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Approximating Generalized Laplace Transforms

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Abstract

An extension of the Laplace transform (LT) obtained by using Bell polynomials and Laguerre-type exponentials was recently introduced. Computational techniques for evaluating the transformed functions numerically are developed. To this end, Tricomi's method is applied in combination with the matrix pencil method for approximating the kernel of generalized Laplace transforms.

Key words: Bell polynomials, Laguerre-type exponentials, Laplace transform.

AMS classification: 44A10, 05A40, 11B83

1 Introduction

It is not necessary to mention the importance of the Laplace transform (shortly LT) [12, 29], which is one of the most widely used tools in many areas of applied mathematics:

$$\mathcal{L}(f) := \int_0^\infty \exp^{-1}(s\,t) f(t)\, dt = F(s)\,, \tag{1}$$

Some extensions have been made, for example, in [30] and connections with the Fourier transform have long been a classic topic [1].

These transforms play an important role in the signals analysis, imagine processing, and all kinds of physical phenomena, as well in the theory of ordinary and partial differential equations. The problem of computing the LT was considered in many articles, [26, 27, 28] and more recently in [20].

New extensions of the LT have been recently introduced [19], replacing the traditional kernel with suitable functions giving back the classical one for particular

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2456-8686, 6(1), 2022:015-040
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parameter choices. A first generalization has been made using the Laguerre exponentials which are obtained by substituting the Laguerre derivative, or its iterations, with the ordinary exponential. This type of functions have been exploited in order to define new classes of special functions [3, 6, 8, 21] and to show their applications in the field of population dynamics [7, 4]. However, a first definition dates back to a work of Le Roy [17] and they have been previously studied also by V. Kyriakova [15, 16] in a more general framework.

A broader class of kernels was also introduced in [19] by constructing the reciprocal function of a kernel related to Laguerre-type exponentials by means of the Blissard formula [22]. Since it is a typical formula of the umbral calculus [24, 23], it was possible to associate a generalized transform to each sequence of numbers, represented by the symbol $a = \{a_k\}_{k=1,2,\dots}$, subject to appropriate restrictions so as to preserve the fundamental properties of the LT.

The problem of constructing a useful algorithm to compute the transforms introduced in [19] is addressed in this paper by extending to LT generalizations the method of substituting the transform with the partial sum of a Dirichlet series.

In Section 2 we briefly recall some basic formulas related to Bell polynomials. Section 3 introduces extensions of the LT whose kernel makes use of Laguerre-type exponentials. In Section 4 further generalizations are presented, obtained by substituting the coefficients of the expansions of such exponentials with a sequences subject only to a reasonable hypothesis of convergence.

A concise description of the matrix pencil method is given at the beginning of Section 5, which presents the the numerical proofs carried out using the computer algebra program Mathematica© that have confirmed the correctness of our theoretical approach.

In our opinion the method presented here shows how Tricomi's technique can be applied to more general transforms than the LT, which involve a large extension of the choice of the kernel. And this is the main motivation of the present paper.

2 Recalling The Bell Polynomials

Considering the n-times differentiable functions $x = g(t)$ and $y = f(x)$, defined in given intervals of the real axis, the composite function $\Phi(t) := f(g(t))$, can be

2456-8686, 6(1), 2022:015-040
https://doi.org/10.26524/cm118

differentiated with respect to t, up to the nth order, by using the chain rule.

We use the notations:

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t).$$

Then the nth derivative of $\Phi(t)$ is represented by:

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n),$$

where Y_n denotes the nth Bell polynomial [2].

The traditional form of the Bell polynomials [5] is given by:

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) f_k \,, \tag{2}$$

where the $B_{n,k}$ satisfy the recursion [5]:

$$B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \ldots, , g_{n-h-k+1})\, g_{h+1} \,. \tag{3}$$

The $B_{n,k}$ functions for any $k = 1, 2, \ldots, n$ are polynomials in the $g_1, g_2, \ldots, g_n$ variables homogeneous of degree k and isobaric of weight n (i.e. they are linear combinations of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$), so that:

$$B_{n,k}(\alpha\beta\, g_1, \alpha\beta^2 g_2, \ldots, \alpha\beta^{n-k+1} g_{n-k+1}) = \alpha^k \beta^n B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\,, \tag{4}$$

and:

$$Y_n(f_1, \beta\, g_1; f_2, \beta^2 g_2; \ldots; f_n, \beta^n g_n) = \beta^n\, Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n)\,. \tag{5}$$

2456-8686, 6(1), 2022:015-040
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3 Laguerre-Laplace Transforms

Let us consider the Laguerre-type exponential of integer order $r \geq 0$:

$$e_r(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{r+1}} \,. \tag{6}$$

In [19] the generalized LT that we will now call Laguerre-Laplace transforms have been introduced:

$$\mathcal{L}_1(f) := \int_0^{\infty} [e_1(s\,t)]^{-1} f(t)\,dt = \int_0^{\infty} \left[\sum_{k=0}^{\infty} \frac{(s\,t)^k}{(k!)^2}\right]^{-1} f(t)\,dt = F_1(s)\,, \tag{7}$$

and in general:

$$\mathcal{L}_r(f) := \int_0^{\infty} [e_r(s\,t)]^{-1} f(t)\,dt = \int_0^{\infty} \left[\sum_{k=0}^{\infty} \frac{(s\,t)^k}{(k!)^r}\right]^{-1} f(t)\,dt = F_r(s)\,. \tag{8}$$

As in the ordinary Laplace transform, the integrals in (7)-(8) exist for all real numbers $\mathrm{Re}(s) > \sigma_c$, where the constant σ_c, called the convergence abscissa, depends on the function f and determines the region of convergence.

Obviously, for $r = 0$, it is $e_0(x) \equiv \exp(x)$ and, hence, $F_0(s) \equiv F(s)$.

