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Bounds for λ-Domination Number $\gamma_\lambda(G)$ of a Graph

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Abstract

In this paper, we introduce a new domination parameter $\gamma_\lambda(G)$ where $0 \leq \lambda \leq 1$, and initiate a study on $\gamma_{\frac{1}{2}}(G)$. We obtain certain bounds $\gamma_{\frac{1}{2}}(G)$

Key words: Dominating set, Domination number, Lambda dominating set, Lambda domination number.

AMS Subject classification: 05C.

1.Introduction

We consider only finite-simple undirected graphs. If $G = (V, E)$ is a graph, a vertex $v \in V$ is said to dominate itself and its adjacent vertices. In other words, a vertex v dominates a vertex u iff $u \in N[v]$, where $N[v]$ is the closed neighborhood of v. A subset D of $V(G)$ is said to be a dominating set of G iff $V = \cup_{u \in D} N[u]$.The minimum cardinality of a dominating set D of G is denoted by $\gamma(G)$ and is called the domination number of G. If v is a vertex of a graph G, for a positive integer i, $N_i(v)$ denotes the set $N_i(v) = u \in V(G) : d(u, v) = i$

Definition 1.1 *(Slater)*

Given a finite simple graph $G = (V, E)$, a subset B of V is called a k-basis($k \geq 1$), when for each vertex $v \in V$, there is at least one vertex u of B such that the distance between u and v in G, denoted by $d_G(u, v)$, is $\leq k$. Thus a dominating set is a 1 - basis.

Slater also gave an interpretation in terms of communication networks. We quote his interpretation: If V represent a collection of cities and an edge represents a

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communication link, then one may be interested in selecting a minimum number of cities as sites for transmitting stations so that every city either contains a transmitter or can receive messages from at least one of the transmitting stations through links. If only direct transmissions are acceptable, then one wishes to find a minimum 1-basis.

If communication over paths of k links (but not of $k+1$ links) is adequate inquality and rapidity, the problem becomes that of determining a minimum k-basis, i.e., a k-basis with the fewest possible vertices.

Again consider the communication network discussed by Slater. Assume that a transmitting station is situated at a vertex u (a city) in V. Suppose that v_1 and $v_2 \in V$ such that $d(u, v_1) = 1$ and $d(u, v_2) = 2$. If the message signals are transmitted from the transmitting station at u, the quality/strength of the signals received at v_1 and v_2 may not be same, as $d(u, v_2)$ is greater than $d(u, v_1)$. If we take the quality/strength of the received signal at v_1 as unity, the quality/strength of the received signal at v_2 will be ≤ 1. In fact, in real situations, it will be less than 1. The quality of the received signal at v decreases as $d(u, v)$ increases. As all the transmitting stations are transmitting same information, in most of the practical cases, we are satisfied if for every non transmitting city v, the sum of the received signals at v from all the transmitting stations is greater than or equal to unity. This motivates us to define a new domination parameter.

Let G be a connected graph with diameter k. Let $1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \lambda_k \geq 0$. To each vertex $u \in V(G)$, we define a weight function f_u defined on $V(G)$ as follows

$$f_u(v) = \begin{cases} 1 & \text{if} \quad v \in N[u] \\ \lambda_i & \text{if} \quad d(u,v) = i, \quad \text{for} \quad 2 \leq i \leq k. \end{cases}$$

we say that a subset D of $V(G)$ is a $(\lambda_1, \lambda_2, ..., \lambda_k)$-dominating set of G if $\sum_{u \in D} f_u(v) \geq 1$ holds for every vertex $v \in V(G)$. The minimum cardinality of a $(\lambda_1, \lambda_2, ..., \lambda_k)$- dominating set of G is said to be the $(\lambda_1, \lambda_2, ..., \lambda_k)$- domination number of G and is denoted by $\gamma_{(\lambda_1, \lambda_2, ..., \lambda_k)}(G)$.

Remark 1.2 (1) If $\lambda_i = 0$ for all $i \geq 2$, then we have the usual domination number $\gamma(G)$. If $\lambda_1 = \lambda_2 = ... = \lambda_r = 1$ and $\lambda_i = 0$ for $i > r$, then we have the r-domination number introduced by Slater.

