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Advanced Fibonacci Sequence and its Sum by Second Order Variable Co-efficient Difference Operator

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Abstract

We introduce a second order difference operator with specific powers of variable co-efficient and its inverse in this study, which allows us to derive the $(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})$-Fibonacci sequence and its summation. This series is known as the Fibonacci sequence with variable co-efficients (VCFS). On the sum of the terms of the variable co-efficient Fibonacci sequence, some theorems and intriguing findings are generated. To demonstrate our findings, appropriate instances are presented.

Key words: Difference operator variable co-efficient, Fibonacci sequence, Closed form solution, Fibonacci summation.

AMS classification: 39A70, 39A10, 47B39, 65J10, 65Q10.

1 Introduction

Jerzy Popenda [6] created a new form of difference operator on $u(t)$ in 1984, $\Delta_\alpha u(t) = u(k+1) - \alpha u(t)$. Miller and Rose [8] proposed the discrete counterpart and inverse $\Delta_h^{-\nu} f(t)$ of the Riemann-Liouville fractional derivative in 1989, ([1, 5]).

. In 2012, G.Britto Antony Xavier, et.al, [2] extended the operator $\Delta_\alpha(\ell)$ to Forward $(\alpha,\beta)-$difference operator as $h\Delta_{(\alpha,\beta)h} v(t) = \beta v(k+h) - \alpha v(t)$ for the real valued function $v(t)$. G.Britto Antony Xavier, et.al, [3] presented the second order $\alpha-$difference operator as $\Delta_{\alpha(\ell,m)} v(t) = v(k+2\ell) - \alpha[v(k+\ell) + v(k+m)] - \alpha^2 v(t)$ in 2014, and discovered a finite series solution to the associated generalised second order difference equation $\Delta_{\alpha(\ell,m)} v(t) = u(t)$. With this background, we used the difference Operator with variable co-efficients to produce advanced Fibonacci sequence and its sum in this study.

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2 Variable Co-efficient Fibonacci Sequence and its Sum

Fibonacci and Lucas numbers appear in the extensive books of Koshy [7] and Vajda [10], and they cover a wide range of interest in modern mathematics. Falcon and Plaza [4] developed the $k-$Fibonacci sequence, which has only one integer parameter k and is defined as

$$F_{k,0}=0, \quad F_{k,1}=1 \quad \text{and} \quad F_{k,n+1}=kF_{k,n}+F_{k,n-1}, \quad \text{where} \quad n\geq 1,\, k\geq 1.$$

For example, if $k=2$, the Pell sequence is $P_0=0$, $P_1=1$ and $P_{n+1}=2P_n+P_{n-1}$ for $n\geq 1$. The second order difference operator with variable co-efficients $\underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\Delta} v(t)=v(t)-\alpha_1 t^{r_1}v(t-1)-\alpha_2 t^{r_2}v(t-2)$ is used in this section to generate the $(\alpha_1 t^{r_1},\alpha_2 t^{r_2})-$Fibonacci sequence and its summation.

Definition 2.1 Let $k\in[0,\infty)$, $(\alpha_1 t^{r_1},\alpha_2 t^{r_2})$-Fibonacci sequence is defined as

$$F_0=1,\;\; F_1=\alpha_1 t^{r_1},\;\; F_n=\alpha_1[t-(n-1)]^{r_1}F_{n-1}+\alpha_2[t-(n-2)]^{r_2}F_{n-2},\; n\geq 2. \tag{1}$$

The sequence (1) gives the typical Fibonacci sequence when $\alpha_1=\alpha_2=r_1=r_2=1$.

Example 2.2 (i) Taking $t=7$,$\alpha_1=10$, $\alpha_2=7$, $r_1=3$ and $r_2=2$ in (1), we get a Fibonacci sequence with variable co-efficients $\{1,490,193207,12173560,\cdots\}$.
(ii) When $t=9$,$\alpha_1=0.8$, $\alpha_2=0.3$, $r_1=2$ and $r_2=4$ in (1), we have a Fibonacci sequence with variable co-efficients $\{1,583.2,238903.02,65566186.13,\cdots\}$.

Similarly, Fibonacci sequences corresponding to each pair $(\alpha_1 t^{r_1},\alpha_2 t^{r_2})\in\mathbb{R}^2$ can be found.

