2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

ORIGINAL RESEARCH **Open Access**

Sum Divisor Cordial Labeling in the Context of Graph Operations on Grötzsch

U. M. Prajapati [1], P. A. Patel [2]

Received: 02 January 2022 / Accepted: 01 March 2022 / Published online: 22 March 2022

Abstract

A Sum divisor cordial labeling of a graph G with vertex set V is a bijection r from V to $\{1,2,3,\ldots,|V(G)|\}$ such that an edge uv is assigned the label 1 if 2 divides $r(u)+r(v)$ and 0 otherwise; and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a sum divisor cordial labeling is called sum divisor cordial graph. In this research paper, we investigate the sum divisor cordial labeling bahevior for Grötzsch graph, fusion of any two vertices in Grötzsch graph, duplication of an arbitrary vertex in Grötzsch graph, duplication of an arbitrary vertex by an edge in Grötzsch graph, switching of an arbitrary vertex of degree four in Grötzsch graph, switching of an arbitrary vertex of degree three in Grötzsch graph and path union of two copies of Grötzsch.

Key Words: sum divisor cordial labeling, fusion, duplication, switching, path union.

AMS Classification: 05A05, 05A17, 11B25.

1 Introduction

Let $G=(V,E)$ be a simple, finite, undirected and non-trivial graph with the vertex set V. The number of elements of V, denoted as $|V(G)|$ is called the order of G while the number of elements of E, denoted as $|E(G)|$ is called the size of G. More detail of graph labeling results and its applications can be found in Gallian [2]. We provide brief summary of definitions and other related information which are useful for the further investigations.

The present work is aimed to discuss one such labeling known as sum divisor cordial labeling.

Note: Vartharajan et al. [3] introduced the concept of divisor cordial labeling. Lawrence Rozario Raj and Lawrence Joseph Monoharan [2] proved that $S'(K_{\{2,m\}})$, $S'(K_{\{1,n,n\}})$, double fan , cone , Jewel graph admits divisor cordial labeling. Bosmia and Kanani [4] proved that bistar $B_{m,n}$, splitting graph of bistar $B_{m,n}$, degree splitting graph of bistar $B_{m,n}$, shadow graph

[1] St. Xavier's College (Autonomous), Ahmedabad 380009, India, E-mail: udayan64@yahoo.com
[2] Research scholar, Department of Mathematics , Gujarat University, Ahmedabad 380009, India, E-mail: patelprerak111@gmail.com

of bistar $B_{m,n}$, restricted square graph of bistar $B_{m,n}$, barycentric subdivision of bistar $B_{m,n}$ and corona product of bistar $B_{m,n}$ with K_1 admit divisor cordial labeling. Lourdusamy and Patrick [8] introduced the concept of sum divisor cordial labeling and proved that $K_2+\{mk\}_1$, bistar, jewel, path, comb, star, crown, flower, gear, subdivision of the star and square graph of $B_{m,n}$ are sum divisor cordial graphs. Prajapati and Patel [5] proved that friendship graph F_n, duplication of the a vertex by an edge in F_n, duplication of the an edge by a vertex in F_n and duplication of the a vertex by a vertex in F_n are divisor cordial labeling.

2 Definitions

Definition 2.1: [2] A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0)-v_f(1)|\le 1$ and $|e_f(0)-e_f(1)|\le 1$. A graph G is cordial if it admits cordial labeling.

Definition 2.2: [3] A divisor cordial labeling of a graph G with vertex set V is a bijection r from V to $\{1,2,3,...,|V(G)|\}$ such that if each edge uv is assigned the label 1 if $r(u)$ divides $r(v)$ or $r(v)$ divides $r(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph with a divisor cordial labeling, then it is called a divisor cordial graph.

Definition 2.3: [8] A sum divisor cordial labeling of a graph G with vertex set V is a bijection r from V to $\{1,2,3,...,|V(G)|\}$ such that an edge uv is assigned the label 1 if 2 divides $r(u)+r(v)$ and 0 otherwise; and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits sum divisor cordial labeling is called a sum divisor cordial graph.

Definition 2.4: Let u and v be two distinct vertices of graph G. A new graph G' is constructed by fusing(identifying) two vertices u and v by a single vertex w in G' such that every edge which was incident with either u (or) v in G now incident with w in G'.