(2) one can take $\lambda_i = \frac{1}{i}$, so that

$$f_u(v) = \begin{cases} 1 & \text{if} \quad v = u \\ \frac{1}{d(u,v)} & \text{if} \quad v \neq u. \end{cases}$$

We initiate a study on this new parameter by restricting ourselves to the case

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$0 < \lambda_2 < 1$ and $\lambda_i = 0$ for all $i \geq 3$. We reformulate our definition as follows

Definition 1.3 ***(λ domination)*** Let λ be such that $0 < \lambda < 1$. Let G be a graph (G need not be connected). To each $u \in G$, define f_u on $V(G)$ as follows:

$$f_u(v) = \begin{cases} 1 & \text{if} \quad v \in N[u] \\ \lambda & \text{if} \quad d(u,v) = 2 \\ 0 & \text{otherwise} \end{cases}$$

A subset D of V is said to be a λ - dominating set if for each $v \in V(G)$, $\sum_{u \in D} f_u(v) \geq 1$ holds. The minimum cardinality of a λ - dominating set is called the λ - domination number of G and is denoted by $\gamma_\lambda(G)$. A λ -dominating set with cardinality $\gamma_\lambda(G)$ is said to be a γ_λ - set of G. Let $0 < \lambda < 1$. Find an integer $n \geq 2$ such that

Let $0 < \lambda < 1$. Find an integer $n \geq 2$ such that $\frac{1}{n} \leq \lambda < \frac{1}{n-1}$.A subset D of $V(G)$ is a λ - dominating set of G iff to each vertex $v \in V$, either $v \in N[D]$ or $|N_2(u) \cap D| \geq n$, where $N_2(u)$ is the second neighborhood of u. $(N_2(u) = v \in V(G)/d(u,v) = 2)$. Thus D is a λ - dominating set for G iff D is an $\frac{1}{n}$- dominating set for G, and hence $\gamma_\lambda(G) = \gamma_{\frac{1}{n}}(G)$, whenever $\frac{1}{n} \leq \lambda < \frac{1}{n-1}$. Thus it is enough to study the parameters $\gamma_{\frac{1}{n}}(G)$, for $n \geq 2$.

2. $\lambda_{\frac{1}{n}}$(G) For Some Graphs

First we observe that $1 \leq \gamma_{\frac{1}{2}}(G) \leq \gamma_{\frac{1}{3}}(G) \leq ..\gamma_{\frac{1}{n}}(G) \leq \gamma(G)$. Hence if $\gamma_{\frac{1}{2}}(G) = \gamma(G)$, then $\gamma_{\frac{1}{n}}(G) = \gamma(G)$ for all $n \geq 2$. In particular $\gamma(G) = 1$ iff $\gamma_{\frac{1}{n}}(G) = 1$ forall $n \geq 2$ iff $\Delta(G) = n - 1$, where $|V(G)| = n$. We Know that $\gamma(G) \leq \dfrac{n}{2}$, for all graphs G with $\delta(G) \geq 1$. It follows that $\gamma_{\frac{1}{k}}(G) \leq \dfrac{n}{2}$ for all $k \geq 2$ and for all graphs G with $\delta(G) \geq 1$ and hence the set $\{\gamma_{\frac{1}{k}}(G)/k = 2, 3, 4....\}$ can contain at the most $\dfrac{n}{2} - 1$ distinct integers, for all graphs with $\delta(G) \geq 1$. For the graph $K_n \circ K_1$, the corona of the complete graph K_n , we have $\gamma_{\frac{1}{k}}(G) = k$ for all $2 \leq k \leq n$. Thus there are graphs G for which the set $\gamma_{\frac{1}{k}}/k \geq 2$ has exactly $\left[\dfrac{n}{2}\right] - 1$ elements.
$\gamma_{\frac{1}{k}}(G)$ for some standard graphs:

(1) $\gamma_{\frac{1}{k}}(K_n) = 1$, for all $k \geq 2$.
(2) If $G = K_{m,n}, (2 \leq m \leq n)$, is a complete bipartite graph, then $\gamma_{\frac{1}{k}}(K_{m,n}) = 2$ for all $k \geq 2$
(3) If C_n is a cycle on n vertices, then

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$$\gamma_{\frac{1}{2}}(C_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil & \text{if} \quad n \neq 4 \\ 2 & \text{if} \quad n = 4 \end{cases}$$

and $\gamma_{\frac{1}{k}}(C_n) = \gamma(C_n)$ for all $k \geq 3$.

(4) For the path P_n on n vertices, $\gamma_{\frac{1}{2}}(P_n) = \left\lfloor \frac{n}{4} \right\rfloor + 1$ and $\gamma_{\frac{1}{k}}(P_n) = \gamma(P_n)$ for all $k \geq 3$.

(5) For the Peterson graph P, $\gamma_{\frac{1}{2}}(P) = 2$.

