)2^{\hat{\ell}_2(k-r_1\ell_1+\ell_2)}}{(2^{\ell_2}-1)} - \frac{\ell_2 2^{\ell_2}2^{\hat{\ell}_2(k-r_1\ell_1+\ell_2)}}{(2^{\ell_2}-1)^2}\Big). \quad (8)$$

Proof: By using the lemmas (3.4) and (3.5), we get the proof of (8).

Example 3.7 Taking $k = 4$, $\ell_1 = 1$ and $\ell_2 = 2$ in(3.6), we get the summation is equal to 107.5

Theorem 3.8 Let $k \in (-\infty, \infty)$ and $\ell_1, \ell_2 > 0$, then we have

$$\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1} k \sin k = \sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]} \sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]} (k-r_2\ell_2-r_1\ell_1)\sin(k-r_2\ell_2-r_1\ell_1)$$

$$= \frac{1}{2(1-\cos\ell_1)}\Big[k\frac{(\sin(k-\ell_1-\ell_2)-\sin(k-\ell_1)-\sin(k-\ell_2)+\sin\ k)}{2(1-\cos\ell_2)}$$

$$-\frac{\ell_2}{4(1-\cos\ell_2)^2}\Big(\sin(k-\ell_1-\ell_2)-2\sin(k-\ell_1)+\sin(k-\ell_1+\ell_2)+\sin(k-\ell_2)$$

$$2\sin k+\sin(k+\ell_2)\Big)\Big] - \frac{\ell_1}{4(1-\cos\ell_1)^2 2(1-\cos\ell_2)}\Big[\sin(k-\ell_1-\ell_2)-\sin(k-\ell_1)$$

$$-\sin(k-\ell_2)+\sin k-\sin(k-\ell_2)+\sin k+\sin(k+\ell_1-\ell_2)-\sin(k+\ell_1)\Big]. \quad (9)$$

Proof: Taking $u(k) = k$ and $v(k) = \sin k$ in (3.5), we get the proof of (9).

Corollary 3.9 Let $\ell_1, \ell_2 > 0$, $k \in (-\infty, \infty)$ then we have

$$\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1}\sin pk = \frac{\sin p(k-\ell_2-\ell_1) - \sin p(k-\ell_1) - \sin p(k-\ell_2) + \sin pk}{4(1-\cos p\ell_1)(1-\cos p\ell_2)}$$

Definition 3.10 The generalized finite Fourier series with two parameter $(GFFS_2)$ is defined by

$$u(k) = \frac{a_0}{2} + \sum_{\omega=1}^{\eta-1}(a_\omega cos\omega k + b_\omega sin\omega k) + \frac{a_\eta}{2}cos\eta k, \quad k \in \{a+r\ell\}_{r=0}^{2\eta-1}, \tag{10}$$

where
$a_\omega = \frac{\ell_1+\ell_2}{\pi}\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1}u(k)cos\omega k\Big|_a^{a+2\pi}\Big|_a^{a+2\pi}$, $b_\omega = \frac{\ell_1+\ell_2}{\pi}\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1}u(k)sin\omega k\Big|_a^{a+2\pi}\Big|_a^{a+2\pi}$.

Theorem 3.11 Let ℓ_1 and $\ell_2 > 0$ and $k \in (-\infty, \infty)$, then $GFFS_2$ of $k2^k$ is

$$k2^k = \frac{\ell_1+\ell_2}{\pi}\Big\{\Big(\frac{(k+\ell_2+\ell_1)2^{k+\ell_2+\ell_1}}{(2^{\ell_1}-1)(2^{\ell_2}-1)} - \frac{\ell_2 2^{\ell_2}2^{k+\ell_2+\ell_1}}{(2^{\ell_2}-1)^2(2^{\ell_1}-1)} - \frac{\ell_1 2^{\ell_1}2^{k+\ell_2+\ell_1}}{(2^{\ell_1}-1)^2(2^{\ell_2}-1)}\Big)$$
$$-\Big(\frac{\ell_2+\hat{\ell_1}(k)2^{\ell_2+\hat{\ell_1}(k)}}{(2^{\ell_1}-1)(2^{\ell_2}-1)} - \frac{\ell_2 2^{\ell_2}2^{\ell_2+\hat{\ell_1}(k)}}{(2^{\ell_2}-1)^2(2^{\ell_1}-1)} - \frac{\ell_1 2^{\ell_1}2^{\ell_2+\hat{\ell_1}(k)}}{(2^{\ell_1}-1)^2(2^{\ell_2}-1)}\Big)$$
$$-\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\Big(\frac{\hat{\ell_2}(k-r_1\ell_1+\ell_2)2^{\hat{\ell_2}(k-r_1\ell_1+\ell_2)}}{(2^{\ell_2}-1)} - \frac{\ell_2 2^{\ell_2}2^{\hat{\ell_2}(k-r_1\ell_1+\ell_2)}}{(2^{\ell_2}-1)^2}\Big) + \sum_{\omega=1}^{\eta-1}\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}$$
$$\Big(\mu(k)\sin\omega(k-r_2\ell_2-r_1\ell_1)\sin\omega k + \mu(k)\cos\omega(k-r_2\ell_2-r_1\ell_1)\cos\omega k\Big)\Big\}. \tag{11}$$

Where $\mu(k) = (k-r_2\ell_2-r_1\ell_1)2^{(k-r_2\ell_2-r_1\ell_1)}$

Proof: The proof of (11) obtained by using (10) and (9).

