



On the Oscillation of Impulsive Vector Partial Differential Equations with Damping Term

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Abstract

This article is an attempt to study the class of impulsive vector partial functional differential equations with continuous distributed deviating arguments. The main aim of this paper is to present some sufficient conditions for the H -oscillation of solutions, using impulsive differential inequalities and an averaging technique with two different boundary conditions. Our main results are point up with a suitable example.

Key words: Neutral partial differential equations, Oscillation, Vector, Impulse, Distributed deviating arguments.

AMS Subject classification: 34A37, 35L70, 35R10, 35R12.

1. Introduction

Impulsive differential equations arise in many biological, chemical, physical systems and electrical networks. They appear in these systems as a natural description of observed evolution phenomena, in several real world problems. The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects and has many real world applications, since many models involve forces acting abruptly, almost instantly, and at different times. The theory of impulsive differential equations has its origins in Mil'man and Mishkys paper, in [12].

Early work on the oscillation theory of impulsive differential equations appeared in 1989 [5]. The first paper on impulsive partial differential equations [3] was published in 1991. Following these early results, several authors worked on the oscillatory behavior of impulsive partial differential equations with delays [4, 9, 10, 11, 26, 17, 18, 19, 20, 21]

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In 1970, Domšlak introduced the concept of H-oscillation to study the oscillation of the solutions of vector differential equations, where H is a unit vector in \mathbb{R}^M . We refer the reader to [1, 2, 6] for vector ordinary differential equations, [8, 13, 14, 15, 16, 28] for vector partial differential equations and [7, 27] for impulsive vector partial differential equations. For the essential background on the oscillation theory of differential equations, we refer the reader to the monographs [22, 25, 29, 30, 31, 32] and the references cited therein. There has been very little work on the study of impulsive vector partial differential equations with functional arguments. This scarcity has been the motivation that has led us to attempt to initiate a research effort and make some progress, in the study of impulsive nonlinear vector partial differential equations with continuous distributed deviating arguments.

In this paper, we consider impulsive nonlinear neutral delay vector partial differential equations with damping term of the form

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left[p(t) \frac{\partial}{\partial t} (U(x, t) + g(t)U(x, \rho(t))) \right] + r(t) \frac{\partial}{\partial t} (U(x, t) + g(t)U(x, \rho(t))) \\ + \int_c^d q(x, t, \zeta) U(x, \tau(t, \zeta)) d\eta(\zeta) = a(t) \Delta U(x, t) \\ + \int_c^d a(t, \zeta) \Delta U(x, \theta(t, \zeta)) d\eta(\zeta) + F(x, t), \quad \text{for } t \neq t_j \\ U(x, t_j^+) = a_j(x, t_j, U(x, t_j)), \\ \frac{\partial U(x, t_j^+)}{\partial t} = b_j \left(x, t_j, \frac{\partial U(x, t_j)}{\partial t} \right), \quad \text{for } t = t_j, j = 1, 2, \dots, \\ \text{and } (x, t) \in \Omega \times \mathbb{R}_+ \equiv G, \end{array} \right. \quad (1)$$

where Ω is a bounded domain in \mathbb{R}^N with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space \mathbb{R}^N . Moreover, we consider the following boundary conditions:

$$\frac{\partial}{\partial \gamma} U(x, t) + \mu(x, t)U(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+ \quad (2)$$

$$U(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+ \quad (3)$$

where γ is the outward unit normal vector to $\partial\Omega$, $\mu(x, t) \in C(\partial\Omega \times \mathbb{R}_+, \mathbb{R}_+)$ and $\mathbb{R}_+ = [0, +\infty)$.

Next, we present the following set of conditions which we assume to hold, throughout

the paper.

(H₁) $r(t) \in C(\mathbb{R}_+, \mathbb{R})$, $p(t) \in C^1(\mathbb{R}_+, (0, +\infty))$ with $p'(t) \geq 0$, $\int_{t_0}^{+\infty} \frac{1}{P(s)} ds = +\infty$,

where $P(t) = \exp\left(\int_{t_0}^t \frac{p'(s) + r(s)}{p(s)} ds\right)$, and $g(t) \in C^2(\mathbb{R}_+ \times [c, d], \mathbb{R}_+)$.

(H₂) $q(x, t, \zeta) \in C(\bar{G} \times [c, d], \mathbb{R}_+)$, $Q(t, \zeta) = \min_{x \in \bar{\Omega}} q(x, t, \zeta)$, $\tau(t, \zeta)$, $\theta(t, \zeta) \in C(\mathbb{R}_+ \times [c, d], \mathbb{R})$, $\tau(t, \zeta) \leq t$, $\theta(t, \zeta) \leq t$ for $\zeta \in [c, d]$, $\tau(t, \zeta)$ and $\theta(t, \zeta)$ are non-decreasing with respect to t and ζ respectively and

$$\liminf_{t \rightarrow +\infty, \zeta \in [c, d]} \tau(t, \zeta) = \liminf_{t \rightarrow +\infty, \zeta \in [c, d]} \theta(t, \zeta) = +\infty.$$

There exists a function $\sigma(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\sigma(t) \leq \tau(t, c)$, with $\sigma'(t) > 0$ and $\lim_{t \rightarrow +\infty} \sigma(t) = +\infty$.

(H₃) $a(t, \zeta) \in C(\mathbb{R}_+ \times [c, d], \mathbb{R})$, $a(t) \in PC(\mathbb{R}_+, \mathbb{R}_+)$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind, only at $t = t_j$, $j = 1, 2, \dots$, and left continuous at $t = t_j$, $j = 1, 2, \dots$.

(H₄) $\rho(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\lim_{t \rightarrow +\infty} \rho(t) = +\infty$, $\eta(\zeta) : [c, d] \rightarrow \mathbb{R}$ is nondecreasing, integral of (1) is a Stieltjes integral, $F \in C(\bar{G}, \mathbb{R}^M)$, $f_H(x, t) \in C(\bar{G}, \mathbb{R})$ and $\int_{\Omega} f_H(x, t) dx \leq 0$.

