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A Study on Compactness in Intuitionistic Fuzzy Topological Spaces

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Abstract

The purpose of this paper is to introduce and study the compactness in intuitionistic fuzzy topological spaces. Here we define two new notions of intuitionistic fuzzy compactness in intuitionistic fuzzy topological space and find their relation. Also we find the relationship between intuitionistic general compactness and intuitionistic fuzzy compactness. Here we see that our notions satisfy hereditary and productive property.

Key words: Fuzzy set, Compactness, Intuitionistic Fuzzy, Topological Spaces

AMS classification: 32C18, 5A420, 03E72.

1 Introduction

In this paper, we study the compactness in intuitionistic fuzzy topological spaces. Here we define two new notions of intuitionistic fuzzy compactness in intuitionistic fuzzy topological space and find their relation. Also we find the relationship between intuitionistic general compactness and intuitionistic fuzzy compactness. Finally we observe that our notions preserve under one-one, onto andcontinuous mapping.

2 Compactness in intuitionistic fuzzy topological space

Definition 2.1 Let (X, τ) be an intuitionistic fuzzy topological space. A family $\{(\mu_{G_i}, \nu_{G_i}) : j \in J\}$ of IFOS in X is called open cover of X if $\cup\mu_{G_i} = 1$ and $\cap\nu_{G_i} = 0$. If every open cover of X has a finite subcover then X is said to be intuitionistic fuzzy compact (IF-compact, in short).

Definition 2.2 A family $\{(\mu_{G_i}, \nu_{G_i}) : j \in J\}$ of IFOS in X is called (α, β)-level open cover of X if $\cup\mu_{G_i} \geq \alpha$ and $\cap\nu_{G_i} \leq \beta$ with $\alpha + \beta \leq 1$. If every (α, β)-level open cover of X has a finite subcover then X is said to be (α, β) -level IF-compact.

Theorem 2.3 Let (X, T) be a topological space and (X, τ) be its corresponding IFTS, where $\tau = \{(1_{A_j}, 1^c_{Aj}), j \in J : A_j \in T\}$. Then (X, T) is compact if (X, τ) is

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IF-compact.

Proof Let (X,T) be compact. Consider $\{G_i | i \in J\}$ be the open cover of X,

$$i.e. \cup G_i = X \tag{1}$$

Since X is compact then $\exists G_{i_1}, G_{i_2}, ..., G_{i_n} \in T$ such that

$$G_{i_1} \cup G_{i_2} \cup .... \cup G_{i_n} = X \tag{2}$$

Now it is clear that $(1_{G_i}, 1^c_{G_i}) \in T$ (by the definition).
Also we have,

$$\begin{aligned} \cup(1_{G_i}, 1^c_{G_i}) &= (\cup 1_{G_i}, \cap 1^c_{G_i}) \\ &= (1_{\cup G_i}, 1^c_{\cap G_i}) \\ &= (1_X, 1^c_{\cap G_i}) \end{aligned}$$

But we have, $1_X + 1^c_{\cap G_i} \leq 1$ then it must be $1^c_{\cap G_i} = 0$. Therefore we get,

$$\cup(1_{G_i}, 1^c_{G_i}) = (1_X, 0).$$

Also by (ii) we get,

$$\begin{aligned} (1_X, 0) &= (1_{G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n}}, 0) \\ &= (\cup_{j=1}^n 1_{G_{ij}}, 0) \\ &= \cup(1_{G_{ij}}, 0) \end{aligned}$$

Hence it is clear that the IFTS (X,τ) is IF-compact.

Corollary 2.4 Let (X,T) be a topological space and (X,τ) be its corresponding IFTS, where $\tau = \{(1_{A_j}, 1^c_{A_j}), j \in J : A_j \in T.\}$ Then (X,T) is compact if (X,τ) is (α,β)-level IF-compact.

Proof: Here it is clear that for any $\alpha, \beta \in I$ with $\alpha + \beta \leq 1 \Rightarrow 1 \geq \alpha$ and $\beta \geq 0$. So, (X,τ) is (α,β)-level IF-compact.

Theorem 2.5 Let $(X,\mathcal{J})$ be an intuitionistic topological space and (X,τ) be its corresponding IFTS, where $\tau = \{1_{A_j} = (1_{A_{j_1}}, 1_{A_{j_2}}), j \in J : A_j = (A_{j_1}, A_{j_2}) \in \mathcal{J}\}$. Then $(X,\mathcal{J})$ is intuitionistic compact if (X,τ) is IF-compact.