Definition 2.5: [9] Duplication of a vertex u_k of a graph G produces a new graph G' by adding a new vertex u_k' such that $N(u_k)=N(u_k')$. In other words a vertex u_k' is said to be a duplication of u_k if all the vertices which are adjacent to u_k in G are adjacent to u_k' in G'.

Definition 2.6: A vertex switching G_u of a graph G is obtained by taking a vertex u of G, removing the entire edges incident with u and adding edges joining u to every vertex which are non-adjacent to u in G.

Definition 2.7: [10] The path union of a graph G is the graph obtained by adding an edge between corresponding vertices of G_j to G_{j+1}, $1\le j\le n-1$ where $G_1,G_2,G_3,...,G_n$ ($n\ge 2$) are n copies of G. It is denoted by $p(n.G)$.

2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

Definition 2.8: A Grötzsch graph G_z is a triangle-free bipartite undirected graph with 11 vertices and 20 edges, chromatic number 4, and crossing number 5.

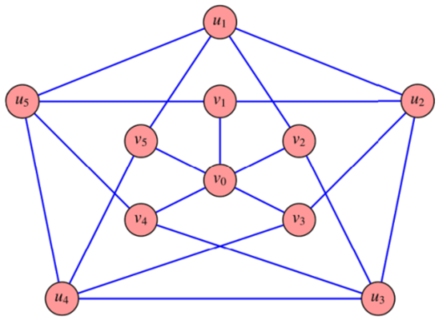


Figure A: Grotzch graph G_z

In this research paper, we always fix the position of vertices $v_1, v_2, v_3, v_5, u_1, u_2, u_3, u_4, u_5$ of G_Z as mentioned in the above figure A, unless or otherwise specified.

3 Main Results

Theorem 2.1: The graph G_Z is a sum divisor cordial graph.

Proof**:** Let G_Z be the Grötzsch graph and Let v_0 be the central vertex and $v_1, v_2, v_3, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of the G_Z. Then $|V(G_Z)|=11$ and $|E(G_Z)|=20$.

Define $r: V(G_Z) \rightarrow \{1, 2, 3, ..., |V(G_Z)|\}$ as follows:

$$r(p)=\begin{cases} 1, & if\ p=v_0; \\ 2j+1, & if\ p=v_j, 1\leq j\leq 5; \\ 2j, & if\ p=u_j, 1\leq j\leq 5. \end{cases}$$

From the above labeling pattern, we have $e_r(1)=e_r(0)=10$.

Hence, we observe that $|e_r(1)-e_r(0)|\leq 1$, so G_Z is a sum divisor cordial graph.

Example 2.1. A sum divisor cordial labeling of Grötzsch graph G_Z is shown in Figure B.

2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

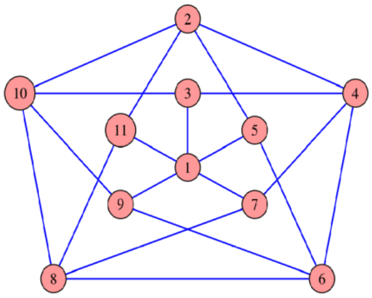


Figure:B

Theorem 2.2: A graph made from fusion of any two vertices in G_z is a sum divisor cordial graph.

Proof: Let G be the graph made from G_z by fusion of any two vertices in G_Z . Then $|V(G)|=10$.

Case 1: Without loss of generality, we assume that the vertices u_1 and u_2 are fussed to the new vertex u and $u=u_1u_2$.

Define $r:V(G)\to\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2i+1, & if\ p=v_i,\ 0\le i\le 5;\\ 2, & if\ p=u;\\ 2i+2, & if\ p=u_{i+2}, 1\le i\le 3.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=9$ and $e_r(1)=10$.

Case 2: Without loss of generality, we assume that the vertices u_1 and u_3 are fussed to the new vertex u and $u=u_1u_3$.

Define $r:V(G)\to\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}3, & if\ p=v_1;\\ 10, & if\ p=v_2;\\ 2i+1, & if\ p=v_i,\ 3\le i\le 4;\\ 5, & if\ p=v_5;\\ 2, & if\ p=u;\\ 4, & if\ p=u_2;\\ 2i+2, & if\ p=u_{i+2},\ 2\le i\le 3.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=9$ and $e_r(1)=10.$

Case 3: Without loss of generality, we assume that the vertices u_1 and v_1 are fussed to the new vertex u and $u=u_1v_1$.