4 Odd and Even function

Definition 4.1 (Odd and Even function) Even functions are the functions for which the left half of the plane looks like the mirror image of the right of the plane. Odd functions are the functions where the left half of the plane looks like the mirror image of the right of the plane,only upside-down.
The function $u(k)$ is an even function if $u(-k) = u(k)$ and odd if $u(-k) = -u(k)$.

Definition 4.2 The generalized finite half-range Fourier series $(GFFS_2)$ of odd and even functions are is defined as If $u(k)$ is even function, then (10) becomes,

$$u(k) = \frac{a_0}{2} + \sum_{\omega=1}^{\eta-1} a_\omega cos\omega k + \frac{a_\eta}{2} cos\eta k, \tag{12}$$

where $a_\omega = \frac{\ell_1+\ell_2}{\pi} \Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1} u(k) cos\omega k\Big|_a^{a+2\pi}\Big|_a^{a+2\pi}$.

If $u(k)$ is odd function, then (10) becomes,

$$u(k) = \sum_{\omega=1}^{\eta-1} b_\omega \sin\omega k, \tag{13}$$

where $b_\omega = \frac{\ell_1+\ell_2}{\pi} \Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1} u(k) sin\omega k\Big|_a^{a+2\pi}\Big|_a^{a+2\pi}$.

Theorem 4.3 Let $u(k) = k$, ℓ_1 and $\ell_2 > 0$, then $GFFS_2$ of k is

$$k = \frac{\ell_1+\ell_2}{\pi} \sum_{\omega=1}^{\eta-1} \frac{1}{2(1-\cos\ell_1)} \Big[k \frac{(\sin(k-\ell_1-\ell_2) - \sin(k-\ell_1) - \sin(k-\ell_2) + \sin k)}{2(1-\cos\ell_2)}$$
$$-\frac{\ell_2}{4(1-\cos\ell_2)^2}\Big(\sin(k-\ell_1-\ell_2) - 2\sin(k-\ell_1) + \sin(k-\ell_1+\ell_2) + \sin(k-\ell_2)$$
$$-2\sin k + \sin(k+\ell_2)\Big)\Big] - \frac{\ell_1}{4(1-\cos\ell_1)^2 2(1-\cos\ell_2)}\Big[\sin(k-\ell_1-\ell_2) - \sin(k-\ell_1)$$
$$-\sin(k-\ell_2) + \sin k - \sin(k-\ell_2) + \sin k + \sin(k+\ell_1-\ell_2) - \sin(k+\ell_1)\Big] \sin\omega k. \tag{14}$$

Proof: The proof of (14) obtained by using (13) and (3).

Example 4.4 Taking $k = \pi$, $\ell_1 = \pi$ and $\ell_2 = \frac{\pi}{2}$, we get

$$b_\omega = \frac{\ell_1+\ell_2}{\pi} \sum_{r_2=0}^{2} (\pi - r_2\frac{\pi}{2}) \sin\omega(\pi - r_2\tfrac{\pi}{2}) + \sum_{r_2=0}^{0} (\pi - r_2\frac{\pi}{2} - \pi)$$
$$\sin\omega(\pi - r_2\frac{\pi}{2} - \pi) - \sum_{r_2=0} (-r_2\tfrac{\pi}{2}) \sin\omega(-r_2\frac{\pi}{2}) = \frac{\ell_1+\ell_2}{\pi}\pi\sin\omega(\pi) + \frac{\pi}{2}$$

Particularly taking $\eta = 3$, we obtain $b_1 = \frac{\ell_1+\ell_2}{\pi}$, $b_2 = 0$ and $k = \frac{l_1+l_2}{2} sink$.

Theorem 4.5 Let $u(k) = k^2$, ℓ_1 and $\ell_2 > 0$, then $GFFS_2$ of k^2 is

$$k^2 = \frac{\ell_1+\ell_2}{\pi}\sum_{\omega=1}^{\eta}\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}(k-r_2\ell_2-r_1\ell_1)^2\Big(\frac{1}{2}+\cos\omega(k-r_2\ell_2-r_1\ell_1)cos\omega k\Big). \quad (15)$$

Proof: By using (12) and (3), we get the proof of (15).