Definition 2.3 A second order difference operator with variable co-efficients on $v(t)$, denoted as $\underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\Delta} v(t)$, is defined as

$$\underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\Delta} v(t)=v(t)-\alpha_1 t^{r_1}v(t-1)-\alpha_2 t^{r_2}v(t-2), \quad t\in[0,\infty),, \tag{2}$$

for each pair $(\alpha_1 t^{r_1},\alpha_2 t^{r_2})\in\mathbb{R}^2$, and its inverse is defined as follows:

$$\text{if} \quad \underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\Delta} v(t)=u(t), \quad \text{we write} \quad v(t)=\underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\overset{-1}{\Delta}} u(t). \tag{3}$$

Lemma 2.4 Consider $v(t)$ be a functions of $t \in (-\infty, \infty)$. Then we get

$$\mathop{\Delta}\limits^{-1}_{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})} a^{st}\Big[1 - \frac{\alpha_1 t^r}{a^s} - \frac{\alpha_2 t^s}{a^{2s}}\Big] = a^{st}. \tag{4}$$

Proof: Taking $u(t) = a^{st}\Big[1 - \frac{\alpha_1 t^{r_1}}{a^s} - \frac{\alpha_2 t^{r_2}}{a^{2s}}\Big]$ in (2), we obtained $\mathop{\Delta}\limits_{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})} a^{st} = a^{st}\Big[1 - \frac{\alpha_1 t^{r_1}}{a^s} - \frac{\alpha_2 t^{r_2}}{a^{2s}}\Big]$. Now (4) follows from (3).

Remark 2.5 If $\alpha_1 = 1 = \alpha_2$ in lemma 2.4, then we obtained

$$\mathop{\Delta}\limits^{-1}_{(t^{r_1}, t^{r_2})} a^{st}\Big[1 - \frac{t^{r_1}}{a^s} - \frac{t^{r_2}}{a^{2s}}\Big] = a^{st}. \tag{5}$$

Lemma 2.6 For the function e^{-st} of $t \in (-\infty, \infty)$, we obtained

$$\mathop{\Delta}\limits^{-1}_{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})} e^{-st}\Big[1 - \alpha_1 t^{r_1} e^s - \alpha_2 t^{r_2} e^{2s}\Big] = e^{-st}. \tag{6}$$

Proof: The proof begins with the assumption that $a = e^{-1}$ in (4).

Remark 2.7 For the function e^{-st} of $t \in (-\infty, \infty)$, we obtained

$$\mathop{\Delta}\limits^{-1}_{(t^{r_1}, t^{r_2})} e^{-st}\Big[1 - t^{r_1} e^s - t^{r_2} e^{2s}\Big] = e^{-st}. \tag{7}$$

Theorem 2.8 If $v(t) = \mathop{\Delta}\limits^{-1}_{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})} u(t)$, $F_0 = 1, F_1 = \alpha_1 t^{r_1}$ and $F_{i+1} = \alpha_1 (t-n)^{r_1} F_n + \alpha_2 (t-(n-1))^{r_2} F_{n-1}$, for $i = 0, 1, 2, ...$ then

$$v(t) - F_{n+1} v(t-(n+1)) - \alpha_2 (t-n)^{r_2} F_n v(t-(n+2)) = \sum_{i=0}^{n} \underset{i}{F}\, u(t-i). \tag{8}$$

Proof: From (2) and (3), we arrive

$$v(t) = u(t) + \alpha_1 t^{r_1} v(t-1) + \alpha_2 t^{r_2} v(t-2). \tag{9}$$

By replacing t with $t-1$ and then inserting the value of $v(t-1)$ in (9), we get

$$v(t) = u(t)+F_1u(t-1)+(\alpha_1^2 t^{r_1}(t-1)^{r_1}+\alpha_2 t^{r_2})v(t-2)+\alpha_1\alpha_2 t^{r_1}(t-1)^{r_2}v(t-3) \quad (10)$$

which gives

$$v(t) = F_0u(t) + F_1u(t-1) + F_2v(t-2) + \alpha_2(t-1)^{r_2}F_1v(t-3), \quad (11)$$

where F_0, F_1 and F_2 are given in (1). By replacing t by $t-2$ in (9) and then inserting $v(t-2)$ in (11), we get
$v(t) = F_0u(t) + F_1u(t-1) + F_2u(t-2) + F_3v(t-3) + \alpha_2(t-2)^{r_2}F_2v(t-4)$, where F_3 is given in (1). We acquire (8) by repeating this technique over and over.