(6) For the graphs G_1 and G_2 given in Fig.1, we have $\gamma(G_1) = 5, \gamma_{\frac{1}{2}}(G_1) = 3$ and $\gamma(G_2) = 4$ while $\gamma_{\frac{1}{2}}(G_2) = 3$.

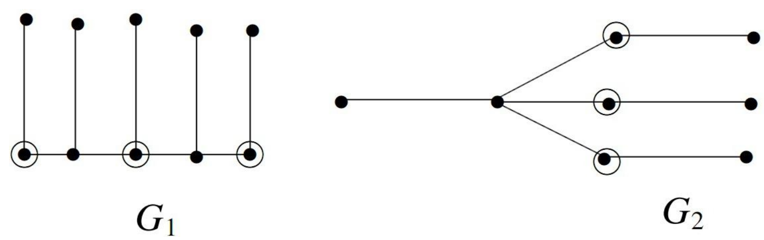

Figure 1: Two graphs G_1 and G_2 with $\gamma_{\frac{1}{2}}(G_1) = \gamma_{\frac{1}{2}}(G_2) = 3$

Theorem 2.1 If G is a graph with $diam(G) = 2$, then $\gamma_{\frac{1}{2}}(G) \leq 2$.

proof: If $\Delta(G) = n1$, then $\gamma_{\frac{1}{2}}(G) = 1$. If $\Delta(G) \neq n-1$, then $\gamma_{\frac{1}{2}}(G) \geq 2$. If $\Delta(G) \neq n-1$, select two vertices u_1 and $u_2 \in V(G)$, with $d(u, v) = 2$. As $V(G) = N_1[u_1] \cup N_1[u_2] \cup (N_2(u) \cap N_2(v))$, it follows that u_1, u_2 is a $\gamma_{\frac{1}{2}}$- set for G.

Remark 2.2 Converse of the above theorem is not true. For the path P_7 on seven vertices, $\gamma_{\frac{1}{2}}(P_7) = 2$ but $diam(P_7) = 6$. One can prove that if G is connected and $\gamma_{\frac{1}{2}}(G) = 2$, then $diam(G) \leq 6$.

3. Bounds for $\gamma_{\frac{1}{2}}$(G)

In this section we obtain some bounds for the parameter $\gamma_{\frac{1}{2}}(G)$. Let $u \in V(G)$. In this section by f_u we mean the map $f_u : V \to \{0, \frac{1}{2}, 1\}$ given by

$$f_u(v) = \begin{cases} 1 & \text{if} \quad v \notin N[u] \\ \frac{1}{2} & \text{if} \quad v \in N_2[u] \\ 0 & otherwise \end{cases}$$

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Theorem 3.1 If G is a graph on n vertices and $\Delta(G) = \Delta$, then $\left\lceil \frac{n}{1+\frac{1}{2}\Delta+\frac{1}{2}\Delta^2} \right\rceil \leq \gamma_{\frac{1}{2}}(G) \leq n - \Delta$.
Proof: Let D be a $\gamma_{\frac{1}{2}}$-set for G.
Then $(\sum_{u\in D} f_u)(v) \geq 1$, for all $v \in V(G)$.
Hence $\sum_{v\in V}(\sum_{u\in D} f_u)(v) \geq n$.

$$(i.e) \sum_{u\in D}(\sum_{v\in V} f_u)(v) \geq n. \qquad (1)$$

As to each $u \in D$,

$$\sum_{v\in V} f_u = 1 + |N_1(u)| + \frac{1}{2}|N_2(u)| \leq 1 + \Delta + \frac{1}{2}\Delta(\Delta-1) = 1 + \frac{1}{2}\Delta + \frac{1}{2}\Delta^2$$

from(1), we obtain $|D|(1 + \frac{1}{2}\Delta + \frac{1}{2}\Delta^2) \geq n$
Thus, $\gamma_{\frac{1}{2}}(G) \geq \left\lceil \frac{n}{1+\frac{1}{2}\Delta+\frac{1}{2}\Delta^2} \right\rceil$.
The upper bound follows from the fact $\gamma_{\frac{1}{2}}(G) \leq \gamma(G) \leq n - \Delta$.
If $\Delta(G) = n-1$ or $n-2$, $\gamma_{\frac{1}{2}}(G) = 1$ or 2 respectively and hene $\gamma_{\frac{1}{2}}(G) = n - \Delta$.
For the graph G with $\Delta(G) \leq n-3$, we can improve the upper bound given in the Theorem 5.