Theorem 4.6 Let $u(k) = k^3$, ℓ_1 and $\ell_2 > 0$, then $GFFS_2$ of k^3 is

$$k^3 = \frac{\ell_1+\ell_2}{\pi}\sum_{\omega=1}^{\eta-1}\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}(k-r_2\ell_2-r_1\ell_1)^3\sin\omega(k-r_2\ell_2-r_1\ell_1)\sin\omega k. \quad (16)$$

Proof: The proof of (16) obtained by using (13) and (3).

Theorem 4.7 Let $u(k) = k\sin k$, ℓ_1 and $\ell_2 > 0$, then $GFFS_2$ of $k\sin k$ is

$$k\sin k = \frac{a_0}{2} + \sum_{\omega=1}^{\eta-1} a_\omega\cos\omega k + \frac{a_\eta}{2}\cos\eta k. \quad (17)$$

Here $a_\omega = \frac{\ell_1+\ell_2}{2\pi}\Delta_{\ell_2}^{-1}\Delta_{\ell_1}^{-1}k\sin k\cos\omega\Big|_0^\pi$

Proof: The proof of (17) obtained by using (13) and (3).

Example 4.8 Taking $\ell_1 = \frac{\pi}{2}$, $\ell_2 = \frac{\pi}{4}$in (17),we arrive

$$a_\omega = \frac{\ell_1+\ell_2}{2\pi}\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}(k-r_2\ell_2-r_1\ell_1)\sin(k-r_2\ell_2-r_1\ell_1)\cos\omega(k-r_2\ell_2-r_1\ell_1)\Big|_0^\pi$$

$$= \frac{\ell_1+\ell_2}{2\pi}\Big(\sum_{r_2=0}^{4}(\pi-r_2\frac{\pi}{4})\sin(\pi-r_2\frac{\pi}{4})\cos\omega(\pi-r_2\frac{\pi}{4}) + \sum_{r_2=0}^{2}(\pi-r_2\frac{\pi}{4}-\frac{\pi}{2})$$

$$\sin(\pi-r_2\frac{\pi}{4}-\frac{\pi}{2})\cos\omega(\pi-r_2\frac{\pi}{4}-\frac{\pi}{2}) + \sum_{r_2=0}(\pi-r_2\tfrac{\pi}{4}-\pi)sin(\pi-r_2\tfrac{\pi}{4}-\pi)$$

$$\cos\omega(\pi - r_2\tfrac{\pi}{4}\pi) - \sum_{r_2=0}(-r_2\tfrac{\pi}{4})sin(-r_2\tfrac{\pi}{4})\cos\omega(-r_2\frac{\pi}{4})\Big)$$

$$a_\omega = \frac{\ell_1+\ell_2}{2\pi}\Big(0.09686\cos\omega\Big(\frac{3\pi}{4}\Big) + 0.043055\cos\omega\Big(\frac{\pi}{2}\Big)$$
$$+0.010765\cos\omega\Big(\frac{\pi}{4}\Big) + 0.043055\cos\omega\Big(\frac{\pi}{2}\Big)\Big).$$

Particularly taking $\eta = 3$,we get $k\sin k = \frac{\ell_1+\ell_2}{4\pi}(0.193735) + \frac{\ell_1+\ell_2}{2\pi}(0.193608)\cos k$ $+\frac{\ell_1+\ell_2}{2\pi}(0.193269)\cos 2k + \frac{\ell_1+\ell_2}{4\pi}(0.192697)\cos 3k.$

5 Conclusion

The Fourier series and its transforms have wide range of applications specially in the field of digital signal processing. For the odd and even functions no usual finite half range Fourier series expression, we are able to find finite Fourier series expression (decomposition) using generalized inverse difference operator with two parameters.

References

[1] Anguelov.R, Spline-Fourier approximation of discontinuous waves, J. Univ. Comput. Sci. 4(2) (1998).

[2] Britto Antony Xavier.G, Sathya.S, Vasanthakumar.S.U, m-Series of the Generalized Difference Equation to Circular Functions,International Journal of Mathematical Archive-4(7), 200-209 (2013).

[3] Edwards R F, Fourier Series, A Modern Introduction, Vol.1. Holt, Rinehart and Winston, New York, 1967.

[4] Govindan.B, Nandhini.M, Meganathan.M, Generalized Discrete Finite Half Range Fourier Series,Advanced in Mathematics: Scientific Journal, 9, 1225-1237 (2020).

[5] Jerri.A.J, The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet Approximations, Kluwer Academic Publishers, Dordrecht-Bosten-London, 1998.

[6] Maria Susai Manuel.M, Britto Antony Xavier.G and Thandapani.E, Theory of Generalized Difference Operator and Its Applications, Far East Journal of Mathematical Sciences, 20(2), 163 - 171 (2006).

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https://doi.org/10.26524/cm95

[7] Sidi.A, Practical Extrapolation Methods: Theory and Applications, Cambridge University Press, 2005.

[8] Steven W. Smith, The Scientist and Engineer's Guide to Digital Signal Processing, Second Edition, California Technical Publishing San Diego, California, 1999.

[9] Tolstov.G.P, Fourier series, Dover, New York, 1976.