(H₅) All the components of $U(x, t)$ and its derivative $\frac{\partial}{\partial t} U(x, t)$ are piecewise continuous in t with discontinuities of the first kind, only at $t = t_j$, $j = 1, 2, \dots$, and left continuous at $t = t_j$

$$U(x, t_j) = U(x, t_j^-), \quad \frac{\partial}{\partial t} U(x, t_j) = \frac{\partial}{\partial t} U(x, t_j^-), \quad j = 1, 2, \dots.$$

(H₆) $a_j, b_j \in PC(\bar{G} \times \mathbb{R}, \mathbb{R})$ for $j = 1, 2, \dots$, and there exist constants $\alpha_j, \alpha_j^*, \beta_j, \beta_j^*$ such that for $j = 1, 2, \dots$,

$$\alpha_j^* \leq \frac{a_j(x, t_j, U(x, t_j))}{U(x, t_j)} \leq \alpha_j, \quad \beta_j^* \leq \frac{b_j\left(x, t_j, \frac{\partial U(x, t_j)}{\partial t}\right)}{\frac{\partial U(x, t_j)}{\partial t}} \leq \beta_j.$$

In Section 2, we present the definitions and introduce the notation we will use through the paper. In Section 3, we discuss the H -oscillation of problem 1-2. In Section 4, we discuss the H -oscillation of problem 1-3. In Section 5 we present an example to point up the main result.

2. Preliminaries

In this section, we present some definitions and review some noteworthy results, from the literature which we will use throughout the paper.

Definition 2.1 [See [32]] By a solution of (1)-(2)(3) we mean a function $U(x, t) \in C^2(\bar{\Omega} \times [t_1, +\infty), \mathbb{R}^M) \cap C^1(\bar{\Omega} \times [\hat{t}_1, +\infty), \mathbb{R}^M)$ which satisfies 1, where $t_1 := \min \left\{ 0, \min_{1 \leq i \leq m} \left[\inf_{t \geq 0} \theta(t, \zeta) \right], \left\{ \inf_{t \geq 0} \rho(t) \right\} \right\}$ and $\hat{t}_1 := \min \left\{ 0, \min_{\zeta \in [c, d]} \left\{ \inf_{t \geq 0} \tau(t, \zeta) \right\} \right\}$.

Now based on this definition of a solution, we can precisely define what we mean by H -oscillation.

Definition 2.2 [See [32]] Let H be a fixed unit vector in \mathbb{R}^M . A solution $U(x, t)$ of 1 is said to be H -oscillatory in G if the inner product $\langle U(x, t), H \rangle$ has a zero in $\Omega \times [t, +\infty)$ for $t > 0$. Otherwise it is a H -nonoscillatory.

Next, we state two results which will help us establish our results.

Lemma 2.3 [See [5]] If X and Y are nonnegative, then

$$\begin{aligned} X^\delta - \delta XY^{\delta-1} + (\delta - 1)Y^\delta &\geq 0, & \text{if } \delta > 1 \\ X^\delta - \delta XY^{\delta-1} - (1 - \delta)Y^\delta &\leq 0, & \text{if } 0 < \delta < 1. \end{aligned}$$

In both cases, the equality holds, if and only if $X = Y$. It is known [30] that the first eigenvalue λ_0 of the problem

$$\begin{cases} \Delta \nu(x) + \lambda \nu(x) = 0 & \text{in } \Omega \\ \nu(x) = 0 & \text{on } \partial\Omega \end{cases}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω .

For convenience, we introduce the following notations:

$$\begin{aligned} u_H(x, t) &= \langle U(x, t), H \rangle, \quad G(t) = g_0 \int_c^d Q(t, \zeta) d\eta(\zeta), \\ V_H(t) &= \frac{1}{|\Omega|} \int_\Omega u_H(x, t) dx, \quad \tilde{V}_H(t) = K_\varphi \int_\Omega u_H(x, t) \varphi(x) dx \end{aligned}$$

where $g_0 = 1 - g(\tau(t, \zeta))$, $|\Omega| = \int_\Omega dx$, $K_\varphi = \left(\int_\Omega \varphi(x) dx \right)^{-1}$.

3. H -Oscillations of (1)-(2)

In this section, we establish sufficient conditions for the H -oscillation of all solutions of (1)-(2).

Lemma 3.1 Let H be a fixed unit vector in \mathbb{R}^M and $U(x, t)$ be a solution of (1).

(i) If $u_H(x, t)$ is eventually positive, then $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality

$$\begin{cases} \frac{\partial}{\partial t} \left(p(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) \right) \\ + r(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) + \int_c^d Q(t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) \\ - a(t) \Delta u_H(x, t) - \int_c^d a(t, \zeta) \Delta u_H(x, \theta(t, \zeta)) d\eta(\zeta) \leq f_H(x, t), \quad t \neq t_j \\ \alpha_j^* \leq \frac{u_H(x, t_j^+)}{u_H(x, t_j)} \leq \alpha_j, \quad \beta_j^* \leq \frac{u'_H(x, t_j^+)}{u'_H(x, t_j)} \leq \beta_j, \quad j = 1, 2, \dots \end{cases} \quad (4)$$

(ii) If $u_H(x, t)$ is eventually negative, then $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality

$$\begin{cases} \frac{\partial}{\partial t} \left(p(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) \right) \\ + r(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) + \int_c^d Q(t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) \\ - a(t) \Delta u_H(x, t) - \int_c^d a(t, \zeta) \Delta u_H(x, \theta(t, \zeta)) d\eta(\zeta) \geq f_H(x, t), \quad t \neq t_j \\ \alpha_j^* \geq \frac{u_H(x, t_j^+)}{u_H(x, t_j)} \geq \alpha_j, \quad \beta_j^* \geq \frac{u'_H(x, t_j^+)}{u'_H(x, t_j)} \geq \beta_j, \quad j = 1, 2, \dots \end{cases} \quad (5)$$

Proof: (i) Let $u_H(x, t)$ be eventually positive.

