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On a self-adjoint coupled system of second-order differential inclusions

Aurelian Cernea[1],

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Abstract

A self-adjoint coupled system of second-order differential inclusions with nonlocal multi-point boundary conditions is considered. An existence result is established when the set-valued maps have nonconvex values.

Key words: differential inclusion, measurable selection, boundary value problem.

AMS classification: 34A60

1. Introduction

A basic fact in the theory of ordinary differential equations is that any linear second-order differential equation may be written in the self-adjoint form $(r(t)y')' = q(t)y$. This equation with boundary conditions of the form $\alpha_1 y(0) - \alpha_2 y'(0) = 0$, $\beta_1 y(T) - \beta_2 y'(T) = 0$ is known as the Sturm-Liouville problem. In the set-valued framework differential inclusions of the form $(r(t)y')' \in F(t, y)$ are called Sturm-Liouville type differential inclusions without any particular choice for the boundary conditions.

The present note is devoted to the following coupled system of Sturm-Liouville differential inclusions

$$\begin{cases} (p_1(t)x_1')' \in F_1(t, x_1, x_2), \quad a.e.\ t \in [a, b], \\ (p_2(t)x_2')' \in F_2(t, x_1, x_2), \quad a.e.\ t \in [a, b], \end{cases} \tag{1}$$

[1]Faculty of Mathematics and Computer Science, University of Bucharest and Academy of Romanian Scientists, Bucharest, Romania.
Email:acernea@fmi.unibuc.ro

with nonlocal multi-point boundary conditions of the form

$$\begin{cases} x_1'(a) = 0, \quad x_1(b) = \sum_{j=1}^{m} \alpha_j x_2(\xi_j), \\ x_2'(a) = 0, \quad x_2(b) = \sum_{k=1}^{n} \beta_k x_1(\mu_k), \end{cases} \tag{2}$$

where $F_1(.,.,.) : [a,b] \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$, $F_2(.,.,.) : [a,b] \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ are given set-valued maps, $a < \xi_1 < ... < \xi_m < \mu_1 < ... < \mu_n < b$, $\alpha_j, \beta_k \in \mathbf{R}_+$, $j = \overline{1,m}$, $k = \overline{1,n}$ and $p_1(.) : [a,b] \to (0,\infty)$, $p_2(.) : [a,b] \to (0,\infty)$ are continuous.

Our study is motivated by a recent paper [11], where sufficient conditions for the existence and uniqueness of solutions for such type of problem are find in the single-valued case; namely, the right-hand side in (1) is given by (single-valued) mappings. All the results in [11] are proved by using several suitable theorems from fixed point theory.

Our intention is to extend the study in [11] to the set-valued framework. The approach we present here takes into account the case when the values of F_1 and F_2 are not convex; but these set-valued maps are assumed to be Lipschitz in the second and third variable. In this case we establish an existence result for problem (1)–(2). Our result use Filippov's technique ([10]); more exactly, the existence of solutions is obtained by starting from a pair of given "quasi" solutions. In addition, the result provides an estimate between the "quasi" solutions and the solutions obtained.

Similar results for "simple" Sturm-Liouville differential inclusions may be found in the literature [2, 3, 4, 5, 6]. As far as we know the present paper is the first in literature which contains an existence result of Filippov type for coupled systems of Sturm-Liouville differential inclusions. We also mention that the technique presented here may be seen at coupled system of fractional differential inclusions [7, 8, 9]. Even if the method we use here is known in the theory of differential inclusions it is largely ignored by the authors that are dealing with such problems in favor of fixed point approaches, most probably, because it is much easier to handle the applications of classical fixed point theorems.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

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2. Preliminaries

We set by I the interval $[a,b]$. We denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.):I\to\mathbf{R}$ endowed with the norm $|x(.)|_C=\sup_{t\in I}|x(t)|$ and by $L^1(I,\mathbf{R})$ the Banach space of all integrable functions $x(.):I\to\mathbf{R}$ endowed with the norm $|x(.)|_1=\int_a^b|x(t)|dt$.