Computation of Laguerre-Laplace Transforms

By applying the matrix pencil method detailed in [25], the kernel of the Laguerre-Laplace transform (8) can be represented as:

$$[e_r(x)]^{-1} \simeq [\tilde{e}_r(x)]^{-1} = \sum_{n=1}^{N_r} R_{n,r}\, e^{-\alpha_{n,r}\, x}\,, \tag{9}$$

where $R_{n,r}$ and $\alpha_{n,r} > 0$ denote, respectively, the $n-$th residue and damping factor of the approximated kernel. In (9), the expansion order N_r is selected in such a way as to satisfy, for $\forall x \geq 0$, the condition:

$$\left|\, [e_r(x)]^{-1} - [\tilde{e}_r(x)]^{-1} \,\right| \leq \delta\,, \tag{10}$$

where δ denotes a given sufficiently small positive parameter used to control the accuracy of the representation of $[e_r(x)]^{-1}$.

Therefore, upon substituting (9) into (8), it is straightforward to conclude that:

$$F_r(s) \simeq \tilde{F}_r(s) = \sum_{n=1}^{N_r} R_{n,r}\, F(\alpha_{n,r}\, s)\,. \tag{11}$$

In particular, assuming:

$$f(t) = \sum_{k=0}^{\infty} \gamma_k\, L_k(t)\,, \tag{12}$$

it follows, from Tricomi's theory [20], that:

$$F(s) = \int_0^{\infty} e^{-s\,t} f(t)\, dt = \frac{1}{s} \sum_{k=0}^{\infty} \gamma_k \left(\frac{s-1}{s} \right)^k . \tag{13}$$

In this way, one can readily find:

$$\tilde{F}_r(s) = \frac{1}{s} \sum_{k=0}^{\infty} \gamma_k\, \Psi_k^{(r)}(s)\,, \tag{14}$$

with:

$$\Psi_k^{(r)}(s) = \sum_{n=1}^{N_r} \frac{R_{n,r}}{\alpha_{n,r}} \left(1 - \frac{1}{\alpha_{n,r}\, s} \right)^k . \tag{15}$$

Numerical examples of this procedure are shown in the subsections 5.1 and 5.2.

4 Bell-Laplace Transforms

In [19] a broader class of generalized Laplace transforms was introduced. We refer to them as Bell-Laplace transforms, since they are defined by using Bell polynomials.

This type of transform is presented as an immediate extension of those considered in Section 3, since the Taylor expansion coefficients $1/(k!)^r$, appearing in the kernel, are replaced by positive numbers subject only to the restriction indicated by the

2456-8686, 6(1), 2022:015-040
https://doi.org/10.26524/cm118

hypothesis reported below.

More precisely, given the umbral symbol $a := \{a_k\} = (1, a_1, a_2, a_3, \dots)$, $a_k > 0, \forall k$, we consider the function:

$$\Lambda_a(x) = \frac{1}{1 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots} = \frac{1}{1 + \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}} \quad (x \geq 0)\,, \tag{16}$$

and the associated generalized Laplace transform defined by:

$$\mathcal{L}_a(f) = \int_0^\infty \Lambda_a(s\,t)\, f(t)\, dt = \int_0^\infty f(t) \sum_{k=0}^{\infty} C_k(a)\, \frac{(s\,t)^k}{k!}\, dt = F_a(s)\,, \tag{17}$$

where the coefficients $C_k(a)$ are given by:

$$C_k(a) := \sum_{h=1}^{k} (-1)^h h!\, B_{k,h}(a_1, a_2, \dots, a_{k-h+1})\,, \quad C_0(a) := 1\,, \tag{18}$$

with $B_{k,h}(\cdot)$ being defined in (3). Note that the ordinary Laplace transform corresponds to the sequence: $a = (1, 1, \dots, 1, \dots)$, that is $a_k \equiv 1\,, \forall k \geq 1$. Therefore, it results:

$$\mathcal{L}_{(1,1,\dots,1,\dots)}(f) \equiv \mathcal{L}(f)\,.$$

A number of sums defining the coefficients $C_k(a)$ corresponding to different sequences $a = \{a_k\}$ can be found in [18], however, in what follows, we will assume the fundamental hypothesis that preserves the property of the ordinary Laplace transform:

HP. For every fixed s in the region of convergence, the power series $\sum_{k=0}^{\infty} C_k(a)\,(s\,t)^k/k!$, in equation (17) has an exponential decay when $t \to \infty$.

In this context, another possibility is to assume $a_k = k!\,, \forall k \geq 1$. In this case, the kernel of the Bell-Laplace transform becomes:

$$\Lambda_a(x) = \frac{1}{1 + x + x^2 + x^3 + \dots} \quad (x \geq 0)\,. \tag{19}$$

2456-8686, 6(1), 2022:015-040
https://doi.org/10.26524/cm118

Computation of Bell-Laplace Transforms

By applying the matrix pencil method detailed in [25], the kernel of the Bell-Laplace transform (17) can be represented as:

$$\Lambda_a(x) \simeq \tilde{\Lambda}_a(x) = \sum_{n=1}^{N_a} R_{n,a}\, e^{-\alpha_{n,a}\, x}\,, \tag{20}$$

where $R_{n,a}$ and $\alpha_{n,a} > 0$ denote, respectively, the $n-$th residue and damping factor of the approximated kernel. In (20), the expansion order N_a is selected in such a way as to satisfy, $\forall x \geq 0$, the condition:

$$\left| \Lambda_a(x) - \tilde{\Lambda}_a(x) \right| \leq \delta\,, \tag{21}$$

where δ denotes a given sufficiently small positive parameter used to control the accuracy of the representation of $\Lambda_a(x)$.

Therefore, upon substituting (20) into (17), it is straightforward to conclude that:

$$F_a(s) \simeq \tilde{F}_a(s) = \sum_{n=1}^{N_a} R_{n,a}\, F(\alpha_{n,a}\, s)\,. \tag{22}$$

In particular, assuming:

$$f(t) = \sum_{k=0}^{\infty} \gamma_k\, L_k(t)\,, \tag{23}$$

it follows, from Tricomi's theory [20], that:

$$F(s) = \int_0^{\infty} e^{-s\,t} f(t)\, dt = \frac{1}{s} \sum_{k=0}^{\infty} \gamma_k \left(\frac{s-1}{s} \right)^k . \tag{24}$$

In this way, one can readily find:

$$\tilde{F}_a(s) = \frac{1}{s} \sum_{k=0}^{\infty} \gamma_k\, \Psi_k^{(a)}(s)\,, \tag{25}$$

with:

$$\Psi_k^{(a)}(s) = \sum_{n=1}^{N_a} \frac{R_{n,a}}{\alpha_{n,a}} \left(1 - \frac{1}{\alpha_{n,a}\, s}\right)^k . \tag{26}$$

Numerical examples of Bell-Laplace Transforms, for particular sequences $\{a_k\}$, are shown in the subsections 5.3 and 5.4.