Case:(i) $t \neq t_j, j = 1, 2, \dots$. Taking the inner product of 1 and H , we get

$$\begin{aligned} & \frac{\partial}{\partial t} \left(p(t) \frac{\partial}{\partial t} (\langle U(x, t), H \rangle + g(t) \langle U(x, \rho(t)), H \rangle) \right) \\ & + r(t) \frac{\partial}{\partial t} (\langle U(x, t), H \rangle + g(t) \langle U(x, \rho(t)), H \rangle) + \int_c^d q(x, t, \zeta) \langle U(x, \tau(t, \zeta)), H \rangle d\eta(\zeta) \\ & = a(t) \Delta \langle U(x, t), H \rangle + \int_c^d a(t, \zeta) \Delta \langle U(x, \theta(t, \zeta)), H \rangle d\eta(\zeta) + \langle F(x, t), H \rangle, \end{aligned}$$

that is

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left(p(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) \right) \\ + r(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) + \int_c^d q(x, t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) \\ = a(t) \Delta u_H(x, t) + \int_c^d a(t, \zeta) \Delta u_H(x, \theta(t, \zeta)) d\eta(\zeta) + f_H(x, t). \end{array} \right. \quad (6)$$

Using condition (H_2) , we have

$$\int_c^d q(x, t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) \geq \int_c^d Q(t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta). \quad (7)$$

From 6 and 7, it follows that

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left(p(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) \right) \\ + r(t) \frac{\partial}{\partial t} (u_H(x, t) + g(t)u_H(x, \rho(t))) + \int_c^d Q(t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) \\ - a(t) \Delta u_H(x, t) - \int_c^d a(t, \zeta) \Delta u_H(x, \theta(t, \zeta)) d\eta(\zeta) \leq f_H(x, t), \quad t \neq t_j. \end{array} \right. \quad (8)$$

Case:(ii) $t = t_j, j = 1, 2, \dots$. Taking the inner product of (1) and H , and using (H_6) , we obtain

$$\begin{aligned} \alpha_j^* &\leq \frac{U(x, t_j^+)}{U(x, t_j)} \leq \alpha_j, & \beta_j^* &\leq \frac{U'(x, t_j^+)}{U'(x, t_j)} \leq \beta_j \\ \alpha_j^* &\leq \frac{\langle U(x, t_j^+), H \rangle}{\langle U(x, t_j), H \rangle} \leq \alpha_j, & \beta_j^* &\leq \frac{\langle U'(x, t_j^+), H \rangle}{\langle U'(x, t_j), H \rangle} \leq \beta_j \end{aligned}$$

that is

$$\alpha_j^* \leq \frac{u_H(x, t_j^+)}{u_H(x, t_j)} \leq \alpha_j, \quad \beta_j^* \leq \frac{u'_H(x, t_j^+)}{u'_H(x, t_j)} \leq \beta_j, \quad j = 1, 2, \dots \quad (9)$$

Therefore, combining 8 and 9 we immediately obtain 4, which shows that $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality 4.

(ii) The proof is similar to that of (i) and we omit it. The proof is complete.

Let H be a fixed unit vector in \mathbb{R}^M . The inner product of boundary conditions 2 [3] and H yields the following boundary conditions:

$$\frac{\partial}{\partial \gamma} u_H(x, t) + \mu(x, t) u_H(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+, \quad (2')$$

$$u_H(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+. \quad (3')$$

Lemma 3.2 Let H be a fixed unit vector in \mathbb{R}^M . If the scalar impulsive partial differential inequality (4) has no eventually positive solutions and the scalar impulsive partial differential inequality (5) has no eventually negative solutions, satisfying the boundary conditions (2') [(3')], then every solution $U(x, t)$ of (1)-(2)(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Proof: Suppose to the contrary that there is a H -nonoscillatory solution $U(x, t)$ of (1)-(2)(3) in G , then $u_H(x, t)$ is eventually positive or eventually negative. If $u_H(x, t)$ is eventually positive, by Lemma 3.1, easily, we obtain that $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality (4). On the other hand, it is easy to see that $u_H(x, t)$ satisfies the boundary conditions (2')[(3')]. This is a contradiction to the hypothesis.

Similarly, if $u_H(x, t)$ is eventually negative using Lemma 3.1, easily, we obtain that $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality (5). It is obvious that $u_H(x, t)$ satisfies, the boundary conditions (2')[(3')]. This is a contradiction. The proof is complete.

Theorem 3.3 Let H be a fixed unit vector in \mathbb{R}^M . If the impulsive differential inequality

$$\begin{cases} (p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \leq 0, & t \neq t_j \\ \alpha_j^* \leq \frac{Z_H(t_j^+)}{Z_H(t_j)} \leq \alpha_j, & \beta_j^* \leq \frac{Z'_H(t_j^+)}{Z'_H(t_j)} \leq \beta_j, \quad j = 1, 2, \dots, \end{cases} \quad (10)$$

has no eventually positive solutions and the impulsive differential inequality

$$\begin{cases} (p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \geq 0, & t \neq t_j \\ \alpha_j^* \geq \frac{Z_H(t_j^+)}{Z_H(t_j)} \geq \alpha_j, & \beta_j^* \geq \frac{Z'_H(t_j^+)}{Z'_H(t_j)} \geq \beta_j, \quad j = 1, 2, \dots, \end{cases} \quad (11)$$

has no eventually negative solutions, then every solution $U(x, t)$ of (1)-(2) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Proof: Suppose to the contrary that there exists a solution $U(x, t)$ of (1)-(2) which is not H -oscillatory in $\Omega \times \mathbb{R}_+$. Without loss of generality, we may assume that $u_H(x, t) > 0$ in $\Omega \times [t_0, +\infty)$, for some $t_0 > 0$. By the assumption that there exists a $t_1 > t_0$ such that $\rho(t) \geq t_0$ for $t \geq t_0$, $\tau(t, \zeta) \geq t_0$ and $\theta(t, \zeta) \geq t_0$ for $(t, \zeta) \in [t_1, +\infty) \times [c, d]$, we have that

$$u_H(x, \rho(t)) > 0, \quad u_H(x, \tau(t, \zeta)) > 0 \quad \text{and} \quad u_H(x, \theta(t, \zeta)) > 0,$$

for $x \in \Omega$, $t \in [t_1, +\infty)$, $\zeta \in [c, d]$.