The Pompeiu-Hausdorff distance of the closed subsets $A,B\subset\mathbf{R}$ is defined by $d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}$, where $d^*(A,B)=\sup\{d(a,B);a\in A\}$ and $d(x,B)=\inf_{y\in B}d(x,y)$.

The next technical result is proved in [11].

Lemma 2.1 Let $f_1(.):[a,b]\to\mathbf{R}$, $f_2(.):[a,b]\to\mathbf{R}$ be continuous mappings. Then the solution of the linear system

$$\begin{cases}(p_1(t)x_1')'=f_1(t) & t\in[a,b],\\ (p_2(t)x_2')'=f_2(t) & t\in[a,b]\end{cases}$$

with boundary conditions (2) is given by

$$\begin{cases} x_1(t)=\int_a^t(\frac{1}{p_1(s)}\int_a^s f_1(\tau)d\tau)ds+\frac{1}{C}[-\int_a^b(\frac{1}{p_1(s)}\int_a^s f_1(\tau)d\tau)ds+\\ \sum_{j=1}^m\alpha_j\int_a^{\xi_j}(\frac{1}{p_2(s)}\int_a^s f_2(\tau)d\tau)ds-\int_a^b(\frac{\sum_{j=1}^m\alpha_j}{p_2(s)}\int_a^s f_2(\tau)d\tau)ds+\\ (\sum_{j=1}^m\alpha_j)(\sum_{k=1}^n\beta_k\int_a^{\mu_k}(\frac{1}{p_1(s)}\int_a^s f_1(\tau)d\tau)ds)]\\ x_2(t)=\int_a^t(\frac{1}{p_2(s)}\int_a^s f_2(\tau)d\tau)ds+\frac{1}{C}[-\int_a^b(\frac{1}{p_2(s)}\int_a^s f_2(\tau)d\tau)ds+\\ \sum_{k=1}^n\beta_k\int_a^{\mu_k}(\frac{1}{p_1(s)}\int_a^s f_1(\tau)d\tau)ds-\int_a^b(\frac{\sum_{k=1}^n\beta_k}{p_1(s)}\int_a^s f_1(\tau)d\tau)ds+\\ (\sum_{k=1}^n\beta_k)(\sum_{j=1}^m\alpha_j\int_a^{\xi_j}(\frac{1}{p_2(s)}\int_a^s f_2(\tau)d\tau)ds)], \end{cases} \tag{3}$$

where $C=1-(\sum_{j=1}^m\alpha_j)(\sum_{k=1}^n\beta_k)\neq 0$.

Definition 2.2 The mappings $x_1(.),x_2(.)\in C(I,\mathbf{R})$ are said to be solutions of problem (1)–(2) if there exists $f_1(.),f_2(.)\in L^1(I,\mathbf{R})$ such that $f_1(t)\in F_1(t,x_1(t),x_2(t))$ *a.e.* (I), $f_2(t)\in F_2(t,x_1(t),x_2(t))$ *a.e.* (I) and $x_1(.)$ and $x_2(.)$ are given by (3).

In what follows $\chi_A(\cdot)$ denotes the characteristic function of the set $A \subset \mathbf{R}$.

Remark 2.3 Let us introduce the following notations

$$\mathcal{K}_1(t,\tau) = (\int_\tau^t \frac{1}{p_1(s)}ds)\chi_{[a,t]}(\tau) - \frac{1}{C}(\int_\tau^b \frac{1}{p_1(s)}ds) +$$
$$\frac{1}{C}\sum_{j=1}^m \alpha_j \sum_{k=1}^n \beta_k (\int_\tau^{\mu_k} \frac{1}{p_1(s)}ds)\chi_{[a,\mu_k]}(\tau),$$

$$\mathcal{K}_2(t,\tau) = \frac{1}{C}\sum_{j=1}^m \alpha_j (\int_\tau^{\xi_j} \frac{1}{p_2(s)}ds)\chi_{[a,\xi_j]}(\tau) - \frac{1}{C}\sum_{j=1}^m \alpha_j (\int_\tau^b \frac{1}{p_2(s)}ds),$$