Theorem 3.2 If G is a connected graph with $\Delta(G) \leq n-3$, then $\gamma_{\frac{1}{2}}(G) \leq n-\Delta-1$.
Proof: Let u be a vertex of degree Δ. Let T be a spanning tree of G in which $degT(u) = \Delta(G)$. As $\Delta(G), n-1$, T is not a star, and hence we have,

$$2 \leq \gamma_{\frac{1}{2}}(G) \leq \gamma_{\frac{1}{2}}(T) \leq n - \Delta(T) = n - \Delta(G). \qquad (2)$$

Assume that $\gamma_{\frac{1}{2}}(G) = n - \Delta$.
Then by (1), $\gamma_{\frac{1}{2}}(G) = \gamma_{\frac{1}{2}}(T) = \gamma(T) = n - \Delta$.
Hence by Theorem 2.14(page 51 in [2]), T is a wounded spider. As $\Delta < n-2$, the wounded spider T has at least two non wounded legs(edges).
Let $D = \{v \in V/v, u \text{ and } \deg_T(v) = 2\}$. Then D is a $\frac{1}{2}$- dominating set for T and for G. As $|D| = n - \Delta - 1$, we get a contradiction to our assumption that $\gamma_{\frac{1}{2}}(G) = n - \Delta$.
Thus $\gamma_{\frac{1}{2}}(G) \leq n - \Delta - 1$.

Remark 3.3 For a wounded spider T with $\Delta(T) \leq n-3$, $\gamma_{\frac{1}{2}}(T) = n - \Delta - 1$. For the graph G given in the Fig.2, $\gamma_{\frac{1}{2}}(G) = n - \Delta - 1$.

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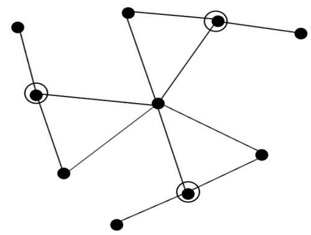

Figure 2: A graph G, which is not a tree, with

$$\Delta(G) = n - 4 \text{ and } \gamma_{\frac{1}{2}}(G) = n - \Delta - 1.$$

In the following theorem, we characterize trees with $\Delta(T) \leq n-3$ and $\gamma_{\frac{1}{2}}(G) \leq n - \Delta - 1$.

Theorem 3.4 Let T be a tree with $\Delta(T) \leq n-3$. Then $\gamma_{\frac{1}{2}}(T) = n - \Delta(T) - 1$ iff T is either the path P_5 on five vertices or it is obtained from the star $K_{1,t}$ for some $t \geq 3$, by any one of the following operations.

:(i) subdivide at least two edges of $K_{1,t}$.
:(ii) subdivide exactly one edge of $K_{1,t}$ twice (i.e. exactly one edge of $K_{1,t}$ is replaced by a path of length three.)
:(iii) subdivide exactly one edge of $K_{1,t}$ twice and subdivide another edge once.
:(iv) attach two pendant vertices at a pendant vertex of $K_{1,t}$

(These operations are illustrated in the Fig.3) $\Delta(G) = n - 4$

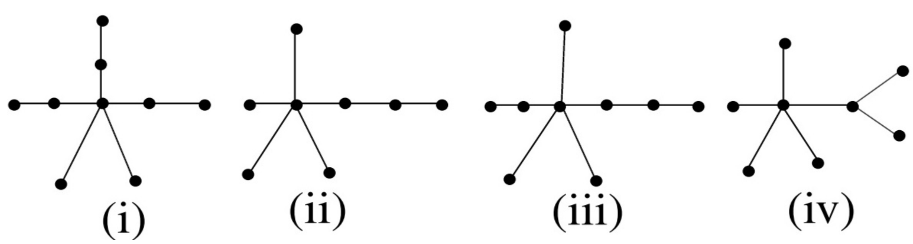

Figure 3: Trees obtained from $K_{1,5}$ using

and $\gamma_{\frac{1}{2}}(G) = n - \Delta - 1$.
Proof: Note that $\gamma_{\frac{1}{2}}(P_5) = 2 = n - \Delta - 1$. Let u be the vertex of $K_{1,t}$,($t \geq 3$) with $deg(u) = t$.

(i) If T is obtained from $K_{1,t}$ by subdividing at least two edges of $K_{1,t}$, then $D = v \in T/degT(v) = 2$ is a $\gamma_{\frac{1}{2}}$ - set for T. (T need not be a wounded spider).