For $t \geq t_0$ and $t \neq t_j$ for $j = 1, 2, \dots$, we multiply both sides of inequality (4) by $\frac{1}{|\Omega|}$ and integrate with respect to x over the domain Ω to attain

$$\left\{ \begin{array}{l} \frac{d}{dt} \left(p(t) \frac{d}{dt} \left(\frac{1}{|\Omega|} \int_{\Omega} u_H(x, t) dx + \frac{1}{|\Omega|} \int_{\Omega} g(t) u_H(x, \rho(t)) dx \right) \right) \\ + r(t) \frac{d}{dt} \left(\frac{1}{|\Omega|} \int_{\Omega} u_H(x, t) dx + \frac{1}{|\Omega|} \int_{\Omega} g(t) u_H(x, \rho(t)) dx \right) \\ + \frac{1}{|\Omega|} \int_{\Omega} \int_c^d Q(t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) dx - a(t) \frac{1}{|\Omega|} \int_{\Omega} \Delta u_H(x, t) dx \\ - \frac{1}{|\Omega|} \int_{\Omega} \int_c^d a(t, \zeta) \Delta u_H(x, \theta(t, \zeta)) d\eta(\zeta) dx \leq \frac{1}{|\Omega|} \int_{\Omega} f_H(x, t) dx, \quad t \neq t_j \end{array} \right. \quad (12)$$

Using Green's formula and boundary condition (2'), we have that

$$\int_{\Omega} \Delta u_H(x, t) dx = \int_{\partial\Omega} \frac{\partial}{\partial\gamma} u_H(x, t) dS = - \int_{\partial\Omega} \mu(x, t) u_H(x, t) dS \leq 0, \quad (13)$$

and

$$\begin{aligned} \int_{\Omega} \Delta u_H(x, \theta(t, \zeta)) dx &= \int_{\partial\Omega} \frac{\partial}{\partial\gamma} u_H(x, \theta(t, \zeta)) dS \\ &= - \int_{\partial\Omega} \mu(x, \theta(t, \zeta)) u_H(x, \theta(t, \zeta)) dS \leq 0, \quad t \geq t_0 \end{aligned} \quad (14)$$

where dS is the surface component on $\partial\Omega$. Furthermore, by (H_4) , $\int_{\Omega} f_H(x, t) dx \leq 0$.

Combining (12)-(14) we get

$$\frac{d}{dt} \left[p(t) \frac{d}{dt} (V_H(t) + g(t)V_H(\rho(t))) \right] + r(t) \frac{d}{dt} (V_H(t) + g(t)V_H(\rho(t))) + \int_c^d Q(t, \zeta) V_H(\tau(t, \zeta)) d\eta(\zeta) \leq 0, \quad t \geq t_0.$$

Setting $Z_H(t) = V_H(t) + g(t)V_H(\rho(t))$, we have

$$(p(t)Z_H'(t))' + r(t)Z_H'(t) + \int_c^d Q(t, \zeta) V_H(\tau(t, \zeta)) d\eta(\zeta) \leq 0. \quad (15)$$

Clearly $Z_H(t) > 0$ for $t \geq t_1$. Next we prove that $Z_H'(t) > 0$ for $t \geq t_2$. In fact assume there exists $T \geq t_2$ such that $Z_H'(T) \leq 0$. We have

$$p(t)Z_H''(t) + (p'(t) + r(t)) Z_H'(t) \leq 0, \quad t \geq t_2 \quad (16)$$

and from (H_1) it follows that $P'(t) = P(t) \left(\frac{p'(t)+r(t)}{p(t)} \right)$, $P(t) > 0$ and $P'(t) \geq 0$ for $t \geq t_2$. Multiply both sides of this inequality by $\frac{P(t)}{p(t)}$, we obtain

$$P(t)Z_H''(t) + P'(t)Z_H'(t) = (P(t)Z_H'(t))' \leq 0. \quad (17)$$

From (17) we have $P(t)Z_H'(t) \leq P(T)Z_H'(T) \leq 0$, $t \geq T$. Thus

$$Z_H(t) \leq Z_H(T) + P(T)Z_H'(T) \int_T^t \frac{ds}{P(s)} \quad \text{for } t \geq T.$$

Again, from (H_1) we have $\lim_{t \rightarrow \infty} Z_H(t) = -\infty$ which contradicts the fact that $Z_H(t) > 0$ for $t > 0$. Hence $Z_H'(t) > 0$ and since $\rho(t) \leq t$ for $t \geq t_1$, we have

$$\begin{aligned} V_H(t) &= Z_H(t) - g(t)V_H(\rho(t)) \geq Z_H(t) - g(t)Z_H(\rho(t)) \\ &\geq Z_H(t) (1 - g(t)) \end{aligned}$$

$$\text{and } V_H(\tau(t, \zeta)) \geq g_0 Z_H(\tau(t, \zeta)).$$

Therefore from (15), we have

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + g_0 \int_c^d Q(t, \zeta)Z_H(\tau(t, \zeta))d\eta(\zeta) \leq 0, \quad t \geq t_0$$

From (H_2) and $Z'_H > 0$, we have

$$Z_H(\tau(t, \zeta)) \geq Z_H(\tau(t, c)) > 0, \quad \zeta \in [c, d] \quad \text{and} \quad \sigma(t) \leq \tau(t, c) \leq t.$$