$$\mathcal{K}_3(t,\tau) = \frac{1}{C}\sum_{k=1}^n \beta_k (\int_\tau^{\mu_k} \frac{1}{p_1(s)}ds)\chi_{[a,\mu_k]}(\tau) - \frac{1}{C}\sum_{k=1}^n \beta_k (\int_\tau^b \frac{1}{p_1(s)}ds),$$

$$\mathcal{K}_4(t,\tau) = (\int_\tau^t \frac{1}{p_2(s)}ds)\chi_{[a,t]}(\tau) - \frac{1}{C}(\int_\tau^b \frac{1}{p_2(s)}ds) +$$
$$\frac{1}{C}\sum_{k=1}^n \beta_k \sum_{j=1}^m \alpha_j (\int_\tau^{\xi_j} \frac{1}{p_2(s)}ds)\chi_{[a,\xi_j]}(\tau).$$

Then the solutions $(x_1(.), x_2(.))$ in Lemma 2.1 may be put as

$$x_1(t) = \int_a^b \mathcal{K}_1(t,\tau)f_1(\tau)d\tau + \int_a^b \mathcal{K}_2(t,\tau)f_2(\tau)d\tau, \quad t \in I$$
$$x_2(t) = \int_a^b \mathcal{K}_3(t,\tau)f_1(\tau)d\tau + \int_a^b \mathcal{K}_4(t,\tau)f_2(\tau)d\tau, \quad t \in I.$$

Moreover, if we define $M_1 := \max_{t\in I} \frac{1}{|p_1(s)|}$, $M_2 := \max_{t\in I} \frac{1}{|p_2(s)|}$, for any $t, \tau \in I$ we have the following estimates

$$|\mathcal{K}_1(t,\tau)| \le M_1(b-a)(1+\frac{1}{|C|}) + \frac{1}{|C|}\sum_{j=1}^m \alpha_j \sum_{k=1}^n \beta_k M_1(\mu_k - a) =: k_1,$$

$$|\mathcal{K}_2(t,\tau)| \le \frac{1}{|C|}\sum_{j=1}^m \alpha_j M_2(\xi_j - a) + \frac{1}{|C|}\sum_{j=1}^m \alpha_j M_2(b-a) =: k_2$$

$$|\mathcal{K}_3(t,\tau)| \le \frac{1}{|C|}\sum_{k=1}^n \beta_k M_1(\mu_k - a) + \frac{1}{|C|}\sum_{k=1}^n \beta_k M_1(b-a) =: k_3$$

$$|\mathcal{K}_4(t,\tau)| \le M_2(b-a)(1+\frac{1}{|C|}) + \frac{1}{|C|}\sum_{k=1}^n \beta_k \sum_{j=1}^m \alpha_j M_2(\xi_j - a) =: k_4,$$

Finally, in the proof of our main result we need the following classical selection result for set-valued maps (e.g., [1]).

Lemma 2.4 Let Z be a separable Banach space, B its closed unit ball, $A : I \to \mathcal{P}(Z)$ is a set-valued map whose values are nonempty closed and $b : I \to Z, c : I \to \mathbf{R}_+$ are two measurable functions. If

$$A(t) \cap (b(t) + c(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to A(t) \cap (b(t) + c(t)B)$ admits a measurable selection.

3. Main result

Our results are proved under the following hypotheses.

Hypothesis i) $F_1 : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $F_2 : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ have nonempty closed values and the set-valued maps $F_1(., y_1, y_2)$, $F_2(., y_1, y_2)$ are measurable for any $y_1, y_2 \in \mathbf{R}$.

ii) There exist $l_1(.), l_2(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F_1(t, ., .)$ is $l_1(t)$-Lipschitz and $F_2(t, ., .)$ is $l_2(t)$-Lipschitz; i.e.,

$$d_H(F_1(t, y_1, z_1), F_1(t, y_2, z_2)) \leq l_1(t)(|y_1 - y_2| + |z_1 - z_2|) \quad \forall\, y_1, y_2, z_1, z_2 \in \mathbf{R}.$$

$$d_H(F_2(t, y_1, z_1), F_2(t, y_2, z_2)) \leq l_2(t)(|y_1 - y_2| + |z_1 - z_2|) \quad \forall\, y_1, y_2, z_1, z_2 \in \mathbf{R}.$$

In what follows $l(t) = k_1 l_1(t) + k_2 l_2(t) + k_3 l_1(t) + k_4 l_2(t)$, $t \in I$.