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(ii) If T is obtained from $K_{1,t}$ by subdividing exactly one edge of $K_{1,t}$ twice, then $\gamma_{\frac{1}{2}}(T) = \gamma(T) = 2 = n - \Delta - 1$.

(iii) If T is obtained from $K_{1,t}$ by subdividing on edge twice and another edge at once, then $D = v \in V(T)/deg_T(v) = 2$ is a $\gamma_{\frac{1}{2}}$- set of T, and hence $\gamma_{\frac{1}{2}}(T) = 3 = n - \Delta - 1$.

(iv) If T is obtained from $K_{1,t}$ by attaching two pendant vertices at a pendant vertex of $K_{1,t}$, then also $\gamma_{\frac{1}{2}}(T) = \gamma(T) = 2 = n - \Delta - 1$.

Thus all these operations on $K_{1,t}$ yield a tree with $\gamma_{\frac{1}{2}}(T) = n - \Delta - 1$.

Conversely, let T be a tree with $\Delta(T) \leq n-3$ and $\gamma_{\frac{1}{2}}(T) = n - \Delta - 1$. If T is a path, then $\gamma_{\frac{1}{2}}(T) = n - 3$. As $\gamma_{\frac{1}{2}}(P_n) = \left\lfloor \frac{n}{4} \right\rfloor + 1$, we have,

$$2 = \Delta(T) \leq n - 3 = \left\lfloor \frac{n}{4} \right\rfloor + 1.$$

Therefore $5 \leq n \leq \frac{n}{4} + 4$. i.e., $5 \leq n \leq \frac{16}{3}$. Thus $n = 5$ and $T = P_5$.
Now, assume that T is not a path. So $\Delta(T) \geq 3$. Let u be a vertex with degree Δ. The induced graph $\langle N[u] \rangle$ is the star $K_{1,t}$, where $t = \Delta(T) = \deg u$.
We observe the following:

(1) In the induced graph $\langle V - N[u] \rangle$, degree of each vertex is ≤ 1. [If possible, let w be a vertex in $\langle V - N[u] \rangle$ with $deg(w) \geq 2$. Select two vertices w_1 and w_2 in $\langle V - N[u] \rangle$ such that $w_1 w w_2$ is a path in $\langle V - N[u] \rangle$. As $D = u \cup ((VN[u])w_1, w_2)$ is a dominating set for T, with cardinality $n - \Delta - 2, \gamma_{\frac{1}{2}}(T), n - \Delta - 1$, a contradtction].

(2) From (1), it follows that $d(w, u) \leq 3$ in T, for all $w \in V - N[u]$.

(3) There can be at most one vertex w in T such that $d(u, w) = 3$. [For if $w_1, w_2 \in V(T)$ such that $d(u, w_1) = d(u, w_2) = 3$, then $(VN(u))w_1, w_2$ is a dominating set for T with $n - \Delta - 2$ elements, which is a contradiction, as $n - \Delta - 1 = \gamma_{\frac{1}{2}}(T) = \gamma(T)$].

(4) If $n - \Delta = 3$, then T is obtained from $K_{1,t}$ by either subdividing ex-actly two edges once, or subdividing one edge twice, or by attaching two pendant vertices at a pendant vertex of $K_{1,t}$. Thus in this case T is obtained from $K_{1,t}$ by using one of the operations (i), (ii) and (iv). We observe the following, by assuming $n - \Delta - 1 \geq 3$. (i.e.)$|VN[u]| \geq 3$.

(5) No vertex of $N[u]$ is adjacent to two distinct vertices of $VN[u]$. [For, if a vertex $w \in N(u)$ is adjacent to more than one vertex of $VN[u]$. consider $D' = v \in Vu/deg_T(v), 1$. If $|D'| \geq 2$, then D' is a $\frac{1}{2}$- dominating set for T and if $|D'| = 1$, (i.e. $D' = w$), then u, w is a $\gamma_{\frac{1}{2}}$- set for T. Any how $\gamma_{\frac{1}{2}}, n - \Delta - 1$]

(6) If $d(u, v) \leq 2$, for all $v \in V(T)$, then from (5), it follows that T is obtained from $K_{1,t}$, by subdividing exactly $n - \Delta - 1$ edges of $K_{1,t}$.

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(7) If there is a vertex $w \in T$ such that $d(u,w) = 3$, then $|V - N[u]| = n - \Delta - 1 \leq 3$. [If $|V - N[u]| \geq 4$, then $D = w \cup \{v/deg(v) = 2$ and v is not on the $u - w$ path in T$\}$ is a $\frac{1}{2}$- dominating set of T with $n - \Delta - 2$ elements, which is a contradiction].
(8) From (1),(5) and (7), it follows that if there is a vertex w such that $d(u,w) = 3$, then it follows that T is obtained from $K_{1,t}$ by using the operation (iii).