Thus, $Z_H(\sigma(t)) \leq Z_H(\tau(t, c))$ and therefore

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \leq 0, \quad t \geq t_1. \quad (18)$$

For $t \geq t_0$, $t = t_j$, $j = 1, 2, \dots$, multiplying both sides of inequality (4) by $1/|\Omega|$ and integrating with respect to x over the domain Ω , we obtain

$$\alpha_j^* \leq \frac{V_H(t_j^+)}{V_H(t_j)} \leq \alpha_j, \beta_j^* \leq \frac{V'_H(t_j^+)}{V'_H(t_j)} \leq \beta_j.$$

Since $Z_H(t) = V_H(t) + g(t)V_H(\rho(t))$, we have that

$$\alpha_j^* \leq \frac{Z_H(t_j^+)}{Z_H(t_j)} \leq \alpha_j, \quad \beta_j^* \leq \frac{Z'_H(t_j^+)}{Z'_H(t_j)} \leq \beta_j. \quad (19)$$

Therefore (18) and (19) show that $Z_H(t) > 0$ is a positive solution of the impulsive differential inequality (10). This is a contradiction.

Suppose, now, that $u_H(x, t) < 0$ is a negative solution of the impulsive partial differential inequality (5) satisfying the boundary condition (2), $(x, t) \in \Omega \times [t_0, +\infty)$, $t_0 > 0$. Using the above procedure, easily, we can reach a contradiction. The proof is complete.

Theorem 3.4 Assume that if there exist a function $\Phi(t) \in C^1(\mathbb{R}_+, (0, +\infty))$ which is nondecreasing with respect to t , such that

$$\int_{t_0}^{+\infty} \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left[\Phi(s)G(s) - \frac{D^2(s)}{4E(s)} \right] ds = +\infty, \quad (20)$$

where

$$D(t) = \frac{\Phi'(t)}{\Phi(t)} - \frac{r(t)}{p(t)} \quad \text{and} \quad E(t) = \frac{\sigma'(t)}{\Phi(\sigma(t))p(\sigma(t))},$$

then every solution of (1)-(2) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Proof: We show that the inequality (10) has no eventually positive solution if the conditions of Theorem (3.3) hold. Suppose that $Z_H(t)$ is an eventually positive solution of the inequality (10). Then there exists a number $t_1 > t_0$ such that $Z_H(\sigma(t)) > 0$ for $t > t_1$. Thus we have

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \leq 0. \quad (21)$$

Define

$$W(t) := \Phi(t) \frac{p(t)Z'_H(t)}{Z_H(\sigma(t))},$$

then $W(t) \geq 0$ and

$$W'(t) \leq \left(\frac{\Phi'(t)}{\Phi(t)} - \frac{r(t)}{p(t)} \right) W(t) - \Phi(t)G(t) - \frac{W^2(t)}{\Phi(\sigma(t))} \frac{\sigma'(t)}{p(\sigma(t))}.$$

Thus $W'(t) \leq D(t)W(t) - G(t)\Phi(t) - W^2(t)E(t)$ and $W(t_j^+) \leq \frac{\beta_j}{\alpha_j^*} W(t_j)$. Define

$$A(t) = \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} W(t).$$

It is clear that $W(t)$ is continuous in each interval $(t_j, t_{k+1}]$, and since $W(t_j^+) \leq \frac{\beta_j}{\alpha_j^*} W(t_j)$, it follows that

$$A(t_j^+) = \prod_{t_0 \leq t_i \leq t_j} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} W(t_j^+) \leq \prod_{t_0 \leq t_i < t_j} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} W(t_j) = A(t_j)$$

and for all $t \geq t_0$,

$$A(t_j^-) = \prod_{t_0 \leq t_i \leq t_{j-1}} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} W(t_j^-) \leq \prod_{t_0 \leq t_i < t_j} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} W(t_j) = A(t_j)$$

which implies that $A(t)$ is continuous on $[t_0, +\infty)$.

$$\begin{aligned} & A'(t) + \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right) A^2(t)E(t) + \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(t)\Phi(t) - A(t)D(t) \\ &= \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} [W'(t) + W^2(t)E(t) - W(t)D(t) + G(t)\Phi(t)] \leq 0, \end{aligned}$$

that is

$$A'(t) \leq - \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right) E(t)A^2(t) + A(t)D(t) - \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(t)\Phi(t). \quad (22)$$

Taking

$$X(t) = \left(\prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right) E(t) \right)^{\frac{1}{2}} A(t) \quad \text{and} \quad Y(t) = \frac{D(t)}{2} \left(\prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \frac{1}{E(t)} \right)^{\frac{1}{2}},$$

using Lemma (2.3), we have

$$D(t)A(t) - \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right) E(t)A^2(t) \leq \frac{D^2(t)}{4E(t)} \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1}.$$

Thus

$$A'(t) \leq - \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left[G(t)\Phi(t) - \frac{D^2(t)}{4E(t)} \right].$$

Integrating both sides from t_0 to t , we have

$$A(t) \leq A(t_0) - \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left[G(s)\Phi(s) - \frac{D^2(s)}{4E(s)} \right] ds.$$

Letting $t \rightarrow \infty$ and using (20) we have $\lim_{t \rightarrow \infty} A(t) = -\infty$, which leads to a contradiction with $A(t) \geq 0$ and completes the proof.