Theorem 3.1 Assume that $C \neq 0$, Hypothesis is satisfied and $|l(.)|_1 < 1$. $(y_1(.), y_2(.)) \in C(I, \mathbf{R})^2$ are considered such that there exist $q_1(.), q_2(.) \in L^1(I, \mathbf{R})$ with $d((p_1(t)y_1(t)')', F_1(t, y_1(t), y_2(t))) \leq q_1(t)$ a.e. $t \in I$, $d((p_2(t)y_2(t)')', F_2(t, y_1(t), y_2(t))) \leq q_2(t)$ a.e. $t \in I$, $y_1'(a) = y_2'(a) = 0$, $y_1(b) = \sum_{j=1}^{m} \alpha_j y_2(\xi_j)$, $y_2(b) = \sum_{k=1}^{n} \beta_k y_1(\mu_k)$.

Then there exists $(x_1(.), x_2(.)) \in C(I, \mathbf{R})^2$ a solution of problem (1)–(2) satisfying for all $t \in I$

$$|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)| \leq \frac{(k_1 + k_3)|q_1(.)|_1 + (k_2 + k_4)|q_2(.)|_1}{1 - |l(.)|_1}. \tag{4}$$

Proof: From the assumptions of the theorem

$$F_1(t, y_1(t), y_2(t)) \cap \{(p_1(t)y_1(t)')' + q_1(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I),$$
$$F_2(t, y_1(t), y_2(t)) \cap \{(p_2(t)y_2(t)')' + q_2(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

By Lemma 2.4 there exist mesurable selections $f_1^1(t) \in F_1(t, y_1(t), y_2(t))$, $f_2^1(t) \in F_2(t, y_1(t), y_2(t))$ $a.e.$ (I) such that

$$|f_1^1(t) - (p_1(t)y_1(t)')'| \leq q_1(t), \quad |f_2^1(t) - (p_2(t)y_2(t)')'| \leq q_2(t) \quad a.e.\ (I).$$

Define

$$x_1^1(t) = \int_a^b \mathcal{K}_1(t, \tau) f_1^1(\tau) d\tau + \int_a^b \mathcal{K}_2(t, \tau) f_2^1(\tau) d\tau, \quad t \in I$$
$$x_2^1(t) = \int_a^b \mathcal{K}_3(t, \tau) f_1^1(\tau) d\tau + \int_a^b \mathcal{K}_4(t, \tau) f_2^1(\tau) d\tau, \quad t \in I.$$

We have the estimates

$$|x_1^1(t) - y_1(t)| \leq k_1|q_1(.)|_1 + k_2|q_2(.)|_1 \quad \forall\, t \in I,$$
$$|x_2^1(t) - y_2(t)| \leq k_3|q_1(.)|_1 + k_4|q_2(.)|_1 \quad \forall\, t \in I,$$

and so,

$$|x_1^1(t) - y_1(t)| + |x_2^1(t) - y_2(t)| \leq (k_1 + k_3)|q_1(.)|_1 + (k_2 + k_4)|q_2(.)|_1 =: k.$$

In the next part of the proof we construct, by induction, the sequences $x_n^1(.), x_n^2(.) \in C(I, \mathbf{R})$ and $f_n^1(.)$, $f_n^2(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_1^n(t) = \int_a^b \mathcal{K}_1(t, \tau) f_1^n(\tau) d\tau + \int_a^b \mathcal{K}_2(t, \tau) f_2^n(\tau) d\tau, \quad t \in I$$
$$x_2^n(t) = \int_a^b \mathcal{K}_3(t, \tau) f_1^n(\tau) d\tau + \int_a^b \mathcal{K}_4(t, \tau) f_2^n(\tau) d\tau, \quad t \in I. \tag{5}$$