Proof Let $(X,\mathcal{J})$ be an intuitionistic compact space, we shall prove that (X,τ) is

IF-compact. Consider $\{1_{A_k}\}$ be an open cover of τ, i.e. $\cup 1_{A_k} = (1,0)$, where $(1,0)$ is intuitionistic fuzzy set. Now

$$\begin{aligned} 1_{A_k} = (1_{A_{k_1}}, 1_{A_{k_2}}) \quad &\Rightarrow \quad \cup 1_{A_k} = \cup(1_{A_{k_1}}, 1_{A_{k_2}}) \\ &\Rightarrow \quad \cup 1_{A_k} = (\cup 1_{A_{k_1}}, \cap 1_{A_{k_2}}) \\ &\Rightarrow 1 \quad 1_{\cup A_k} = (1_{\cup A_{k_1}}, 1_{\cap A_{k_2}}) \\ &\Rightarrow 1 \quad 1_{\cup A_k} = (1_X, 0) \\ &\Rightarrow 1 \quad 1_{\cup A_k} = (1, 0) \end{aligned}$$

By the given definition $\{A_k \in \mathcal{J}\}, k \in \wedge$ is the open cover of X, since $\cup A_k = (X, \phi)$. But we have $(X, \mathcal{J})$ is compact then $\exists A_{k_{i_1}}, A_{k_{i_2}}, ..., A_{k_{i_n}} \in \mathcal{J}$ such that

$$\begin{aligned} &\cup_{j=1}^{n} A_{k_{ij}} = (X, \phi) \\ \Rightarrow\quad &\cup_{j=1}^{n} (A_{k_{ij_1}}, A_{k_{ij_2}}) = (X, \phi) \\ \Rightarrow\quad &\cup_{j=1}^{n} A_{k_{ij_1}}, \cap_{j=1}^{n} A_{k_{ij_2}} = (X, \phi) \\ \Rightarrow\quad &(1_{\cup_{j=1}^{n} A_{k_{ij_1}}}, 1_{\cap_{j=1}^{n} A_{k_{ij_2}}}) = (1_X, 1_\phi) \\ \Rightarrow\quad &(1_{\cup_{j=1}^{n} A_{k_{ij_1}}}, 1_{\cap_{j=1}^{n} A_{k_{ij_2}}}) = (1, 0) \end{aligned}$$

Hence (X, τ) is IF-compact.

Theorem 2.6 Let (X, τ) and (Y, δ) be IFTSs and $f : X \to Y$ is bijective, open and continuous. Then (Y, δ) is IF-compact $\Rightarrow (X, \tau)$ is IF-compact.

Proof Let $A_i = (\mu_i, \nu_i) \in \tau$ with $\cup A_i = (1,0)$. Now $A_i \in \tau \Rightarrow f(A_i) \in \delta$ with $\cup f(A_i) = (1,0)$. For $y \in Y, f(A_i)(y) = (y, f(\mu_{A_i})(y), f(\nu_{A_i})(y))$, where $f(\mu_{A_i})(y) = \sup_{x \in f^{-1}(y)} \mu_{A_i}(x) = \mu_{A_i}(x)$. Similarly we get, $f(\nu_{A_i})(y) = \nu_{A_i}(x)$. Now $\cup f(A_i) = \cup f(\mu_{A_i}), f(\nu_{A_i}) = (\cup f(\mu_{A_i}), \cap f(\nu_{A_i}))$, i.e. $\cup f(\mu_{A_i})(y) = \cup \mu_{A_i} = 1$ and $\cap f(\nu_{A_i})(y) = \cap \nu_{A_i}(x) = 0$, so $\cup f(A_i) = (1,0)$. Since f is an open then $\{f(A_i)\}$ is an open cover of Y. Again Y is compact then there exist $f(A_{1_i}), f(A_{2_i}), ..., f(A_{n_i}) \in \delta$ such that $\cup_{j=1}^{n} f(A_{ji}) = (1,0) \Rightarrow f(\cup_{j=1}^{n} A_{ji}) = (1,0) \Rightarrow f^{-1}(f(\cup_{j=1}^{n} A_{ji})) = f^{-1}(1,0) \Rightarrow f^{-1}(1,0) \subseteq \cup_{j=1}^{n} A_{ji}$(since from Chang $\mu \subseteq f^{-1}(f(\mu))$). Therefore $\cup_{j=1}^{n} A_{ji} = (1,0)$. Hence (X, τ) is IF-compact.