5 Numerical Results

The matrix pencil method

For a better understanding of the numerical technique used, we premise a concise exposition of the matrix pencil method, a computational tool usually exploited in signal theory [25].

Suppose we have N samples $(y_0, y_1, \dots, y_{N-1})$ of a given function $y(t)$ taken at equal intervals of T. Let us model the function as a sum of M complex exponentials:

$$y(t) = \sum_{j=1}^{M} R_j e^{s_j t} , \tag{27}$$

so that:

$$y_k = y(k\,T) = \sum_{j=1}^{M} R_j z_j^k , \tag{28}$$

with $z_j = e^{s_j T}$. The matrix pencil method allows evaluating the best estimates of the parameters M and R_j, z_j $(j = 1, 2, \dots, M)$ in (28). To this end, let us introduce the matrices $[Y_1]$ and $[Y_2]$ as follows:

$$[Y_1] = \begin{bmatrix} y_1 & y_2 & \cdots & y_L \\ y_2 & y_3 & \cdots & y_{L+1} \\ \vdots & \vdots & \cdots & \vdots \\ y_{N-L} & y_{N-L+1} & \cdots & y_{N-1} \end{bmatrix}_{(N-L)\times L} , \tag{29}$$

$$[Y_2] = \begin{bmatrix} y_0 & y_1 & \cdots & y_{L-1} \\ y_1 & y_2 & \cdots & y_L \\ \vdots & \vdots & \cdots & \vdots \\ y_{N-L-1} & y_{N-L} & \cdots & y_{N-2} \end{bmatrix}_{(N-L)\times L} , \tag{30}$$

where $1 \leq L \leq N-1$. Using simple algebra, it can be readily verified that:

$$[Y_1] = [Z_1]\,[R]\,[Z_0]\,[Z_2] \;, \tag{31}$$

$$[Y_2] = [Z_1]\,[R]\,[Z_2] \;, \tag{32}$$

with:

$$[Z_1] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & \cdots & \vdots \\ z_1^{N-L-1} & z_2^{N-L-1} & \cdots & z_M^{N-L-1} \end{bmatrix}_{(N-L)\times M} , \tag{33}$$

$$[Z_2] = \begin{bmatrix} 1 & z_1 & \cdots & z_1^{L-1} \\ 1 & z_2 & \cdots & z_2^{L-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & z_M & \cdots & z_M^{L-1} \end{bmatrix}_{M\times L} , \tag{34}$$

$$[Z_0] = \mathrm{diag}\left(z_1, z_2, \ldots, z_M\right) \;, \tag{35}$$

$$[R] = \mathrm{diag}\left(R_1, R_2, \ldots, R_M\right) \;. \tag{36}$$

Using (31) and (32), the following matrix pencil can be defined:

$$[Y_1] - \lambda [Y_2] = [Z_1] [R] \{[Z_0] - \lambda [I]\} [Z_2] \ , \tag{37}$$

where $[I]$ is the identity matrix of order M.

The rank of the left-hand side of 37 is M if $M \leq L \leq N - M$ [25]. However, if $\lambda = z_j$ $(j = 1, 2, \ldots, M)$, the rank is reduced to $M - 1$ since the $j-$th row and column of $[Z_0] - \lambda[I]$ become zero. This implies that the quantities z_j are the generalized eigenvalues of the matrix pair $\{[Y_1], [Y_2]\}$. Hence:

$$[Y_1] [r_j] = z_j [Y_2] [r_j] \ , \tag{38}$$

with $[r_j]$ denoting the generalized eigenvectors corresponding to z_j so that:

$$\Big\{[Y_2]^{\dagger} [Y_1] - z_j [I]\Big\} [r_j] = [0] \ , \tag{39}$$

where $[Y_2]^{\dagger}$ is the Moore-Penrose pseudo-inverse of $[Y_2]$, that is:

$$[Y_2]^{\dagger} = \Big\{[Y_1]^H [Y_1]\Big\}^{-1} [Y_1]^H \ , \tag{40}$$

with the superscript ^{H} defining the complex conjugate matrix transposition. It is straightforward to conclude that the parameters z_j for $j = 1, 2, \ldots, M$ can be evaluated as the eigenvalues of $[Y_2]^{\dagger} [Y_1]$.

Once M and z_j $(j = 1, 2, \ldots, M)$ are knwon, the complex amplitudes R_j can be easily determined by solving the least-square problem:

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & \cdots & \vdots \\ z_2^{N-1} & z_2^{N-1} & \cdots & z_M^{N-1} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_M \end{bmatrix} . \tag{41}$$

The matrix equation (41) can be solved using the QR decomposition with Householder transformations [25]. In the following, we assume $L = \lfloor N/2 \rfloor$.

In next sections we show some numerical examples of the computational procedures described in the previous sections.

Example: Laguerre-Laplace Transform of Order 1

The kernel of the Laguerre-Laplace transform of order $r = 1$ is given by:

$$[e_1(x)]^{-1} = \left[\sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}\right]^{-1} = I_0\left(2\sqrt{x}\right)^{-1}\,, \tag{42}$$

where $I_0(\cdot)$ denotes the modified Bessel function of the first kind and order 0.