Our observations 1 to 8 completes the proof for the converse part. Examples for graphs G which attain the lowerbound $\left\lceil \frac{n}{1+\frac{1}{2}\Delta+\frac{1}{2}\Delta^2} \right\rceil$ for $\gamma_{\frac{1}{2}}(G)$.
(This lower bound is given in the Theorem 5).

(1) The cycle C_{4k} , for all $k \geq 1$.
$n = 4k; \Delta = 2$ and $\gamma_{\frac{1}{2}}(C_{4k}) \lceil \frac{4k}{k} \rceil = k = \left\lceil \frac{4k}{1+1+2} \right\rceil$
(2) Peterson graph P.
(3) The graph G given the Fig.4.

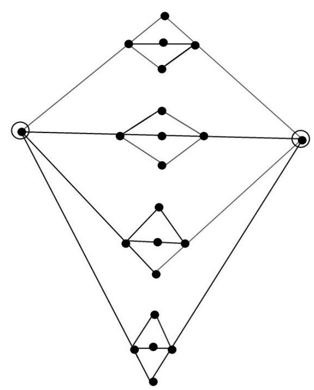

Figure 4:

Definition 3.5 Let D be a $\frac{1}{2}$- dominating set of G. To each $u \in D$, define $PN_{\frac{1}{2}}(u,D)$, private neighborhood of u in D as $PN_{\frac{1}{2}}(u,D) = \{x \in V/N[x] \cap D = u$ and $|N_2(x) \cap D| \leq 1\} \cup \{x \in V/N[x] \cap D = \emptyset$ and $|N_2(x) \cap (Du)| = 1\}$

Remark 3.6 An $\frac{1}{2}$- dominating set of D of G is a minimal $\frac{1}{2}$- dominating set of G iff $PN_{\frac{1}{2}}(u,D), \emptyset$, for every $u \in D$.

Theorem 3.7 For any connected graph G, $\left\lceil \frac{1+diam(G)}{4} \right\rceil \leq \gamma_{\frac{1}{2}}(G)$. Proof: Let D be a $\gamma_{\frac{1}{2}}$- set of G. Let u and $v \in V(G)$ such that $d(u,v) = diam(G)$ Let

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P be a $u - v$ - shortest path. So P is a path on $1 + diam(G)$ - vertices. Let $D_1 = D \cap V(P)$ and $D_2 = D - D_1$. If $a \in D_1$, then $|N[a] \cap V(P)| \leq 3$ and $|N_2(a) \cap V(P)| \leq 2$. Let $a \in D_2$. Then $|N[a] \cap V(P)| \leq 3$. If $|N(a) \cap V(P)| = 3$, then $|N_2(a) \cap V(P)| \leq 2$. If $|N(a) \cap V(P)| = 2$, then $|N_2(a) \cap V(P)| \leq 3$. If $|N(a) \cap V(P)| = 1$, then $|N_2(a) \cap V(P)| \leq 4$, and if $|N(a) \cap V(P)| = 0$, then $|N_2(a) \cap V(P)| \leq 5$.
Thus if $a \in D = D_1 \cup D_2$, we have $\sum\limits_{x \in V(P)} = f_a(x) \leq 4$.

$$\sum_{a \in D} \Big(\sum_{x \in V(P)} f_a(x) \leq 4 f_a(x) \Big) \leq 4\gamma_{\frac{1}{2}}(G). \tag{3}$$

As D is a $\gamma_{\frac{1}{2}}$-set $\sum_{a \in D} = f_a(x) \geq 1$ for all $x \in V(P)$.
Therefore, from (3), we have $1 + diam(G) = |V(P)| \geq 4\gamma_{\frac{1}{2}}(G)$.

$$\therefore \quad \left\lceil \frac{1 + diam(G)}{4} \right\rceil \leq \gamma_{\frac{1}{2}}(G)$$

Examples for graph G for which $\gamma_{\frac{1}{2}}(G) = \left\lceil \frac{1+diam(G)}{4} \right\rceil$

(1) If $n \neq 0 (mod 4)$, the path P_n will attain this lower bound.
(2) The graph G given in Fig.5