Theorem 2.7 Let (X, τ) and (A, δ) be IFTSs and $f : X \to Y$ is one-one, onto and continuous. Then (X, τ) is IF-compact $\Rightarrow (A, \delta)$ is IF-compact.

Proof Let $A_i = (\mu_i, \nu_i) \in \delta$ with $\cup A_i = (1,0)$. Since δ is a topology so $\cup A_i \in \delta \Rightarrow f^{-1}(\cup A_i) \in \tau$ with $f^{-1}(\cup A_i) = (1,0)$ (as f is continuous) $\Rightarrow \cup f^{-1}(A_i) = (1,0)$. But $\cup f^{-1}(A_i) = \cup f^{-1}(\mu_i, \nu_i) = \cup(f^{-1}(\mu_i), f^{-1}(\nu_i)) \in \tau$ with

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$\cup(f^{-1}(\mu_i), f^{-1}(\nu_i)) = (1,0)$. Since (X,τ) is IF-compact then $\exists\ A_{i_1}, A_{i_2}, ..., A_{i_m} \in \delta$ where $(f^{-1}(\mu_{i_1}), f^{-1}(\nu_{i_1})), (f^{-1}(\mu_{i_2}),$
$f^{-1}(\nu_{i_2})), ..., (f^{-1}(\mu_{i_m}),\ f^{-1}(\nu_{i_m}))\ \in \tau$ such that $(f^{-1}(\mu_{i_1}), f^{-1}(\nu_{i_1})) \cup (f^{-1}(\mu_{i_2}),$
$f^{-1}(\nu_{i_2}))\cup, ..., \cup(f^{-1}(\mu_{i_m}), f^{-1}(\nu_{i_m})) = (1,0)$

$$\begin{aligned}
&\Rightarrow\ \cup_{j=1}^{m}(f^{-1}(\mu_{ij}, f^{-1}(\nu_{ij}))) = (1,0)\\
&\Rightarrow\ (\cup_{j=1}^{m} f^{-1}(\mu_{ij}), \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = (1,0)\\
&\Rightarrow\ f(\cup_{j=1}^{m} f^{-1}(\mu_{ij}), \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = f(1,0)\\
&\Rightarrow\ (\cup_{j=1}^{m} f(f^{-1}(\mu_{ij})), \cap_{j=1}^{m} f(f^{-1}(\nu_{ij}))) = (1,0),
\end{aligned}$$

since f is one-one and onto, so $f(1,0) = (1,0)$. Therefore $(\cup_{j=1}^{m}\mu_{ij}, \cap_{j=1}^{m}\nu_{ij}) = (1,0)$, i.e. $\cup_{j=i}^{m}(\mu_{ij}, \nu_{ij}) = (1,0)$. Hence (Y,δ) is IF-compact.

Theorem 2.8 Let (X,τ) be an IFTS and (V,τ_v) be a subspace of (X,τ) with (X,τ) is IF-compact. Let $f : (X,\tau) \to (V,\tau_v)$ be continuous, open and onto, then (V,τ_v) is IF compact.

Proof Let $\mathcal{M} = \{B_i : i \in J\}$ be an open cover of (V,τ_v) with $\cup B_i = (1_v, 0)$. By the definition of subspace topology, let $B_i = U_i|V$, where $U_i \in \tau$. Since f is continuous then $f^{-1}(B_i) \in \tau$ implies that $f^{-1}(U_i|V) \in \tau$.
As, (X,τ) is IF-compact then $\cup_{i\in J} f^{-1}(U_i|V)(x) = (1_X, 0)$. Thus we see that, $\{f^{-1}(U_i|V) : i \in J\}$ is an open cover of (X,τ). Hence there exist

$$f^{-1}(U_{i1}|V), f^{-1}(U_{i2}|V), ..., f^{-1}(U_{in}|V) \in f^{-1}(U_i|V)$$

such that

$$\cap_{k=1}^{n} f^{-1}(U_{ik}|V) = (1_X, 0).$$

Put $B_{ik} = U_{ik}|V$, then it is clear that $B_{ik} \in \tau_v$ with $\cup_{k=1}^{n} f^{-1}(B_{ik}) = (1_X, 0)$

$$\begin{aligned}
&\Rightarrow\ f(\cup_{k=1}^{n} f^{-1}(B_{ik})) = f(1_X, 0)\\
&\Rightarrow\ \cup_{k=1}^{n} f(f^{-1}(B_{ik})) = (f(1_X), 0)
\end{aligned}$$

$\Rightarrow \cup_{k=1}^{n}(B_{ik}) = (1_V, 0)$ as f is open. Hence (V,τ_v) is IF-compact.