The approximation $[\tilde{e}_1(x)]^{-1}$ of (42) as a sum of exponential terms can be derived using the matrix pencil method. Upon enforcing a maximal deviation $\delta = 10^{-7}$ between $[e_1(x)]^{-1}$ and $[\tilde{e}_1(x)]^{-1}$, one can verify that:

$$[\tilde{e}_1(x)]^{-1} = \sum_{n=1}^{N_1} R_{n,1}\, e^{-\alpha_{n,1}\, x}\,, \tag{43}$$

with $N_1 = 9$ and:

$$\begin{aligned}
R_{1,1} &\simeq 0.0009429027783080 & \alpha_{1,1} &\simeq 0.108582859433469\\
R_{2,1} &\simeq 0.0181559779019476 & \alpha_{2,1} &\simeq 0.177489238529601\\
R_{3,1} &\simeq 0.0888169808858299 & \alpha_{3,1} &\simeq 0.282879032690632\\
R_{4,1} &\simeq 0.2004047551154255 & \alpha_{4,1} &\simeq 0.450634510115705\\
R_{5,1} &\simeq 0.2700764651727050 & \alpha_{5,1} &\simeq 0.723357009576486\\
R_{6,1} &\simeq 0.2418054045450106 & \alpha_{6,1} &\simeq 1.171695052150548\\
R_{7,1} &\simeq 0.1375503239020312 & \alpha_{7,1} &\simeq 1.910904909626434\\
R_{8,1} &\simeq 0.0391426835884303 & \alpha_{8,1} &\simeq 3.145439171078639\\
R_{9,1} &\simeq 0.0031044119784250 & \alpha_{9,1} &\simeq 5.342320873581147
\end{aligned}$$

The accuracy of the representation (43) can be assessed by visual inspection of Figure 1.

Let us now consider the function:

$$f(t) = e^t \, \Gamma(t) \,, \tag{44}$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{k=0}^{K} \gamma_k \, L_k(t) \,, \tag{45}$$

with:

$$\gamma_k = \int_0^\infty e^{-t} \, f(t) \, L_k(t) \, dt = \frac{1}{k+1} \,. \tag{46}$$

Hence, using Tricomi's method, the Laguerre-Laplace transform of order 1 of $f(t)$ can be approximated as:

$$\tilde{F}_1(s) \simeq \frac{1}{s} \sum_{k=0}^{K} \gamma_k \, \Psi_k^{(1)}(s) \,, \tag{47}$$

where $\Psi_k^{(1)}(s)$ is given by (15).

Upon selecting the expansion order $K = 100$, one can readily verify that the approximant $\tilde{f}(t)$ in (45)-(46) is characterized by the distribution shown in Figure 2.

As it can be noticed from the cut sections for $\omega = \mathrm{Im}\{s\} = 1$ and $\sigma = \mathrm{Re}\{s\} = 5$ reported in Figures 3 and 4, respectively, the agreement between the Laguerre-Laplace transform $F_1(s)$ defined by the integral expression in (8) and the relevant Tricomi's approximation $\tilde{F}_1(s)$ as given by (47) is excellent.

Example: Laguerre-Laplace Transform of Order 2

The kernel of the Laguerre-Laplace transform of order $r = 2$ is given by:

$$[e_2(x)]^{-1} = \left[\sum_{k=0}^{\infty} \frac{x^k}{(k!)^3} \right]^{-1} = {}_0F_2(;1,1;x)^{-1} \,, \tag{48}$$

where ${}_0F_2(\cdot)$ denotes the generalized hypergeometric function.

The approximation $[\tilde{e}_2(x)]^{-1}$ of (48) as a sum of exponential terms can be derived using the matrix pencil method. Upon enforcing a maximal deviation $\delta = 10^{-6}$ between $[e_2(x)]^{-1}$ and $[\tilde{e}_2(x)]^{-1}$, one can verify that:

$$[\tilde{e}_2(x)]^{-1} = \sum_{n=1}^{N_2} R_{n,2}\, e^{-\alpha_{n,2}\, x}\,, \tag{49}$$

with $N_2 = 11$ and:

$$\begin{aligned}
&R_{1,2} \simeq 0.00152163969696305 && \alpha_{1,2} \simeq 0.0372180363355215\\
&R_{2,2} \simeq 0.01563911790562768 && \alpha_{2,2} \simeq 0.0770740761072941\\
&R_{3,2} \simeq 0.05380919441791532 && \alpha_{3,2} \simeq 0.1408642812562712\\
&R_{4,2} \simeq 0.10932022287377088 && \alpha_{4,2} \simeq 0.2412633572033314\\
&R_{5,2} \simeq 0.16416939247664572 && \alpha_{5,2} \simeq 0.3989144624401706\\
&R_{6,2} \simeq 0.20058291401447395 && \alpha_{6,2} \simeq 0.6479616510292703\\
&R_{7,2} \simeq 0.20159196904486149 && \alpha_{7,2} \simeq 1.0426736061266058\\
&R_{8,2} \simeq 0.15480259883548350 && \alpha_{8,2} \simeq 1.6690941593410118\\
&R_{9,2} \simeq 0.07779035885705353 && \alpha_{9,2} \simeq 2.6693668765913148\\
&R_{10,2} \simeq 0.01943575641773551 && \alpha_{10,2} \simeq 4.2988208567373025\\
&R_{11,2} \simeq 0.00133681355905532 && \alpha_{11,2} \simeq 7.1400795130657120
\end{aligned}$$

The accuracy of the representation (49) can be assessed by visual inspection of Figure 5.

Let us now consider the function:

$$f(t) = J_0\left(2\sqrt{t}\right)\,, \tag{50}$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{k=0}^{K} \gamma_k\, L_k(t)\,, \tag{51}$$

2456-8686, 6(1), 2022:015-040
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with:

$$\gamma_k = \int_0^\infty e^{-t}\, f(t)\, L_k(t)\, dt = \frac{1}{e\, k!}\,. \tag{52}$$

Hence, using Tricomi's method, the Laguerre-Laplace transform of order 2 of $f(t)$ can be approximated as:

$$\tilde{F}_2(s) \simeq \frac{1}{s} \sum_{k=0}^{K} \gamma_k\, \Psi_k^{(2)}(s)\,, \tag{53}$$

where $\Psi_k^{(2)}(s)$ is given by (15).

Upon selecting the expansion order $K = 100$, one can readily verify that the approximant $\tilde{f}(t)$ in (51)-(52) is characterized by the distribution shown in Figure 6.