$$f_1^n(t) \in F_1(t, x_1^{n-1}(t), x_2^{n-1}(t)), \quad f_2^n(t) \in F_2(t, x_1^{n-1}(t), x_2^{n-1}(t)) \quad a.e.\,(I), \tag{6}$$

$$\begin{aligned} &|f_1^{n+1}(t) - f_1^n(t)| \leq l_1(t)(|x_1^n(t) - x_1^{n-1}(t)| + |x_2^n(t) - x_2^{n-1}(t)|) \quad a.e.\ (I), \\ &|f_2^{n+1}(t) - f_2^n(t)| \leq l_2(t)(|x_1^n(t) - x_1^{n-1}(t)| + |x_2^n(t) - x_2^{n-1}(t)|) \quad a.e.\ (I). \end{aligned} \tag{7}$$

We point out that from (5)–(7) it follows

$$|x_1^{n+1}(t) - x_1^n(t)| + |x_2^{n+1}(t) - x_2^n(t)| \leq k(|l(.)|_1)^n \quad a.e.\ (I) \quad \forall n \in \mathbf{N}. \tag{8}$$

The case $n = 0$ is already proved. Now, we assume (8) valid for $n-1$. For almost all $t \in I$,

$$\begin{aligned} &|x_1^{n+1}(t) - x_1^n(t)| \leq \int_a^b |\mathcal{K}_1(t,\tau)|.|f_1^{n+1}(\tau) - f_1^n(\tau)|d\tau + \int_a^b |\mathcal{K}_2(t,\tau)|.|f_2^{n+1}(\tau) \\ &-f_2^n(\tau)|d\tau \leq k_1 \int_a^b |f_1^{n+1}(\tau) - f_1^n(\tau)|d\tau + k_2 \int_a^b |f_2^{n+1}(\tau) - f_2^n(\tau)|d\tau \leq \\ &k_1 \int_a^b l_1(\tau)(|x_1^n(\tau) - x_1^{n-1}(\tau)| + |x_2^n(\tau) - x_2^{n-1}(\tau)|)d\tau + k_2 \int_a^b l_2(\tau)(|x_1^n(\tau) - \\ &x_1^{n-1}(\tau)| + |x_2^n(\tau) - x_2^{n-1}(\tau)|)d\tau \leq k(|l(.)|_1)^{n-1}(k_1 \int_a^b l_1(\tau)d\tau + k_2 \int_a^b l_2(\tau)d\tau). \end{aligned}$$

In a similar way, we obtain for almost all $t \in I$,

$$|x_2^{n+1}(t) - x_2^n(t)| \leq k(|l(.)|_1)^{n-1}(k_3 \int_a^b l_1(\tau)d\tau + k_4 \int_a^b l_2(\tau)d\tau).$$

Therefore, (8) is true for n.

Inequality (8) shows that the sequences $\{x_1^n(.)\}, \{x_2^n(.)\}$ are Cauchy in the space $C(I, \mathbf{R})$. Let $x_1(.) \in C(I, \mathbf{R})$ and $x_2(.) \in C(I, \mathbf{R})$ be their limits in $C(I, \mathbf{R})$. Also, from (7) we deduce that, for almost all $t \in I$, the sequences $\{f_1^n(t)\}$, $\{f_2^n(t)\}$ are Cauchy in $\mathbf{R}$. We consider $f_1(.)$, $f_2(.)$ their pointwise limit.

At the same time, inequality (8) and Hypothesis give

$$\begin{aligned} &|x_1^n(t) - y_1(t)| + |x_2^n(t) - y_2(t)| \leq |x_1^1(t) - y_1(t)| + |x_2^1(t) - y_2(t)| + \\ &\sum_{i=1}^{n-1}(|x_1^{i+1}(t) - x_1^i(t)| + |x_2^{i+1}(t) - x_2^i(t)|) \leq k + \sum_{i=1}^n k(|l(.)|_1)^i \leq \tfrac{k}{1-|l(.)|_1}. \end{aligned} \tag{9}$$