Define $r:V(G)\rightarrow\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}1, & if\ p=v_0;\\ 3, & if\ p=u;\\ 5, & if\ p=v_2;\\ 2, & if\ p=v_3;\\ i+5, & if\ p=v_i,\ 4\le i\le 5;\\ 7, & if\ p=u_2;\\ 2i+2, & if\ p=u_{i+2},\ 1\le i\le 3.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=10$ and $e_r(1)=10.$

Case 4: Without loss of generality, we assume that the vertices u_1 and v_5 are fussed to the new vertex u and $u=u_1v_5$.

Define $r:V(G)\rightarrow\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2i+1, & if\ p=v_i,\ 1\le i\le 4;\\ 10, & if\ p=u;\\ 2i, & if\ p=u_{i+1},\ 1\le i\le 4.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=9$ and $e_r(1)=10.$

Case 5: Without loss of generality, we assume that the vertices v_1 and v_5 are fussed to the new vertex u and $u=v_1v_5$.

Define $r:V(G)\rightarrow\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2i-1, & if\ p=v_i,\ 2\le i\le 4;\\ 9, & if\ p=u;\\ 2i, & if\ p=u_i, 1\le i\le 5.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=9$ and $e_r(1)=10$.

Case 6: Without loss of generality, we assume that the vertices v_1 and v_4 are fussed to the new vertex u and $u=v_1v_4$.

Define $r:V(G)\to\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2i-1, & if\ p=v_i,\ 2\le i\le 3;\\ 7, & if\ p=u;\\ 9, & if\ p=v_5;\\ 2i, & if\ p=u_i, 1\le i\le 5.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=10$ and $e_r(1)=10$.

Case 7: Without loss of generality, we assume that the vertices u_1 and v_0 are fussed to the new vertex u and $u=u_1v_0$.

Define $r:V(G)\to\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2, & if\ p=v_1;\\ 4, & if\ p=v_2;\\ 2i-1, & if\ p=v_i, 2\le i\le 5;\\ 1, & if\ p=u;\\ 3, & if\ p=u_2;\\ 2i, & if\ p=u_i,\ 3\le i\le 5.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=10$ and $e_r(1)=10$.

Case 8: Without loss of generality, we assume that the vertices v_1 and v_0 are fussed to the new vertex u and $u=v_1v_0$.

Define $r:V(G)\to\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2i-1, & if\ p=v_i,\ 2\le i\le 5;\\ 1, & if\ p=u;\\ 2i, & if\ p=u_i,\ 1\le i\le 5.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=10$ and $e_r(1)=9$.

2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

From above all cases, we observe that $|e_r(1)-e_r(0)|\le 1$. So G is a sum divisor cordial graph.

Example 2.2: The graph made from fusion of two vertices u_1 and u_2 in G_z is a sum divisor cordial graph as shown in Figure C.

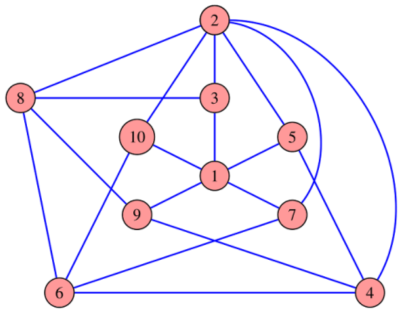


Figure:C

Theorem 2.3: The graph made from duplication of an arbitrary vertex in G_Z is a sum divisor cordial graph.

Proof: Let G_Z be the Grötzsch graph with $|V(G_z)|=11$ and $|E(G_z)|=20$. Let G be the graph made by duplication of an arbitrary vertex w in G_Z. Then $|V(G)|=12$ and $|E(G)|=23$.

Case 1: Without loss of generality, we may take the vertex w=v_1 to be the duplicating vertex and let v_1' be the duplication vertex of v_1.

Define $r:V(G)\rightarrow\{1,2,3,...,|V(G)|\}$ as follows:

$$r(p)=\begin{cases}2k+1, & if\ p=v_k, 0\le k\le 5;\\ 2k, & if\ p=u_k, 1\le k\le 5;\\ 12, & if\ p=v_1'.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=11$ and $e_r(1)=12$.

Case 2: Without loss of generality, we may take the vertex w=u_1 to be the duplicating vertex and let u_1' be the duplication vertex of u_1.