Corollary 2.9 If $\underset{(t^{r_1},t^{r_2})}{\overset{-1}{\Delta}} u(t) = v(t)$, $F_0 = 1, F_1 = t^{r_1}$ and
$F_{n+1} = (t-n)^{r_1}F_n + (t-(n-1))^{r_2}F_{n-1}$, for $i = 0, 1, 2, \ldots$ then

$$v(t) - F_{n+1}v(t-(n+1)) - (t-n)^{r_2}F_nv(t-(n+2)) = \sum_{i=0}^{n} F_i\, u(t-i). \quad (12)$$

Proof: Taking $\alpha_1 = 1 = \alpha_2$ in Theorem(2.8), the proof follows.

Corollary 2.10 If $v(t)$ is a closed form solution of the difference equation with variable co-efficients $\underset{(t^{r_1},t^{r_2})}{\Delta} v(t) = a^{st}[1 - \frac{\alpha_1 t^{r_1}}{a^s} - \frac{\alpha_2 t^{r_2}}{a^{2s}}]$, then we obtain

$$a^{st} - F_{n+1}a^{s[t-(n+1)]} - \alpha_2(t-n)^{r_2}F_na^{s[t-(n+2)]}$$

$$= \sum_{i=0}^{n} F_i a^{s[t-i]}\big[1 - \frac{\alpha_1(t-i)^{r_1}}{a^s} - \frac{\alpha_2(t-i)^{r_2}}{a^{2s}}\big]. \quad (13)$$

Proof: From (4), we have $v(t) = a^{st}$. Now (13) follows by apply $v(t) = a^{st}$ in (8).

The equation (13) is verified in the following example.

Example 2.11 Taking $t = 7$, $s = 1$, $a = 5$, $\alpha_1 = 2$, $\alpha_2 = 3$, $r_1 = 1$ and $r_1 = 2$ in (13), we get $F_0 = 1$, $F_1 = 14$, $F_2 = 315$, $F_3 = 4662$ and we have the solution $5^7 - F_3 5^4 - 3F_2 5^2 5^3 = \sum_{i=0}^{2} F_i 5^{(7-i)} \left[1 - \frac{\alpha_1(t-i)^1}{a} - \frac{\alpha_2(t-i)^2}{a^2}\right] = -5,788,750.$

Remark 2.12 Let $\alpha_1 = \alpha_2 = 1$ in (13). Then we have

$$a^{st} - F_{n+1}a^{s[t-(n+1)]} - (t-n)^{r_2}F_n a^{s[t-(n+2)]} = \sum_{i=0}^{n} F_i a^{t-i}\Big[1 - \frac{(t-i)^{r_1}}{a^s} - \frac{(t-i)^{r_2}}{a^{2s}}\Big]. \quad (14)$$

The equation (14) is verified in the following case.

Example 2.13 Taking $k = 5$, $a = 2$, $s = 1$ and $n = 1$ in (14), we get

$$2^5 - F_2 2^3 - F_1 4^2 2^2 = \sum_{i=0}^{1} F_i\, 2^{(5-i)}\left[1 - \frac{5-i}{a} - \frac{(t-i)^2}{a^2}\right] = -648$$

where $F_0 = 1$, $F_1 = 5$, $F_2 = 45$.

Corollary 2.14 Consider e^{-st} be a function of $t \in (-\infty, \infty)$. Then $e^{-st} - F_{n+1}e^{-s(t-(n+1))} - \alpha_2(t-n)^{r_2}F_n e^{-s(t-(n+2))}$

$$= \sum_{i=0}^{n} F_i e^{-s(t-i)}\Big[1 - \alpha_1(t-i)^{r_1}e^s - \alpha_2(t-i)^{r_2}e^{2s}\Big]. \quad (15)$$

Proof: Taking $v(t) = e^{-st}$ and applying (6) in (8), we get (15).

Example 2.15 Taking $t = 9$, $n = 3$,$s = 1$, $\alpha_1 = 0.8$, $\alpha_2 = 0.3$, $r_1 = 3$ and $r_2 = 2$ in (14), then we obtained $e^{-9} - F_4e^{-5} - (0.3)6^2F_3e^{-4} = \sum_{i=0}^{3} F_i e^{-(9-i)}\left[1 - (0.8)(9-i)^3e - (0.3)(9-i)^2e^2\right] = -89333078.97$ where $F_0 = 1$, $F_1 = 583.2$, $F_2 = 238903.02$, $F_3 = 65566186.13$ and $F_4 = 11333348840$.