As it can be noticed from the cut sections for $\omega = \mathrm{Im}\{s\} = 1$ and $\sigma = \mathrm{Re}\{s\} = 1$ reported in Figures 7 and 8, respectively, the agreement between the Laguerre-Laplace transform $F_2(s)$ defined by the integral expression in (8) and the relevant Tricomi's approximation $\tilde{F}_2(s)$ as given by (53) is excellent.

Example: Bell-Laplace Transform of $e^{\mathrm{i}\pi t}$

The kernel of the Bell-Laplace Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\sin\frac{k\pi}{2}\right|$ $(\forall k \geq 1)$ is given by:

$$\Lambda_a(x) = \frac{1}{1 + \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}} = \frac{1}{1 + \frac{1}{2}\left[I_0\left(2\sqrt{x}\right) - J_0\left(2\sqrt{x}\right)\right]}\,, \tag{54}$$

for $x \geq 0$.

The approximation $\tilde{\Lambda}_a(x)$ of (54) as a sum of exponential terms can be derived using the matrix pencil method. Upon enforcing a maximal deviation $\delta = 10^{-7}$ between $\Lambda_a(x)$ and $\tilde{\Lambda}_a(x)$, one can verify that:

$$\tilde{\Lambda}_a(x) = \sum_{n=1}^{N_a} R_{n,a}\, e^{-\alpha_{n,a}\, x}\,, \tag{55}$$

with $N_a = 13$ and:

$$R_{1,a} \simeq 0.00169530 \qquad \alpha_{1,a} \simeq 0.107767$$

$$R_{2,a} \simeq 0.0275766 \qquad \alpha_{2,a} \simeq 0.170684$$

$$R_{3,a} \simeq 0.120034 \qquad \alpha_{3,a} \simeq 0.258083$$

$$R_{4,a} \simeq 0.250091 \qquad \alpha_{4,a} \simeq 0.380030$$

$$R_{5,a} \simeq 0.330821 \qquad \alpha_{5,a} \simeq 0.543981$$

$$R_{6,a} \simeq -0.374119 \qquad \alpha_{6,a} \simeq 0.997348$$

$$R_{7,a} \simeq 0.591684 \qquad \alpha_{7,a} \simeq 1.32883$$

$$R_{8,a} \simeq -0.00676705 + 0.0404120\,\mathrm{i} \qquad \alpha_{8,a} \simeq 1.98419 - 0.600304\,\mathrm{i}$$

$$R_{9,a} \simeq -0.00676706 - 0.0404120\,\mathrm{i} \qquad \alpha_{9,a} \simeq 1.98419 + 0.600304\,\mathrm{i}$$

$$R_{10,a} \simeq 0.0534155 \qquad \alpha_{10,a} \simeq 3.46158$$

$$R_{11,a} \simeq 0.0110216 \qquad \alpha_{11,a} \simeq 5.33829$$

$$R_{12,a} \simeq 0.00127370 \qquad \alpha_{12,a} \simeq 7.94668$$

$$R_{13,a} \simeq 0.0000399970 \qquad \alpha_{13,a} \simeq 12.0465$$

The accuracy of the representation (55) can be assessed by visual inspection of Figure 9.

Let us now consider the function:

$$f(t) = e^{\mathrm{i}\pi t}\,, \tag{56}$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{k=0}^{K} \gamma_k\, L_k(t)\,, \tag{57}$$

with:

$$\gamma_k = \int_0^{\infty} e^{-t}\, f(t)\, L_k(t)\, dt = \frac{\mathrm{i}}{\pi}\left(1+\frac{\mathrm{i}}{\pi}\right)^{-k-1}. \tag{58}$$

Hence, using Tricomi's method, the Bell-Laplace transform defined by the sequence $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\sin\frac{k\pi}{2}\right|$ of $f(t)$ can be approximated as:

$$\tilde{F}_a(s) \simeq \frac{1}{s}\sum_{k=0}^{K}\gamma_k\, \Psi_k^{(a)}(s)\,, \tag{59}$$

where $\Psi_k^{(a)}(s)$ is given by (26).

Upon selecting the expansion order $K = 200$, one can readily verify that the approximant $\tilde{f}(t)$ in (57)-(58) is characterized by the distribution shown in Figure 10.

As it can be noticed from the cut sections for $\omega = \mathrm{Im}\{s\} = 1$ and $\sigma = \mathrm{Re}\{s\} = 5$ reported in Figures 11 and 12, respectively, the agreement between the Bell-Laplace transform $F_a(s)$ defined by the integral expression in (17) and the relevant Tricomi's approximation $\tilde{F}_a(s)$ as given by (59) is excellent.

Example: Bell-Laplace Transform of e^{-t^2}

The kernel of the Bell-Laplace Laplace transform defined by the finite sequence $a = (1, 2+\sqrt{3}, 2\left(3+2\sqrt{3}\right), 6\left(3+2\sqrt{3}\right), 24\left(2+\sqrt{3}\right), 120, 0, 0, 0, \ldots)$ is given by:

$$\begin{aligned}\Lambda_a(x) &= \frac{1}{1+\sum_{k=1}^{\infty} a_k \frac{x^k}{k!}} \\ &= \frac{1}{1+\left(2+\sqrt{3}\right)x+\left(3+2\sqrt{3}\right)x^2+\left(3+2\sqrt{3}\right)x^3+\left(2+\sqrt{3}\right)x^4+x^5}\,,\end{aligned} \tag{60}$$

for $x \geq 0$.