Define $r:V(G)\rightarrow\{1,2,3,...,|V(G)|\}$ as :

$$r(p)=\begin{cases}2k+1, & if\ p=v_k, 0\le k\le 5;\\ 2k, & if\ p=u_k, 1\le k\le 5;\\ 12, & if\ p=u_1'.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=12$ and $e_r(1)=12$.

2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

Case 3: Without loss of generality, we may take the vertex $w = v_0$ to be the duplicating vertex and let v_0' be the duplication vertex of v_0.

Define $r: V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}$ as :

$$r(p) = \begin{cases} k+1, & if\ \ p = v_k, 0 \le k \le 5; \\ k+6, & if\ \ p = u_k, 1 \le k \le 4; \\ 12, & if\ \ p = u_5; \\ 11, & if\ \ p = v_0'. \end{cases}$$

From the above labeling pattern, we have $e_r(0) = 12$ and $e_r(1) = 11$.

From above all cases, we observe that $|e_r(1) - e_r(0)| \le 1$. So G is a sum divisor cordial graph.

Example 2.3.1: The sum divisor cordial labeling of the graph obtained by duplication of a vertex v_1 in G is shown in Figure D.

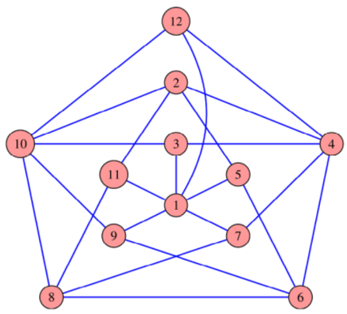


Figure:D

Example 2.3.2: A sum divisor cordial labeling of duplication of a vertex u_1 in G is shown in Figure E.

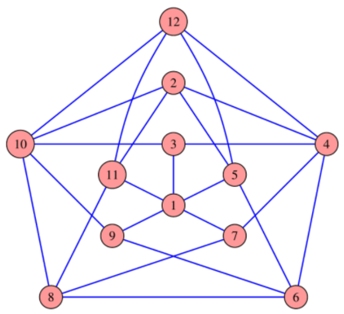


Figure:E

Theorem 2.4: A graph made from duplication of an arbitrary vertex by an edge in G_z is a sum divisor cordial graph.

2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

Proof: Let G_z be a Grötzsch graph and Let v_0 be the central vertex and $v_1, v_2, v_3, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of G_z. Let G be the graph made from duplicating an arbitrary vertex w by an edge e in G_z.

Case 1: Without loss of generality, we may take the duplication of a central vertex $w = v_0$ by an edge $e = v_0'v_0''$ in G_z. Thus $|V(G)| = 13$ and $|E(G)| = 13$

Define $r : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ as:

$$r(p) = \begin{cases} 2t+1, & if\ p = v_t,\ 0 \le t \le 5; \\ 2t, & if\ p = u_t,\ 1 \le t \le 5; \\ 12, & if\ p = v_0'; \\ 13, & if\ p = v_0''. \end{cases}$$

From the above labeling pattern, we have $e_r(0) = 12$ and $e_r(1) = 11$.

Case 2: Without loss of generality, we may take duplication of the central vertex $w = v_1$ by an edge $e = v_1'v_1''$ in G_z. Thus $|V(G)| = 13$ and $|E(G)| = 13$.

Define $r : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ as:

$$r(p) = \begin{cases} 2t+1, & if\ p = v_t,\ 0 \le t \le 5; \\ 2t, & if\ p = u_t,\ 1 \le t \le 5; \\ 12, & if\ p = v_1'; \\ 13, & if\ p = v_1''. \end{cases}$$

From the above labeling pattern, we have $e_r(0) = 12$ and $e_r(1) = 11$.

Case 3: Without loss of generality, we may take duplication of the central vertex $w = v_1$ by an edge $e = v_1'v_1''$ in G_z. Thus $|V(G)| = 13$ and $|E(G)| = 13$.

Define $r : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ as:

$$r(p) = \begin{cases} 2t+1, & if\ p = v_t,\ 0 \le t \le 5; \\ 2t, & if\ p = u_t,\ 1 \le t \le 5; \\ 12, & if\ p = u_1'; \\ 13, & if\ p = u_1''. \end{cases}$$

From the above labeling pattern, we have $e_r(0) = 12$ and $e_r(1) = 11$.