Theorem 3.5 Assume that there exist functions Φ and $\phi \in C^1(\mathbb{R}_+, (0, +\infty))$ where Φ is nondecreasing, and functions $b, B \in C^1(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} = \{(t, s) : t \geq s \geq t_0 > 0\}$

such that

(H₇) $B(t, t) = 0$ and $B(t, s) > 0$ for all $t > s \geq t_0$,

(H₈) $\frac{\partial B(t, s)}{\partial t} \geq 0$ and $\frac{\partial B(t, s)}{\partial s} \leq 0$,

(H₉) $-\frac{\partial B(t, s)}{\partial s} = b(t, s)\sqrt{B(t, s)}$. If

$$\limsup_{t \rightarrow +\infty} \frac{1}{B(t, t_0)} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left(G(s)\Phi(s)B(t, s)\phi(s) - \frac{1}{4} \frac{[b(t, s)\phi(s) - \phi'(s)\sqrt{B(t, s)} - D(s)\phi(s)\sqrt{B(t, s)}]^2}{E(s)\phi(s)} \right) ds = +\infty, \quad (23)$$

then every solution of (1)-(2) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Proof: Let $Z_H(t)$ is an eventually positive solution of (10). Proceeding as in the proof of Theorem (3.4) we obtain

$$A'(t) \leq - \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right) E(t)A^2(t) + A(t)D(t) - \prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(t)\Phi(t)$$

multiplying the above inequality by $B(t, s)\phi(s)$ for $t \geq s \geq T$, and integrating from T to t , we have

$$\begin{aligned} \int_T^t A'(s)B(t, s)\phi(s)ds &\leq - \int_T^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right) E(s)A^2(s)B(t, s)\phi(s)ds \\ &\quad + \int_T^t A(s)D(s)B(t, s)\phi(s)ds \\ &\quad - \int_T^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(s)\Phi(s)B(t, s)\phi(s)ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_T^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(s)\Phi(s)B(t, s)\phi(s)ds &\leq A(T)B(t, T)\phi(T) \\ &\quad - \int_T^t \left[-\frac{\partial B(t, s)}{\partial s}\phi(s) - B(t, s)\phi'(s) - D(s)B(t, s)\phi(s) \right] A(s)ds \\ &\quad - \int_T^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right) E(s)A^2(s)B(t, s)\phi(s)ds. \end{aligned}$$

Therefore

$$\begin{aligned}
 & \int_T^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(s)\Phi(s)B(t,s)\phi(s)ds \\
 & - \frac{1}{4} \int_T^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \frac{\left[b(t,s)\phi(s) - \phi'(s)\sqrt{B(t,s)} - D(s)\phi(s)\sqrt{B(t,s)} \right]^2}{E(s)\phi(s)} \\
 & \leq A(T)B(t,T)\phi(T).
 \end{aligned} \tag{24}$$

From (24) for $t \geq T \geq t_0$, we have

$$\begin{aligned}
 & \frac{1}{B(t,t_0)} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left[G(s)\Phi(s)B(t,s)\phi(s) \right. \\
 & \quad \left. - \frac{1}{4} \frac{\left[b(t,s)\phi(s) - \phi'(s)\sqrt{B(t,s)} - D(s)\phi(s)\sqrt{B(t,s)} \right]^2}{E(s)\phi(s)} \right] ds \\
 & = \frac{1}{B(t,t_0)} \left[\int_{t_0}^T + \int_T^t \right] \left\{ \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left(G(s)\Phi(s)B(t,s)\phi(s) \right. \right. \\
 & \quad \left. \left. - \frac{1}{4} \frac{\left[b(t,s)\phi(s) - \phi'(s)\sqrt{B(t,s)} - D(s)\phi(s)\sqrt{B(t,s)} \right]^2}{E(s)\phi(s)} \right) \right\} ds \\
 & \leq \int_{t_0}^T \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(s)\Phi(s)\phi(s)ds + \phi(T)A(T).
 \end{aligned}$$

Letting $t \rightarrow +\infty$, we have

$$\begin{aligned}
 & \limsup_{t \rightarrow +\infty} \frac{1}{B(t,t_0)} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left(G(s)\Phi(s)B(t,s)\phi(s) \right. \\
 & \quad \left. - \frac{1}{4} \frac{\left[b(t,s)\phi(s) - \phi'(s)\sqrt{B(t,s)} - D(s)\phi(s)\sqrt{B(t,s)} \right]^2}{E(s)\phi(s)} \right) ds \\
 & \leq \int_{t_0}^T \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} G(s)\Phi(s)\phi(s)ds + \phi(T)A(T) \\
 & < +\infty,
 \end{aligned}$$

which leads to a contradiction with (23) and completes the proof.

Choosing $\phi(s) = \Phi(s) \equiv 1$, in Theorem (3.5), we establish the following result.

Corollary 3.6 Assume that the conditions of Theorem (3.5) hold and

$$\limsup_{t \rightarrow +\infty} \frac{1}{B(t, t_0)} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} \left(G(s)B(t, s) - \frac{1}{4} \frac{[b(t, s) - D(s)\sqrt{B(t, s)}]^2}{E(s)} \right) ds = +\infty,$$

then every solution of (1)-(2) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

From Theorem (3.5) and Corollary (3.6), we can have several oscillatory criteria by different choices of the weighted function $B(t, s)$. For example, choosing $B(t, s) = (t - s)^{\lambda-1}$, $t \geq s \geq t_0$, in which $\lambda > 2$ is an integer, then $b(t, s) = (\lambda - 1)(t - s)^{(\lambda-3)/2}$, $t \geq s \geq t_0$. Corollary (3.6) leads to the following result.

Corollary 3.7 If $\lambda > 2$ is an integer such that

$$\limsup_{t \rightarrow +\infty} \frac{1}{(t - t_0)^{\lambda-1}} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} (t - s)^{\lambda-1} \left(G(s) - \frac{1}{4E(s)} \times \left[D^2(s) - \frac{2(\lambda - 1)D(s)}{(t - s)} + \frac{(\lambda - 1)^2}{(t - s)^2} \right] \right) ds = +\infty,$$

then every solution of (1)-(2) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Now we consider $B(t, s) = [P(t) - P(s)]^\lambda$, $t \geq s \geq t_0$, where $P(t) = \int_{t_0}^t \frac{1}{p(s)} ds$ and

$\lim_{t \rightarrow +\infty} P(t) = +\infty$, then $b(t, s) = \lambda[P(t) - P(s)]^{\lambda-2/2}$. This leads to the result in the following Corollary.