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and

$$|f_1^n(t)-(p_1(t)y_1(t)')'|+|f_2^n(t)-(p_2(t)y_2(t)')'|\leq|f_1^1(t)-(p_1(t)y_1(t)')'|+$$
$$|f_2^1(t)-(p_2(t)y_2(t)')'|+\sum_{i=1}^{n-1}(|f_1^{i+1}(t)-f_1^i(t)|+|f_2^{i+1}(t)-f_2^i(t)|)\leq$$
$$|f_1^1(t)-(p_1(t)y_1(t)')'|+|f_2^1(t)-(p_2(t)y_2(t)')'|+\sum_{i=1}^{n-1}(l_1(t)+l_2(t))(|x_1^i(t)$$
$$-x_1^{i-1}(t)|+|x_2^i(t)-x_2^{i-1}(t)|)\leq q_1(t)+q_2(t)+(l_1(t)+l_2(t))\frac{k}{1-|l(.)|_1}$$

for almost all $t\in I$.

This means that the sequences $f_1^n(.)$, $f_2^n(.)$ are integrably bounded and therefore, their limits $f_1(.), f_2(.)$ belong to $L^1(I,\mathbf{R})$.

The next step of the proof contains the construction in (5)–(7). By induction, we suppose that for $M\geq 1$, $x_1^m(.), x_2^m(.)\in C(I,\mathbf{R})$ and $f_1^m(.), f_2^m(.)\in L^1(I,\mathbf{R})$, $m=1,2,...M$ with (5) and (7) for $m=1,2,...M$ and (6) for $m=1,2,...M-1$ are constructed.

Using again Hypothesis

$$F_1(t,x_1^M(t),x_2^M(t))\cap\{f_1^M(t)+(l_1(t)|x_1^M(t)-x_1^{M-1}(t)|+l_1(t)|x_2^M(t)-$$
$$x_2^{M-1}(t)|)[-1,1]\}\neq\emptyset,$$
$$F_2(t,x_1^M(t),x_2^M(t))\cap\{f_2^M(t)+(l_2(t)|x_1^M(t)-x_1^{M-1}(t)|+l_2(t)|x_2^M(t)-$$
$$x_2^{M-1}(t)|)[-1,1]\}\neq\emptyset$$

for almost all $t\in I$.

By Lemma 2.4 we obtain the existence of measurable selections $f_1^{M+1}(.)$ of $F_1(.,x_M^1(.),x_M^2(.))$ and $f_2^{M+1}(.)$ of $F_2(.,x_M^1(.),x_M^2(.))$ such that

$$|f_1^{M+1}(t)-f_1^M(t)|\leq l_1(t)(|x_1^M(t)-x_1^{M-1}(t)|+|x_2^M(t)-x_2^{M-1}(t)|)\quad a.e.\ (I),$$
$$|f_2^{M+1}(t)-f_2^M(t)|\leq l_2(t)(|x_1^M(t)-x_1^{M-1}(t)|+|x_2^M(t)-x_2^{M-1}(t)|)\quad a.e.\ (I).$$

We define $x_1^{M+1}(.), x_2^{M+1}(.)$ as in (5) with $n=M+1$.

Finally, it remains to take $n \to \infty$ in (5) and (9) in order to finish the proof.

Corollary 3.2 Assume that $C \neq 0$, Hypothesis is satisfied, $|l(.)|_1 < 1$, $d(0, F_1(t, 0, 0)) \leq l_1(t)$ a.e. $t \in I$ and $d(0, F_2(t, 0, 0)) \leq l_2(t)$ a.e. $t \in I$.

Then there exists $(x_1(.), x_2(.)) \in C(I, \mathbf{R})^2$ a solution of problem (1)–(2) satisfying for all $t \in I$

$$|x_1(t)| + |x_2(t)| \leq \frac{(k_1 + k_3)|l_1(.)|_1 + (k_2 + k_4)|l_2(.)|_1}{1 - |l(.)|_1}.$$

Proof: We apply Theorem 3.1 with $y_1(.) = y_2(.) = 0$, $q_1(.) = l_1(.)$ and $q_2(.) = l_2(.)$.

Remark 3.3 If in (1) F_1 and F_2 are single-valued maps, Corollary 3.2 provides a generalization to the set-valued framework of Theorem 3 in [11].

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