From above all cases, we observe that $|e_r(1) - e_r(0)| \le 1$, than G is a sum divisor cordial graph.

Example 2.4: A graph made from duplication of vertex v_0 by an edge $e = v_0'v_0''$ in G_z is a sum divisor cordial graph as shown in Figure F.

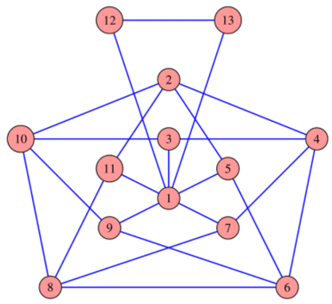


Figure:F

Theorem 2.5: The graph made from switching of an arbitrary vertex of degree four in G_z is a sum divisor cordial graph.

Proof: Let G_z be a Grötzsch graph and let v_0 be the central vertex and $v_1, v_2, v_3, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of the G_z. Let G be the graph made from switching an arbitrary vertex of degree four in G.

Without loss of generality, we may take the switching of a vertex u_1 in G. Thus $|V(G)| = 11$ and $|E(G)| = 21$.

Define $r : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ as follows:

$$r(p) = \begin{cases} 1, & if\ p = v_0; \\ 4, & if\ p = v_1; \\ 2w+1, & if\ p = v_w,\ 2 \le w \le 5; \\ w+1, & if\ p = u_w,\ 1 \le w \le 2; \\ 2w, & if\ p = u_w,\ 3 \le w \le 5. \end{cases}$$

From the above labeling pattern, we have $e_r(0) = 11$ and $e_r(1) = 11$.

Hence, we obeserve that $|e_r(1) - e_r(0)| \le 1$. So G is a sum divisor cordial graph.

Example 2.5: The graph made from switching of vertex u_1 in G_z is a sum divisor cordial graph as shown in Figure G.

2456-8686, 6(1), 2022: 078-090
https://doi.org/10.26524/cm122

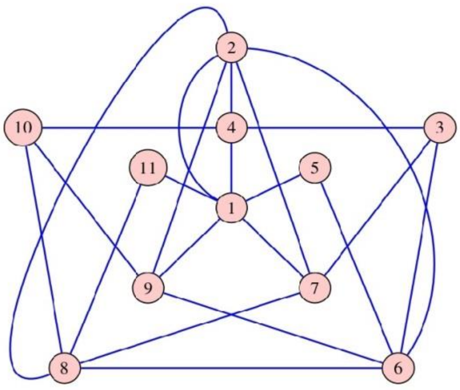


Figure:G

Theorem 2.6: The graph made from switching of an arbitrary vertex of degree three in G_z is a sum divisor cordial graph.

Proof Let G_z be a Grötzsch graph and let v_0 be the central vertex and $v_1,v_2,v_3,v_5,u_1,u_2,u_3,u_4,u_5$ be the remaining vertices of the G_z. Let G be the graph made from switching an arbitrary vertex of degree three in G_z.

Without loss of generality, we may take switching of a vertex v_1 in G. Thus $V(G)=11$ and $E(G)=21$.

Define $r:V(G)\rightarrow\{1,2,3,...,|V(G)|\}$ as :

$$r(p)=\begin{cases}2d+1, & if\ p=v_d,\ 0\leq d\leq 5;\\ d, & if\ p=u_{\frac{d}{2}},\ d=2,4,6,8,10.\end{cases}$$

From the above labeling pattern, we have $e_r(0)=11$ and $e_r(1)=11$

Hence, we observe that $|e_r(1)-e_r(0)|\leq 1$. So G is a sum divisor cordial graph.

Example 2.6: A graph made from switching of an arbitrary vertex v_1 in G_z is a sum divisor cordial graph as shown in Figure H.

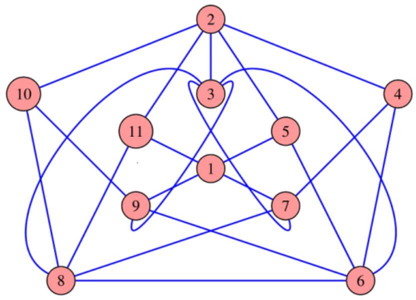


Figure:H

Theorem 2.7: The graph made from path union of two copies of Grötzsch graph G_z is a sum divisor cordial graph.