Theorem 2.16 Let $t \in \mathbb{N}(0)$. Then a closed form solution of the second order difference equation with variable co-efficients $v(t) - \alpha_1t^{r_1}v(t-1) - \alpha_2t^{r_2}v(t-2) = \left[t^k - \alpha_1t^{r_1}(t-1)^k - \alpha_2t^{r_2}(t-2)^k\right]$ is

$$\overset{-1}{\underset{(\alpha_1t^{r_1},\alpha_2t^{r_2})}{\Delta}}\left[t^k - \alpha_1t^{r_1}(t-1)^k - \alpha_2t^{r_2}(t-2)^k\right] = t^k \quad (16)$$

Proof: Taking $v(t) = t^k$ in (2) and using (3), we get (16).

Corollary 2.17 Taking $t = 2$ in Theorem 2.16, we have

$$\underset{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})}{\overset{-1}{\Delta}} \left[t^2 - \alpha_1 t^{r_1}(t-1)^2 - \alpha_2 t^{r_2}(t-2)^2\right] = t^2 \tag{17}$$

which is a closed form solution of the difference equation

$$\underset{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})}{\Delta} v(t) = t^2 - \alpha_1 t^{r_1}(t-1)^2 - \alpha_2 t^{r_2}(t-2)^2.$$

Proof: From (16), replacing $k = 2$, we get 17

Corollary 2.18 If $v(t) = \underset{(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})}{\overset{-1}{\Delta}} \left[t^k - \alpha_1 t^{r_1}(t-1)^k - \alpha_2 t^{r_2}(t-2)^k\right]$ is the closed form solution given in (16), then we have
$v(t) - F_{n+1} v(t-(n+1) - \alpha_2 (t-n)^{r_2} F_n v(t-(n+2))$

$$= \sum_{i=0}^{n} F_i \Big[(t-i)^k - \alpha_1 (t-i)^{r_1}[t-(i+1)]^k - \alpha_1 (t-i)^{r_2}[t-(i+2)]^k\Big]. \tag{18}$$

Proof: Taking $u(t) = t^k - \alpha_1 t^{r_1}(t-1)^k - \alpha_2 t^{r_2}(t-2)^k$ in Theorem 2.8, we have 18.

Example 2.19 Let $t = 7$, $n = 3$, $k = 2$, $r_1 = 3$, $r_2 = 4$ $\alpha_1 = 5$, $\alpha_2 = 3$ in Corollary (2.18). Then $\sum_{i=0}^{3} F^i u(7-i) = v(7) - F_4 v(3) - \alpha_2 (4)^4 F_3 v(2) = -6,988,044,045,122.$ where $u(t) = t^k - \alpha_1 t^r (t-1)^k - \alpha_2 t^s (t-2)^k$, $F_0 = 1$, $F_1 = 1715$, $F_2 = 1,859,403$, $F_3 = 1,168,794,795$ and $F_4 = 377,500,715,025$.

Theorem 2.20 If $v(t)$ is a closed form solution of second order difference equation with variable co-efficients arrive Fibonacci Summation Formula
$v(t) - \alpha_1 t^{r_1} v(t-1) - \alpha_2 t^{r_2} v(t-2) = t^k a^{st} - \alpha_1 t^{r_1}(t-1)^k a^{s(t-1)} - \alpha_2 t^{r_2}(t-2)^k a^{s(t-2)}$, then we have

$$v(t) - F_{n+1} v(t-[n+1]) - \alpha_2 (t-n)^{r_2} v(t-[n+2]) =$$

$$\sum_{i=0}^{n} F_i a^{s(t-i)} \Big[(t-i)^k - \frac{\alpha_1 (t-i)^{r_1}}{a^s}[t-(i+1)]^k - \frac{\alpha_2 (t-i)^{r_2}}{a^{2s}}[t-(i+2)]^k\Big]. \tag{19}$$

Proof: Taking $u(t) = \left[t^k a^{st} - \alpha_1 t^{r_1}(t-1)^k a^{s(t-1)} - \alpha_2 t^{r_2}(t-2)^k a^{s(t-2)}\right]$ in Theorem 2.8 and using (4), we get 19.