The approximation $\tilde{\Lambda}_a(x)$ of (60) as a sum of exponential terms can be derived using the matrix pencil method. Upon enforcing a maximal deviation $\delta = 10^{-7}$ between $\Lambda_a(x)$ and $\tilde{\Lambda}_a(x)$, one can verify that:

$$\tilde{\Lambda}_a(x) = \sum_{n=1}^{N_a} R_{n,a}\, e^{-\alpha_{n,a}\, x}\,, \tag{61}$$

with $N_a = 8$ and:

$$R_{1,a} \simeq 0.0000372863 \qquad \alpha_{1,a} \simeq 0.252288$$
$$R_{2,a} \simeq 0.00144103 \qquad \alpha_{2,a} \simeq 0.596276$$
$$R_{3,a} \simeq 0.00117936 - 0.00519126\,\mathrm{i} \qquad \alpha_{3,a} \simeq 7.99411 - 3.02046\,\mathrm{i}$$
$$R_{4,a} \simeq 0.00117936 + 0.00519126\,\mathrm{i} \qquad \alpha_{4,a} \simeq 7.99411 + 3.02046\,\mathrm{i}$$
$$R_{5,a} \simeq 0.0191846 \qquad \alpha_{5,a} \simeq 1.13904$$

$$R_{6,a} \simeq 0.123688 \qquad \alpha_{6,a} \simeq 1.95865$$
$$R_{7,a} \simeq 0.387925 \qquad \alpha_{7,a} \simeq 3.15101$$
$$R_{8,a} \simeq 0.465366 \qquad \alpha_{8,a} \simeq 4.8504$$

The accuracy of the representation (61) can be assessed by visual inspection of Figure 13.

Let us now consider the function:

$$f(t) = e^{-t^2}, \tag{62}$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{k=0}^{K} \gamma_k \, L_k(t)\,, \tag{63}$$

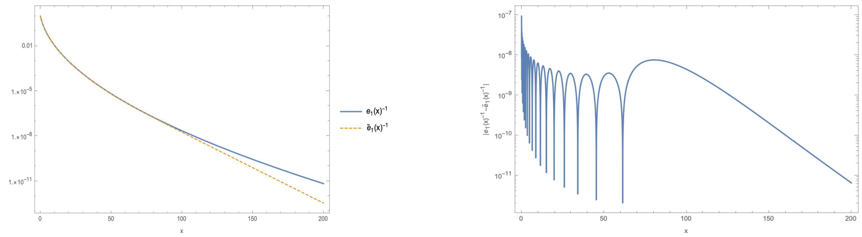
with:

$$\gamma_k = \int_0^\infty e^{-t} \, f(t) \, L_k(t) \, dt\,. \tag{64}$$

Hence, using Tricomi's method, the Bell-Laplace transform defined by the finite sequence $a = \left(1, 2+\sqrt{3}, 2\left(3+2\sqrt{3}\right), 6\left(3+2\sqrt{3}\right), 24\left(2+\sqrt{3}\right), 120, 0, 0, 0, \ldots\right)$ of $f(t)$ can be approximated as:

$$\tilde{F}_a(s) \simeq \frac{1}{s} \sum_{k=0}^{K} \gamma_k \, \Psi_k^{(a)}(s)\,, \tag{65}$$

2456-8686, 6(1), 2022:015-040
https://doi.org/10.26524/cm118



(a) (b)

Figure 1: Distribution of (a) the kernel $[e_1(x)]^{-1}$ relevant to the Laguerre-Laplace transform of order $r = 1$ as compared to its approximant $[\tilde{e}_1(x)]^{-1}$ and of (b) their absolute deviation $\left| [e_1(x)]^{-1} - [\tilde{e}_1(x)]^{-1} \right|$.

where $\Psi_k^{(a)}(s)$ is given by (26).

Upon selecting the expansion order $K = 100$, one can readily verify that the approximant $\tilde{f}(t)$ in (63)-(64) is characterized by the distribution shown in Figure 14.

As it can be noticed from the cut sections for $\omega = \mathrm{Im}\{s\} = 1$ and $\sigma = \mathrm{Re}\{s\} = 2$ reported in Figures 15 and 16, respectively, the agreement between the Bell-Laplace transform $F_a(s)$ defined by the integral expression in (17) and the relevant Tricomi's approximation $\tilde{F}_a(s)$ as given by (65) is excellent.

6 Conclusion

An effective procedure for the numerical evaluation of generalized Laguerre-Laplace and Bell-Laplace transforms has been presented. The matrix pencil method is used to approximate the kernel of the relevant transform as the sum of suitable dumped exponential terms. In this way, the problem can be turned into the computation of the standard Laplace transform. To this end, in the presented study, Tricomi's method is judiciously used. The effectiveness of the proposed methodology has been assessed and demonstrated thorough several test cases involving complex functions typically encountered in Mathematical Physics.

In a forthcoming article will be shown the possibility to use a different procedure based on the generating function.

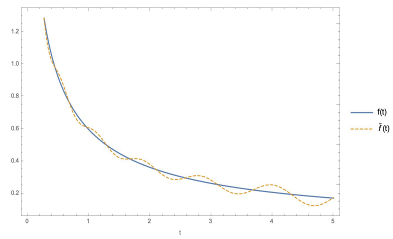
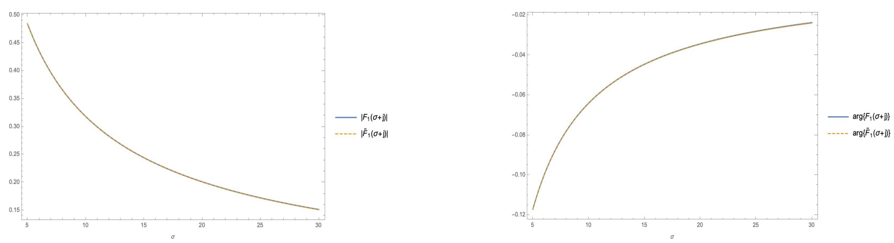
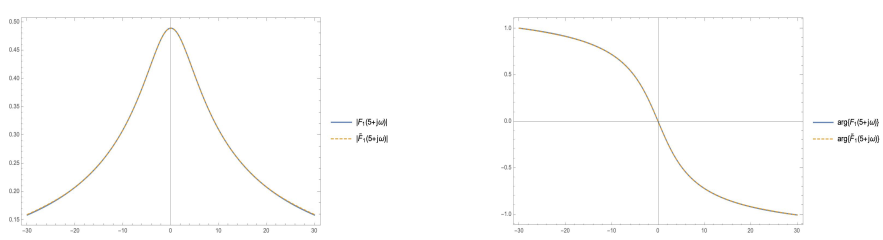


Figure 2: Distribution of the function $f(t) = e^t\,\Gamma(t)$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $K = 100$.