Corollary 3.8 If $\lambda > 2$ is an integer such that

$$\limsup_{t \rightarrow +\infty} \frac{1}{[P(t) - P(t_0)]^\lambda} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \left(\frac{\beta_j}{\alpha_j^*} \right)^{-1} [P(t) - P(s)]^\lambda \left(G(s) - \frac{1}{4E(s)} \times \left[D^2(s) - \frac{2\lambda D(s)}{P(t) - P(s)} + \frac{\lambda^2}{[P(t) - P(s)]^2} \right] \right) ds = +\infty,$$

then every solution of (1)-(2) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

4.H-Oscillation of (1)-(3)

In this section, we establish sufficient conditions for the H -oscillation of all solutions of (1)-(3).

Theorem 4.1 Let H be a fixed unit vector in \mathbb{R}^M . If the impulsive differential inequality

$$\begin{cases} \left(p(t)\tilde{Z}'_H(t) \right)' + r(t)\tilde{Z}'_H(t) + G(t)\tilde{Z}_H(\sigma(t)) \leq 0, & t \neq t_j \\ \alpha_j^* \leq \frac{\tilde{Z}_H(t_j^+)}{\tilde{Z}_H(t_j)} \leq \alpha_j, & \beta_j^* \leq \frac{\tilde{Z}'_H(t_j^+)}{\tilde{Z}'_H(t_j)} \leq \beta_j \quad j = 1, 2, \dots, \end{cases} \quad (25)$$

has no eventually positive solution and the impulsive differential inequality

$$\begin{cases} \left(p(t)\tilde{Z}'_H(t) \right)' + r(t)\tilde{Z}'_H(t) + G(t)\tilde{Z}_H(\sigma(t)) \geq 0, & t \neq t_j \\ \alpha_j^* \geq \frac{\tilde{Z}_H(t_j^+)}{\tilde{Z}_H(t_j)} \geq \alpha_j, & \beta_j^* \geq \frac{\tilde{Z}'_H(t_j^+)}{\tilde{Z}'_H(t_j)} \geq \beta_j \quad j = 1, 2, \dots, \end{cases} \quad (26)$$

has no eventually negative solution, then every solution $U(x, t)$ of (1)-(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Proof: Suppose to the contrary that there exists a solution $U(x, t)$ of (1)-(3) which is not H -oscillatory in G . Without loss of generality, we may assume that $u_H(x, t) > 0$ in $\Omega \times [t_0, +\infty)$, for some $t_0 > 0$.

For $t \geq t_0$ and $t \neq t_j$ for $j = 1, 2, \dots$, we multiply both sides of inequality (4) by $K_\varphi \varphi(x)$ and integrate with respect to x over the domain Ω to attain

$$\begin{cases} \left(\frac{\partial}{\partial t} \left(p(t) \frac{\partial}{\partial t} \left(\int_{\Omega} K_\varphi \varphi(x) u_H(x, t) dx + \int_{\Omega} g(t) K_\varphi \varphi(x) u_H(x, \rho(t)) dx \right) \right) \right. \\ \left. + r(t) \frac{\partial}{\partial t} \left(\int_{\Omega} K_\varphi \varphi(x) u_H(x, t) dx + \int_{\Omega} g(t) K_\varphi \varphi(x) u_H(x, \rho(t)) dx \right) \right. \\ \left. + \int_{\Omega} \int_c^d K_\varphi \varphi(x) Q(t, \zeta) u_H(x, \tau(t, \zeta)) d\eta(\zeta) dx - a(t) \int_{\Omega} K_\varphi \varphi(x) \Delta u_H(x, t) dx \right. \\ \left. - \int_{\Omega} \int_c^d a(t, \zeta) K_\varphi \varphi(x) \Delta u_H(x, \theta(t, \zeta)) d\eta(\zeta) dx \leq \int_{\Omega} K_\varphi \varphi(x) f_H(x, t) dx, \quad t \neq t_j. \right. \end{cases} \quad (27)$$

Using Green's formula and boundary condition (3'), we have that

$$\int_{\Omega} K_{\varphi} \varphi(x) \Delta u_H(x, t) dx = K_{\varphi} \int_{\partial\Omega} \left[\varphi(x) \frac{\partial u_H(x, t)}{\partial \gamma} - u_H(x, t) \frac{\partial \varphi(x)}{\partial \gamma} \right] dS + K_{\varphi} \int_{\Omega} u_H(x, t) \Delta \varphi(x) dx = 0 - \lambda_0 \tilde{V}_H(t) \leq 0, \quad (28)$$

and

$$\int_{\Omega} K_{\varphi} \varphi(x) \Delta u_H(x, \theta(t, \zeta)) dx = K_{\varphi} \int_{\partial\Omega} \left[\varphi(x) \frac{\partial u_H(x, \theta(t, \zeta))}{\partial \gamma} - u_H(x, \theta(t, \zeta)) \frac{\partial \varphi(x)}{\partial \gamma} \right] dS + K_{\varphi} \int_{\Omega} u_H(x, \theta(t, \zeta)) \Delta \varphi(x) dx = 0 - \lambda_0 \tilde{V}_H(\theta(t, \zeta)) \leq 0 \quad (29)$$

where dS is the surface component on $\partial\Omega$. Furthermore, by (H_4) , $\int_{\Omega} f_H(x, t) dx \leq 0$. In view of (27)-(29), we get

$$\frac{d}{dt} \left[p(t) \frac{d}{dt} \left(\tilde{V}_H(t) + g(t) \tilde{V}_H(\rho(t)) \right) \right] + r(t) \frac{d}{dt} \left(\tilde{V}_H(t) + g(t) \tilde{V}_H(\rho(t)) \right) + \int_c^d Q(t, \zeta) \tilde{V}_H(\tau(t, \zeta)) d\eta(\zeta) \leq 0, \quad t \geq t_0.$$

Setting $\tilde{Z}_H(t) = \tilde{V}_H(t) + g(t) \tilde{V}_H(\rho(t))$, we have

$$\left(p(t) \tilde{Z}'_H(t) \right)' + r(t) \tilde{Z}'_H(t) + \int_c^d Q(t, \zeta) \tilde{V}_H(\tau(t, \zeta)) d\eta(\zeta) \leq 0.$$

The rest of the proof is similar to that of Theorem (3.3) and hence the details are omitted. The proof is complete.