Corollary 2.21 A closed form solution of the difference equation

$$\underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\Delta} v(t) = t^3a^t - \alpha_1 t^{r_1}(t-1)^3a^{t-1} - \alpha_2 t^{r_2}(t-2)^3a^{t-2} \text{ is } t^3a^t$$

and hence we arrive the relation

$$t^3a^{st} - F_{n+1}(t-[n+1])^3a^{s(t-[n+1])} - \alpha_2 F_n(t-n)^{r_2}(t-[n+2])^3a^{s(t-[n+2])}$$

$$= \sum_{i=0}^{n} F_i a^{s(t-i)}\Big[(t-i)^3 - \frac{\alpha_1(t-i)^{r_1}}{a^s}[t-(i+1)]^3 - \frac{\alpha_2(t-i)^{r_2}}{a^{2s}}[t-(i+2)]^3\Big]. \quad (20)$$

Proof: The proof follows by taking $k = 3$ in Theorem 2.20.

Example 2.22 Let $t = 5$, $s = 1$, $a = 3$, $n = 4$, $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $r = 3$, $s = 2$ in Corollary (2.21). Then we obtain

$$v(5) - F_5v(0) - (0.03)F_4v(-1) = \sum_{i=0}^{4} F_i[(5-i)^3 3^{t-i} - (0.02)(5-i)^3[5-(i+1)]^3 3^{5-(i+1)} - (.03)(5-i)^2[5-(i+2)]^3 3^{5-(i+2)}] = 30,375.016,$$

where $F_0 = 1$, $F_1 = 2.5$, $F_2 = 3.95$, $F_3 = 3.333$, $F_4 = 1.59978$ and $F_5 = 0.4319556$.

Corollary 2.23 A closed form solution of the second order difference equation $v(t) - \alpha_1 t^{r_1}v(t-1) - \alpha_2 t^{r_2}v(t-2) = t^t e^{-st} - \alpha_1 t^{r_1}(t-1)^k e^{-s(t-1)} - \alpha_2 t^{r_2}(t-2)^k e^{-s(t-2)}$ yields the Fibonacci formula

$$v(t) - F_{n+1}v(t-(n+1)) - \alpha_2 F_n(t-n)^{r_2}v(t-(n+2)) =$$

$$\sum_{i=0}^{n} F_i e^{-st}[(t-i)^k e^{is} - \alpha_1(t-i)^{r_1}[t-(i+1)]^k e^{s(i+1)} - \alpha_2(t-i)^{r_2}[t-(i+2)]^k e^{s(i+2)}]. \quad (21)$$

Proof: Taking $a = e^{-1}$ in (19), we get (21).

Corollary 2.24 If $v(t) = \underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\overset{-1}{\Delta}} e^{-st}[t - \alpha_1 t^{r_1}(t-1)e^s - \alpha_2 t^{r_2}(t-2)e^{2s}]$
yields the relation

$$te^{-st} - F_{n+1}[t-(n+1)]e^{-s[t-(n+1)]} - \alpha_2 F_n(t-n)^{r_2}[t-(n+2)]e^{-s[t-(n+2)]} =$$

$$\sum_{i=0}^{n} F_i e^{-s(t-i)}[(t-i) - \alpha_1(t-i)^{r_1}[k-(i+1)]e^s - \alpha_2(t-i)^{r_2}[k-(i+2)]e^{2s}]. \quad (22)$$

Proof: The proof follows by taking $t = 1$ in Corollory 2.23.

Theorem 2.25 Let $v(t)$ be a solution of the second order difference equation
$v(t)-\alpha_1 t^{r_1}v(t-1)-\alpha_2 t^{r_2}v(t-2)=t^{(k)}a^{st}-\sum\limits_{p=1}^{2}\alpha_p t^{r_p}(t-p)^{(t)}a^{s(t-p)}$,
then we have
$t^{(k)}a^{st}-F_{n+1}(t-[n+1])^{(t)}a^{s(t-[n+1])}-\alpha_2 F_n(t-n)^{r_2}(t-[n+2])a^{s(t-[n+2])}$

$$=\sum_{i=0}^{n}F_i a^{t-i}[(t-i)^{(t)}-\sum_{p=1}^{2}\frac{\alpha_i(t-i)^{r_p}[t-(i+p)]^{(t)}}{a^{ps}} \quad (23)$$

Proof: Taking $v(t)=t^{(k)}a^{st}$ in Theorem 2.8 and using (4), we get 23.