Theorem 2.9 Show that the following statements are equivalent:

(i) X is IF-compact,

(ii) For every F_i where $F_i = (\nu_{F_i}, \mu_{F_i})$ of closed subset of X with $\cap F_i = (0,1)$ implies $\{F_i\}$ contains finite subclass $\{F_{i1}, F_{i2}, ..., F_{im}\}$ with $F_{i1} \cap F_{i2} \cap .... \cap F_{im} = (0,1)$.

Proof $(i) \Rightarrow (ii)$. Suppose $\cap F_i = (0,1)$ then by De Morgan's law

$$\begin{aligned} & (\cap F_i)^c = (0,1)^c \\ \Rightarrow\ & \cup F_i^c = (1,0) \\ \Rightarrow\ & \cup(\nu_{F_i}, \mu_{F_i})^c = (1,0) \\ \Rightarrow\ & \cup(\mu_{F_i}, \nu_{F_i}) = (1,0) \\ \Rightarrow\ & (\cup\mu_{F_i}, \cap\nu_{F_i}) = (1,0). \end{aligned}$$

So, $\{F_i^c\}, (F_i^c = (\mu_{F_i}, \nu_{F_i}))$ is an open cover of X. Since X is IF-compact hence $\exists F_{i1}^c, F_{i2}^c, ..., F_{im}^c \in \{F_i^c\}$ such that $F_{i1}^c \cup F_{i2} \cup .... \cup F_{im}^c = (1,0)$. Then

$$\begin{aligned} (0,1) &= (1,0)^c = (F_{i1}^c \cup F_{i2} \cup .... \cup F_{im}^c)^c \\ &= (F_{i1}^c)^c \cap (F_{i2}^c)^c \cap .... \cap (F_{im}^c)^c \ By De Morgan's law \\ &= F_{i1} \cap F_{i2} \cap .... \cap F_{im}, \end{aligned}$$

so we have shown that $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (i)$. Let $\{G_i\}$ be an open cover of X where $G_i = (\mu_{A_i}, \nu_{G_i})$, i.e. $\cup_i G_i = (1,0)$. By De Morgan's law,

$$(0,1) = (1,0)^c = (\cup_i G_i)^c = \cap_i G_i.$$

Since each G_i is open, so $\{G_i\}$ is a class of closed sets and by (ii) $\exists G_{i1}^c, G_{i2}^c, ..., G_{im}^c \in \{G_i^c\}$ such that

$$G_{i1}^c \cap G_{i2}^c \cap .... \cap G_{im}^c = (0,1).$$

So by De Morgan's law

$$(1,0) = (0,1)^c = (G_{i1}^c \cap G_{i2}^c \cap .... \cap G_{im}^c) = G_{i1} \cup G_{i2} \cup .... \cup G_{im},$$

hence X is IF-compact. So, we have shown that $(ii) \Rightarrow (i)$.

Theorem 2.10 Let the IFTS's (X_1, τ_1) and (X_2, τ_2) be IF-compact. Then the product IFT $\tau_1 \times \tau_2$ on $X_1 \times X_2$ is IF-compact.

Proof Consider, (X_1, τ_1) and (X_2, τ_2) is IF-compact. Let $A_i = (\mu_{A_i}, \nu_{A_i}) \in \tau_1$ with $\cup A_i = (1,0)$ and $B_i = (\mu_{B_i}, \nu_{B_i}) \in \tau_2$ with $\cup B_i = (1,0)$. Now

$$A_i \times B_i = (\mu_{A_i}, \nu_{A_i}) \times (\mu_{B_i}, \nu_{B_i}) = (\mu_{A_i} \times \mu_{B_i}, \nu_{A_i} \times \nu_{B_i})$$

where

$$\begin{aligned}(\mu_{A_i} \times \mu_{B_i})(x, y) &= \min(\mu_{A_i}(x), \mu_{B_i}(y)), \quad where \;\; x \in X_1, y \in X_2 \\ &= \min(1, 1) \\ &= 1.\end{aligned}$$

Similarly,

$$\begin{aligned}(\nu_{A_i} \times \nu_{B_i})(x, y) &= \max(\nu_{A_i}(x), \nu_{B_i}(y)), \\ &= \max(0, 0) = 0\end{aligned}$$