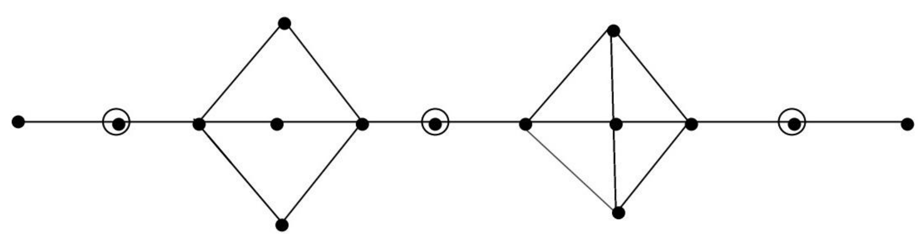

Figure 5: Examples for graph G for which

$$\gamma_{\frac{1}{2}}(G) = \left\lceil \frac{1+diam(G)}{4} \right\rceil.$$

Theorem 3.8 If G is a connected graph with $\delta(G) \geq 2$ and girth $g(G) \geq 9$, then $\gamma_{\frac{1}{2}}(G) \geq 1 + \Delta(G)$. Proof: Let D be a $\gamma_{\frac{1}{2}}$ -set for G and v be a vertex of G with the $degree \Delta(G)$.
As $\delta(G) \geq 2$ and grith $g(G) \geq 9$, the sets $N_1(v), N_2(v)$and $N_3(v)$ are all non-empty independent sets in G. Let $N_1(v) = u_1, u_2, , u_\Delta$. For each $i, 1 \leq i \leq \Delta$, let H_i be the

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component of the induced graph $\langle N_1(v) \cup N_2(v) \cup N_3(v) \rangle$ that contains the vertex u_i . If $i \neq j \in 1,,\Delta, d(x_i, y_i) \geq 3$ for all $x_i \in H_i$ and $y_j \in H_j - u_j$. Select $x_i \in H_i \cap N_2(v)$, for all $i, 1 \leq i \leq \Delta$. (Note that $H_i \cap N_2(v) \neq \phi$.
As D is a $\gamma_{\frac{1}{2}}$- set of G, $\sum_{a \in D} = f_a(x_i) \geq 1$, for all $i, 1 \leq i \leq \Delta$. (Note that if $v \in D, f_v(x_i) = \frac{1}{2} foralli)$. Then $(D \cap (H_i \cup N_2(x_i))) - v,$. As for $i, j, (H_i \cup N_2(x_i)) - v$ and $(H_j \cup N_2(x_j)) - v$ are disjoint, we have $|D| \geq \Delta$. We claim that $|D| = \Delta + 1$. Let $D_i = (D \cap H_i \cup N_2(x_i)) - u$, for $1 \leq i \leq \Delta$. Then $|D_i| \geq 1$, for all i, and $D_i \cap D_j =$ for $i \neq j$.
Case(i): $v \in D$. Then $v \cup (\cup_{i=1}^{\Delta} D_i) \subseteq D$ and hence $1 + \Delta \leq |\Delta|$.
Case(ii): $v < D$. Assume that $|D| = \Delta$. Then $|D_i| = 1$ for all i and $D = D_1 \cup D_2 \cup \ \cup D_k$. As $d(x_i, w) \geq 3$ for all $w \in D_j, i \neq j$, and $|D_i| = 1, D_i \in N[x_i]$, for all i. From $\delta(G) \geq 2$, we have $|N(x_i)| \geq 2$ for all i. Note that $u_i \in N(x_i)$ and for all y , $u_i \in N(x_i)$,
(a) $d(y, u_i) = 2$, as grith $g(G) \geq 9$,
(b) $d(y, w) \geq 3$ for all $w \in D_j, j \neq i$.
It follows that $D_i \in N[y]$ for all $y \neq u_i \in N[x_i]$ and hence $u_i < D_i$, for all i. Also $d(u_i, w) \geq 3$ for all $w \in D_j, i \neq j$ and $D_i \in N(u_i)$. Thus $D_i = x_i$ for all i, $1 \leq i \leq \Delta$ and $D = x_1, x_2, , x_\Delta$. Select $z_i \neq v \in N_2(x_i)$. Then $d(z_i, x_j) \geq 3$ for all $j \neq i$ and $\sum_j f_{x_j}(z_i) = \frac{1}{2}$, which is a contradiction as D is a half - dominating set of G. Thus $1 + \Delta \leq |D|$, even if $v \notin D$.

Remark 3.9 The graphs given in Fig.6 show that theorem 12 is not true if either $\delta(G) = 1$ or the grith $g(G) \leq 8$.