Corollary 2.26 $v(t)=t^{(2)}a^{st}$ given the relation
$t^{(2)}a^{st}-F_{n+1}(t-[n+1])^{(2)}a^{s(t-[n+1])}-\alpha_2 F_n(t-n)^{r_2}(t-[n+2])^{(2)}a^{s(t-[n+2])}$

$$=\sum_{i=0}^{n}F_i a^{t-i}\Big[(t-i)^{(2)}-\sum_{p=1}^{2}\frac{\alpha_p(t-i)^{r_p}}{a^{sp}}[t-(i+p)]^{(2)}\Big]. \quad (24)$$

Proof: The proof follows by taking $k=2$ in Theorem 2.25.

Example 2.27 Let $t=7$, $s=1$, $a=3$, $n=2$, $\alpha_1=0.04$, $\alpha_2=0.06$, $r_1=4$, $r_2=3$ in Corollary (2.26). Then we obtain

$$v(7)-F_3v(4)-(0.03)5^3F_2v(3)=\sum_{i=0}^{2}F_i[(7-i)^{(2)}3^{t-i}-$$
$(0.06)(7-i)^3[7-(i+1)]^{(2)}3^{7-(i+1)}-(.06)(7-i)^{(2)}[7-(i+2)]^3 3^{7-(i+2)}]=-128,674,949.6$,
where $F_0=1$, $F_1=96.04$, $F_2=4999.2936$, $F_3=126,227.0184$.

Corollary 2.28 Let $v(t)$ be a solution of second order difference equation with variable co-efficients
$v(t)-\alpha_1 t^{r_1}v(t-1)-\alpha_2 t^{r_2}v(t-2)=e^{-st}\Big[t^{(k)}-\sum\limits_{p=1}^{2}\alpha_p t^{r_1}(t-1)^{(t)}e^{ps}\Big]$.
Then we have
$t^{(k)}e^{-st}-F_{n+1}(t-[n+1])^{(t)}e^{-s(t-(n+1))}-\alpha_2F_n(t-n)^{r_2}(t-[n+2])^{(t)}e^{-s(t-(n+2))}$

$$=\sum_{i=0}^{n}F_i e^{-(t-i)}\Big[(t-i)^{(t)}-\sum_{p=1}^{2}\alpha_p(t-i)^{r_p}[t-(i+1)]^{(k)}e^{p}.\Big] \quad (25)$$

Proof: Taking $a=e^{-1}$ in Theorem (2.25), we get (25).

Corollary 2.29 A closed form solution of then difference equation with variable co-efficients

$\underset{(\alpha_1 t^{r_1},\alpha_2 t^{r_2})}{\Delta} v(t) = e^{-st}[t^{(3)} - \alpha_1 t^{r_1}(t-1)^{(3)}e^s - \alpha_2 t^{r_2}(t-2)^{(3)}e^{2s}]$ is $t^{(3)}e^{-st}$

and hence we have

$t^{(3)}e^{-st} - F_{n+1}(t-[n+1])^{(3)}e^{-s(t-[n+1])} - \alpha_2(t-n)^{r_2}(t-[n+2])^{(3)}e^{-s(t-[n+2])}$

$$= \sum_{i=0}^{n} F_i e^{-s(t-i)}\Big[(t-i)^{(3)} - \sum_{p=1}^{2}\alpha_p(t-i)^{r_p}[t-(i+p)]^{(3)}e^{ps}\Big]. \quad (26)$$

Proof: The proof follows by taking $t = 3$ in Corollory (2.28), we get (26).

Example 2.30 Let $t = 6$, $s = 1$, $n = 2$, $a = 0.2$, $\alpha_1 = 2$, $\alpha_2 = 0.3$, $r_1 = 3$, $r_2 = 2$ in Corollary (2.29). Then we obtain

$$v(6) - F_3 v(3) - (0.3)F_2 4^2 v(2) = \sum_{i=0}^{2} F_i[(6-i)^{(3)}(0.2)^{t-i} - (2)(6-i)^3 \times$$

$[6-(i+1)]^{(3)}(0.2)^{6-(i+1)} - (0.3)(6-i)^2[6-(i+2)]^{(3)}(0.2)^{6-(i+2)}] = -663773.8675$,
where $F_0 = 1$, $F_1 = 432$, $F_2 = 108010.8$ and $F_3 = 13828622.4$.

3 Conclusion

By introducing the $(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})$-difference operator, we were able to obtain the summation formula for the $(\alpha_1 t^{r_1}, \alpha_2 t^{r_2})$-Fibonacci sequence, and we were able to derive certain results on the closed and summation form solution of the second order difference equation, which will be used in our future research.

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