So, $A_i \times B_i = (1, 0)$.
But by the definition of product topology, $A_i \times B_i \in \tau_1 \times \tau_2$, i.e. $\{A_i \times B_i\}$ is a family of intuitionistic fuzzy open set in $X_1 \times X_2$.
Choose $\cup(A_i \times B_i) = (1, 0)$. Since (X_1, τ_1) is IF-compact, then $\{A_i\}$ has finite subclass $\{A_{ij}\}$ such that $\cup_{j=1}^{n} A_{ij} = (1, 0)$. Similarly, since (X_2, τ_2) is IF-compact, then $\{B_i\}$ has finite subclass $\{B_{ik}\}$ such that $\cup_{k=1}^{m} B_{ij} = (1, 0)$. Therefore

$$\begin{aligned} & \cup_{j=1}^{n} A_{ij} \times \cup_{k=1}^{m} B_{ik} = (1, 0) \\ \Rightarrow\; & \cup_{j=1}^{n} (\mu_{A_{ij}}, \nu_{A_ij}) \times \cup_{k=1}^{m} (\mu_{B_{ik}}, \nu_{B_ik}) = (1, 0) \\ \Rightarrow\; & (\cup_{j=1}^{n} \mu_{A_{ij}}, \cap_{j=1}^{n} \nu_{A_ij}) \times (\cup_{k=1}^{m} (\mu_{B_{ik}}, \cap_{k=1}^{m} \nu_{B_ik})) \\ =\; & (1, 0).\end{aligned}$$

Hence there exist four cases:
Case-I: If $\cup_{j=1}^{n} \mu_{A_ij} = 1, \cup_{k=1}^{m} \mu_{B_ik} = 1$
Case-II: If $\cup_{j=1}^{n} \mu_{A_ij} = 1, \cap_{k=1}^{m} \mu_{B_ik} = 0$
Case-III: If $\cap_{j=1}^{n} \nu_{A_ij} = 0, \cup_{k=1}^{m} \mu_{B_ik} = 1$
Case-IV: If $\cap_{j=1}^{n} \nu_{A_ij} = 0, \cap_{k=1}^{m} \nu_{B_ik} = 0$
Here from four cases, we see that the product topology $(X_1 \times X_2, \tau_1 \times \tau_2)$ is IF-compact.

3 Conclusion

In this paper, we discussed about the compactness in intuitionistic fuzzy topological spaces. Here we define two new notions of intuitionistic fuzzy compactness in intuitionistic fuzzy topological space and find their relation. Also we find the relationship between intuitionistic general compactness and intuitionistic fuzzy compactness.

References

[1] Akray I, On Fuzzy Topological Spaces Induced by a Given Function. Asian Journal of Fuzzy and Applied Mathematics, 1, 44-50(2013).

[2] Atanassov KT, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20, 8796 (1986).

[3] Aygunoglu A, Cetkin V and Aygun H, An Introduction to Fuzzy Soft Topological Spaces. Hacettepe Journal of Mathematics and Statistics, 43, 197-208(2014).

[4] Bayhan S and Coker D, On fuzzy separation axioms in intuitionistic fuzzy topological space, BUSEFAL, 67 , 77-87(1996).

[5] Bayhan S and oker DC, On separation axioms in intuitionistic topological spaces, Int. J. Math. Math. Sci. 27, 10, 621630(2001).

[6] Berge C, Topological Spaces, Oliver and Boyd, Edinburg and London, 54-56, 2010.

[7] Chang CL, Fuzzy Topological Spaces, Journal of Mathematical Analysis and Applications, 110, 141-178(1985).

[8] Coker D, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88, 81-89 (1997).

[9] Ming LY and Kang LY, Fuzzy Topology, World Scientific Publishing Co. Pte. Ltd., 1997.

[10] Oker DC, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 , 8189(1997).

[11] Papageorgiou NS, Fuzzy Topology and Fuzzy Multifunctions. Journal of Mathematical Analysis and Applications, 109, 397-425(1985).

[12] Tamilmani J, Presemi weakly closed set in intuitionistic fuzzy topological spaces, Intern. J. Fuzzy Mathematical Archive, 6(1) , 27-34(2015).

[13] Wong CK, Covering properties of fuzzy topological spaces, J. Math. Anal. Appl. 43 (1973).

[14] Wong CK, Fuzzy topology, product and quotient theorems Research report IBM 1972, J. Math. Anal. Appl. (1974).