Theorem 4.2 Let the conditions of Theorem (3.4) hold, then every solution of (1)-(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Theorem 4.3 Let the conditions of Theorem (3.5) hold, then every solution of (1)-(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Corollary 4.4 Let the conditions of Corollary (3.6) hold, then every solution of (1)-(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Corollary 4.5 Let the conditions of Corollary (3.7) hold, then every solution of (1)-(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

Corollary 4.6 Let the conditions of Corollary (3.8) hold, then every solution of (1)-(3) is H -oscillatory in $\Omega \times \mathbb{R}_+$.

5. Example

In this section we provide an example to point up our result.

Example 5.1 Consider the following impulsive partial differential equations

$$\begin{cases} \frac{\partial}{\partial t} \left(3 \frac{\partial}{\partial t} \left(U(x, t) + \frac{1}{5} U(x, t - \pi) \right) \right) + \left(-\frac{3}{4} \right) \frac{\partial}{\partial t} \left(U(x, t) + \frac{1}{5} U(x, t - \pi) \right) \\ + \frac{4}{5} \int_{\pi/4}^{\pi/2} U(x, t - 2\zeta) d\zeta = 5 \Delta U(x, t) + \frac{22}{5} \int_{\pi/4}^{\pi/2} \Delta U(x, t - 2\zeta) d\zeta + F(x, t), \quad t \neq t_j, \\ U(x, t_j^+) = \frac{j+1}{j} U(x, t_j), \\ \frac{\partial}{\partial t} U(x, t_j^+) = \frac{\partial}{\partial t} U(x, t_j), \quad j = 1, 2, \dots, \end{cases} \quad (30)$$

for $x \in (0, \pi)$, $t \in \mathbb{R}_+$, with the boundary condition

$$U(0, t) = U(\pi, t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \neq t_j, \quad j = 1, 2, \dots. \quad (31)$$

Here $\Omega = (0, \pi)$, $N = 1$, $M = 2$, $\alpha_j = \alpha_j^* = \frac{j+1}{j}$, $\beta_j = \beta_j^* = 1$, $p(t) = 3$, $g(t) = \frac{1}{5}$, $\rho(t) = t - \pi$, $r(t) = -\frac{3}{4}$, $\tau(t, \zeta) = \theta(t, \zeta) = t - 2\zeta$, $\eta(\zeta) = \zeta$, $Q(t, \zeta) = \frac{4}{5}$, $a(t) = 5$, $a(t, \zeta) = \frac{22}{5}$, $[c, d] = [\pi/4, \pi/2]$,

$$F(x, t) = \begin{pmatrix} -\frac{16}{5} \sin x \cos t \\ \sin x e^t \left(\frac{29}{4} + \frac{13}{5} e^{-\pi/2} - \frac{43}{20} e^{-\pi} \right) \end{pmatrix}.$$

Let $H = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we have $f_H(x, t) = f_{e_1}(x, t) = -\frac{16}{5} \sin x \cos t$ and

$$\int_{\Omega} f_{e_1}(x, t) dx = -\frac{16}{5} \int_{\Omega} \sin x \cos t dx = -\frac{32}{5} \cos t \leq 0, \quad \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}.$$

Take $\lambda = 3$, $\sigma(t) = t$, $\sigma'(t) = 1$, $\Phi(t) = 1$. Since $t_0 = 1$, $t_j = 2^j$, $E(s) = \frac{1}{3}$, $D(s) = \frac{1}{4}$, $G(s) = \frac{4\pi}{25}$. Then hypotheses $(H_1) - (H_6)$ hold, and further

$$\begin{aligned} \lim_{t \rightarrow +\infty} \int_{t_0}^t \prod_{t_0 \leq t_j < s} \frac{\beta_j^*}{\alpha_j} ds &= \int_1^{+\infty} \prod_{1 < t_j < s} \frac{j}{j+1} ds \\ &= \int_1^{t_1} \prod_{1 < t_j < s} \frac{j}{j+1} ds + \int_{t_1^+}^{t_2} \prod_{1 < t_j < s} \frac{j}{j+1} ds + \int_{t_2^+}^{t_3} \prod_{1 < t_j < s} \frac{j}{j+1} ds + \dots \\ &= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \dots = \sum_{n=0}^{\infty} \frac{2^n}{n+1} = +\infty. \end{aligned}$$

Thus

$$\limsup_{t \rightarrow +\infty} \frac{1}{(t-1)^2} \left\{ \int_1^t \prod_{1 < t_j < s} \frac{j+1}{j} (t-s)^2 \left[\frac{4\pi}{25} - \frac{3}{64} + \frac{3}{4(t-s)} - \frac{3}{(t-s)^2} \right] ds \right\} = +\infty.$$

Therefore all the conditions of Corollary (4.5) are satisfied and hence every solution $U(x, t)$ of equation (30)-(31) is e_1 -oscillatory in G . In fact

$$U(x, t) = \begin{pmatrix} \sin x \sin t \\ \sin x e^t \end{pmatrix},$$

is one such solution of (30)-(31). We note the above solution $U(x, t)$ is not e_2 -oscillatory in G , where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

6. Conclusion

In this paper, we have established some new oscillation criteria for impulsive vector partial differential equations with damping term. We have derived sufficient conditions for the H-oscillation of solutions, using impulsive differential inequalities

and average technique with two different boundary conditions. The example we have used is illustrative of the application of Corollary (4.5). The present results complement and extend those established for problems without impulses.

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