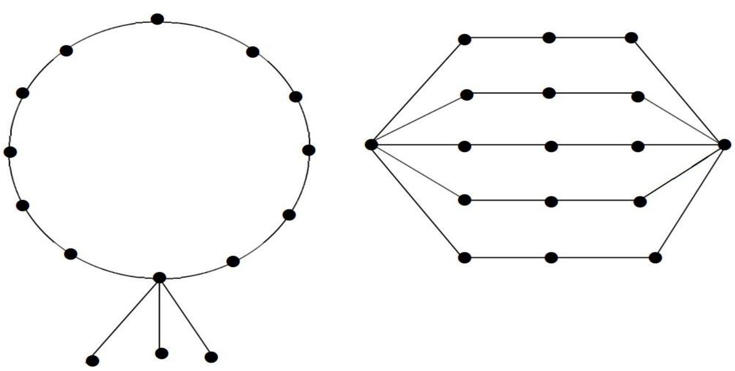

Figure 6: (a) A graph G, with $\delta = 1$, (b) A graph G, with $\delta \geq 2$,

$g(G) = 12$ and $\gamma_{\frac{1}{2}}(G) < \Delta g(G) = 8$ and $\gamma_{\frac{1}{2}}(G) < \Delta$

3.1.Characterization of connected graphs G for which $\gamma_{\frac{1}{2}}(G) = \lfloor \frac{n}{2} \rfloor$:

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If a graph G has no isolated vertex, then $\gamma_{\frac{1}{2}}(G) \leq \gamma(G) \leq \frac{n}{2}$. It is well known that $\gamma(G) = \frac{n}{2}$ if and only if the components of G are cycle C_4 of the corona HK_1 for any connected graph H. If H is a connected graph with$\Delta(H) \geq 2$, select a vertex u in H with $deg(u) = \Delta(H)$. Then $V(H) - u$ is a $\frac{1}{2}$- dominating set for the corona HK_1 and hence $\gamma_{\frac{1}{2}}(HK_1) < \frac{n}{2}$, where $|V(HK_1)| = n$. Thus we have the following theorem

Theorem 3.10 For a graph G of order n, with no isolated vertices, $\gamma_{\frac{1}{2}}(G) = \frac{n}{2}$ if and only if each component of G is either the cycle C_4 or the path P_4 or P_2.
Cockanye, Haynes and Hedetniemi characterized connected graphs G for which $\gamma(G) = \lfloor \frac{n}{2} \rfloor$. They defined six classes $G_i, 1 \leq i \leq 6$, of graphs. (for the description of these classes, we refer pages 44-45 of [2]). They proved the following theorem.

Theorem 3.11 [2] A connected graph G satisfies $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \in G = \cup_{i=1}^{6} G_i$.
Proof: So in order to find all connected graph G with $\gamma_{\frac{1}{2}}(G) = \lfloor \frac{n}{2} \rfloor$, it is enough to search for G in $G = \cup_{i=1}^{6} G_i$. Such a search leads to the following theorem.

Theorem 3.12 A connected graph G satisfies $\gamma_{\frac{1}{2}}(G) = \lfloor \frac{n}{2} \rfloor$ if and only if G is either P_2, P_3, P_4, C_3, C_4 or a connected graph G on five vertices with $\Delta(G) \leq 3$.
Proof: Macuaig and Shephered defined a collection A of graphs consisting of seven graphs (see page 42 in [2]), and obtained the following theorem.

Theorem 3.13 If G is a connected graph with $\delta(G) \geq 2$ and $G \notin A$,then $\gamma(G) \leq \frac{2n}{5}$.
Proof: As $\gamma_{\frac{1}{2}}(G) \leq \gamma(G)$, if G is a connected graph with $\delta(G) \geq 2$and if $G \notin A$, we have $\gamma_{\frac{1}{2}}(G) \leq \frac{2n}{5}$. Except the cycle C_4 , all other six graphs belonging to the class A have $\gamma_{\frac{1}{2}}(G) \leq \frac{2n}{5}$. Thus we have the following theorem.

Theorem 3.14 If G is a connected graph with $\delta(G) \geq 2$ and if G is not the cycle C_4 , then $\gamma_{\frac{1}{2}}(G) \leq \frac{2n}{5}$.
Proof: The bound given in the theorem 17 is sharp, as for any connected graph G on five vertices with $2 \leq \delta(G) \leq \Delta(G) \leq 3, \gamma_{\frac{1}{2}}(G)$ attains this bound.

3.2. Acknowledgements:

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