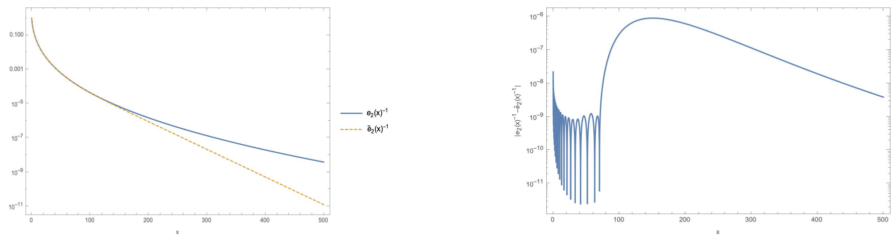
(a) (b)

Figure 3: Magnitude (a) and argument (b) of the Laguerre-Laplace transform of order $r = 1$ relevant to $f(t) = e^t\,\Gamma(t)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F_1(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_1(s)$ of order $K = 100$.



(a) (b)

Figure 4: Magnitude (a) and argument (b) of the Laguerre-Laplace transform of order $r = 1$ relevant to $f(t) = e^t\,\Gamma(t)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 5$ when computed using the exact integral expression $F_1(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_1(s)$ of order $K = 100$.



(a) (b)

Figure 5: Distribution of (a) the kernel $[e_2(x)]^{-1}$ relevant to the Laguerre-Laplace transform of order $r = 2$ as compared to its approximant $[\tilde{e}_2(x)]^{-1}$ and of (b) their absolute deviation $\left| [e_2(x)]^{-1} - [\tilde{e}_2(x)]^{-1} \right|$.

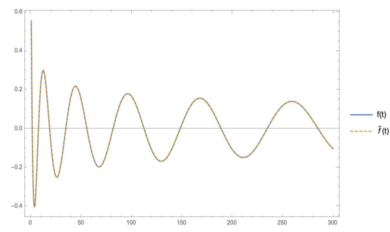
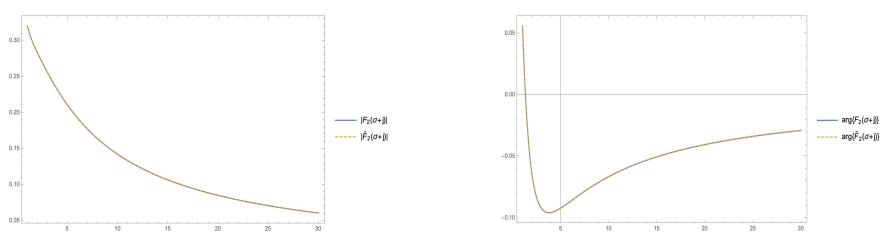


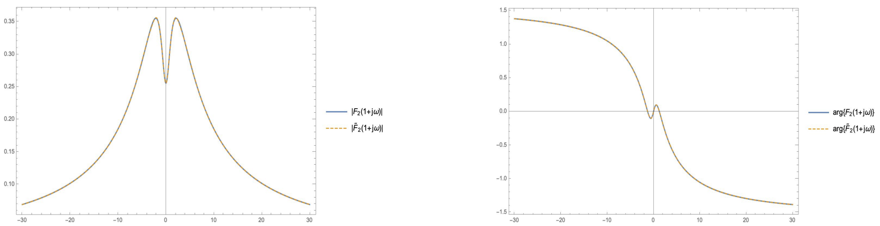
Figure 6: Distribution of the function $f(t) = J_0\left(2\sqrt{t}\right)$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $K = 100$.



(a) (b)

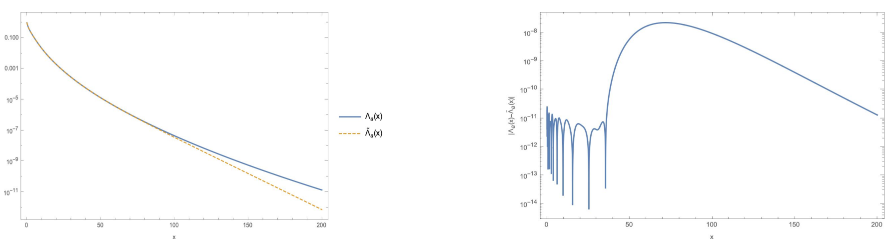
Figure 7: Magnitude (a) and argument (b) of the Laguerre-Laplace transform of order $r = 2$ relevant to $f(t) = J_0\left(2\sqrt{t}\right)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F_2(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_2(s)$ of order $K = 100$.

2456-8686, 6(1), 2022:015-040
https://doi.org/10.26524/cm118



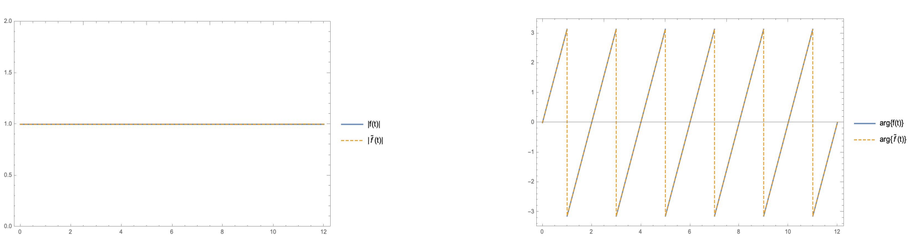
(a) (b)

Figure 8: Magnitude (a) and argument (b) of the Laguerre-Laplace transform of order $r = 2$ relevant to $f(t) = J_0\left(2\sqrt{t}\right)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 1$ when computed using the exact integral expression $F_2(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_2(s)$ of order $K = 100$.