Proof: Consider two copies of Grötzsch graph G_z' and G_z'' respectively. Let $V(G_z') = \{v_0, v_i : 1 \le i \le 10\}$ and $V(G_z'') = \{w, w_i : 1 \le i \le 10\}$ Then $|V(G_z')| = 11$ and $|E(G_z')| = 18$ and $|V(G_z'')| = 11$ and $|E(G_z'')| = 18$. Let G be the graph made from the path union of two copies of Grötzsch graph G_z' and G_z''. Then $V(G) = V(G_z') \cup V(G_z'')$ and $E(G) = E(G_z') \cup E(G_z'') \cup \{v_8 w_8\}$. Note that G has 22 vertices and 37 edges.

Define $r : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ as follows:

$r(v_0) = 1, r(v_1) = 3, r(v_2) = 5, r(v_3) = 7, r(v_4) = 9, r(v_5) = 11, r(v_6) = 2, r(v_7) = 4, r(v_8) = 6, r(v_9) = 8, r(v_{10}) = 10.$

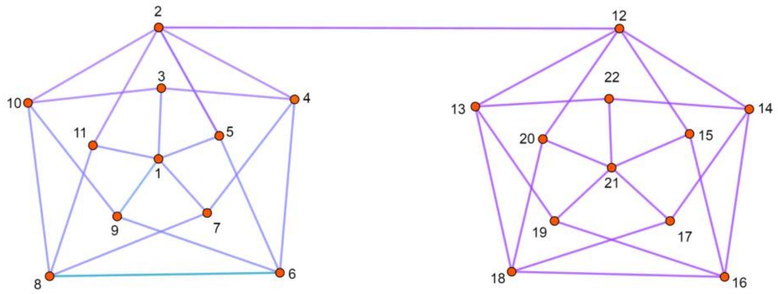
$r(w) = 1,\ r(w_1) = 22,\ r(w_2) = 15,\ r(w_3) = 17,\ r(w_4) = 19, r(w_5) = 20, r(w_6) = 12,$

$r(w_7) = 14,\ r(w_8) = 16,\ r(w_9) = 18,\ r(w_{10}) = 13.$

From the above labeling pattern, we have $e_r(0) = 20$ and $e_r(1) = 21$.

Hence, we obeserve that $|e_r(1) - e_r(0)| \le 1$. So G is a sum divisor cordial graph.

Example 2.7. The graph made from path union of two copies of Grötzsch graph G_z is a sum divisor cordial graph as shown in Figure I.



$G_z{}'$ $\quad$ $G_z{}''$

Figure:I

4 Conclusion

We have derived seven results for sum divisor cordial for Grötzsch graph. From Grötzsch graph, graph obtained by fusion of any two vertices, duplication of an arbitrary vertex, duplication of an arbitrary vertex by an edge, switching of an arbitrary vertex of degree four, switching of an arbitrary vertex of degree three and path union of two copies of Grötzsch graph are sum divisor cordial graph.

5 References

[1] Adalja1 D, Ghodasara G, Subtract Divisor Cordial Labeling of Ring Sum of a Graph With Star Graph, Int. J. of Comp. Sci.and Eng, 6(6), 2018, 1567-1573.
[2] Bosmia M, Kanani K, Divisor Cordial Labeling in the Context of Graph Operations on Bistar, Global Journal of Pure and Applied Mathematics, 12(3), 2016, 2605-2618.
[3] Gallian J, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 16(6), 2021, 1-576.
[4] Harary F, Graph Theory, Addition-Wesley. Reading, Mass, J State Phys, 4, 1972, 103-110.
[5] Kaneria V, Makadia H, Graceful Labeling for Swatik Graph, 5(1), 2015, 111.
[6] Lourdusamy A, Patrick F, Sum Divisor Cordial Graphs, Proyecciones J. of Math, 35(1), 2016, 119-136.
[7] Prajapati U, Patel P, Divisor Cordial Labeling in the Context of Friendship Graph, 5th Edition, Journal of Xidian University, 14(5), 2020, 167-177, https://doi.org/10.37896/jxu14.5/018.
[8] Varatharajan R, Navanaeethakrishnan S, Nagarajan K, Divisor Cordial Graphs, International Journal of Mathematical Combinatorics, 4(4), 2011, 15-25.
[9] West D, Introduction to Graph theory 2nd Edition, Prentice-Hall, Upper Saddle River, 4(4), 2001, 82-83.