(a) (b)

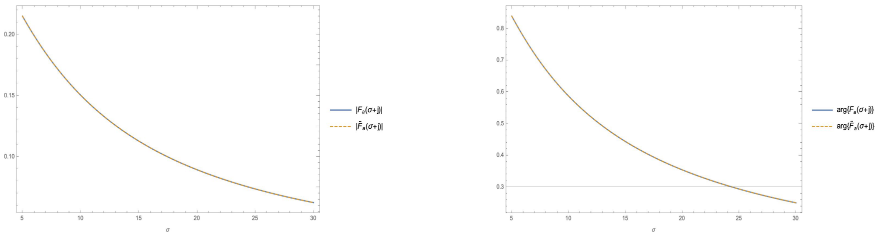
Figure 9: Distribution of (a) the kernel $\Lambda_a(x)$ relevant to the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\sin\frac{k\pi}{2}\right|$ as compared to its approximant $\tilde{\Lambda}_a(x)$ and of (b) their absolute deviation $\left|\Lambda_a(x) - \tilde{\Lambda}_a(x)\right|$.



(a) (b)

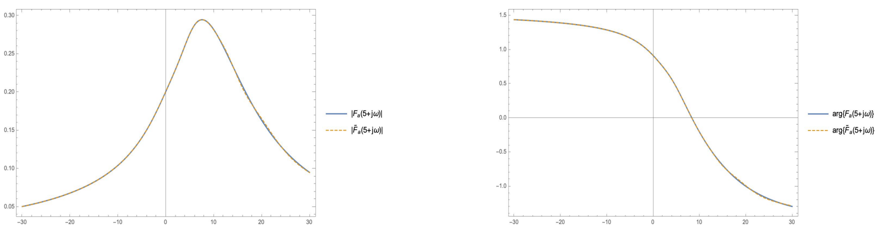
Figure 10: Magnitude (a) and argument (b) of the function $f(t) = e^{\mathrm{i}\pi t}$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $K = 200$.

2456-8686, 6(1), 2022:015-040
https://doi.org/10.26524/cm118



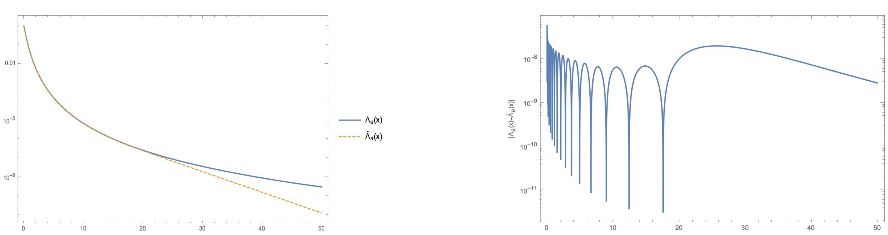
(a) (b)

Figure 11: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\sin\frac{k\pi}{2}\right|$ relevant to $f(t) = e^{\mathrm{i}\pi t}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F_a(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_a(s)$ of order $K = 200$.



(a) (b)

Figure 12: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\sin\frac{k\pi}{2}\right|$ relevant to $f(t) = e^{\mathrm{i}\pi t}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 5$ when computed using the exact integral expression $F_a(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_a(s)$ of order $K = 200$.



(a) (b)

Figure 13: Distribution of (a) the kernel $\Lambda_a(x)$ relevant to the Bell-Laplace transform defined by the finite sequence
$a = \left(1, 2+\sqrt{3}, 2\left(3+2\sqrt{3}\right), 6\left(3+2\sqrt{3}\right), 24\left(2+\sqrt{3}\right), 120, 0, 0, 0, \dots\right)$ as compared to its approximant $\tilde{\Lambda}_a(x)$ and of (b) their absolute deviation $\left|\Lambda_a(x) - \tilde{\Lambda}_a(x)\right|$.

2456-8686, 6(1), 2022:015-040
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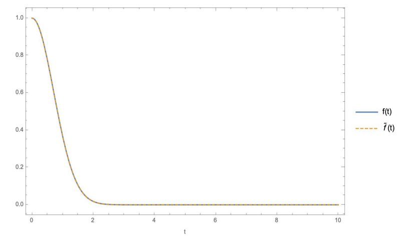
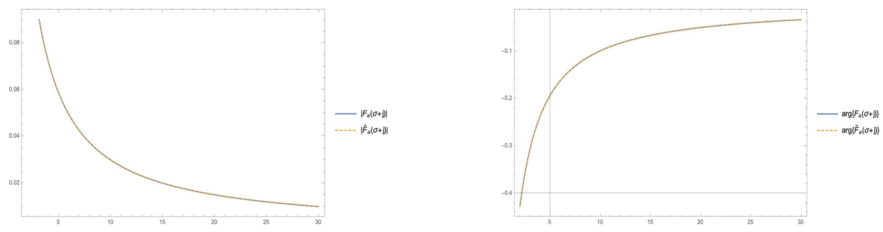
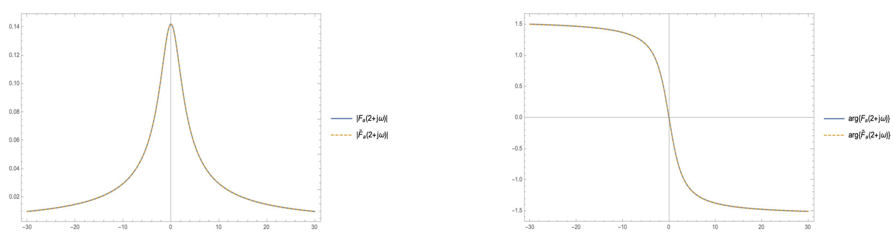


Figure 14: Distribution of the function $f(t) = e^{-t^2}$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $K = 100$.



(a) (b)

Figure 15: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the finite sequence
$a = \left(1, 2+\sqrt{3}, 2\left(3+2\sqrt{3}\right), 6\left(3+2\sqrt{3}\right), 24\left(2+\sqrt{3}\right), 120, 0, 0, 0, \dots\right)$ relevant to $f(t) = e^{-t^2}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F_a(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_a(s)$ of order $K = 100$.



(a) (b)

Figure 16: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the finite sequence
$a = \left(1, 2+\sqrt{3}, 2\left(3+2\sqrt{3}\right), 6\left(3+2\sqrt{3}\right), 24\left(2+\sqrt{3}\right), 120, 0, 0, 0, \dots\right)$ relevant to $f(t) = e^{-t^2}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 2$ when computed using the exact integral expression $F_a(s)$ and the corresponding Tricomi's series approximant $\tilde{F}_a(s)$ of order $K = 100$.

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