

New general solution of a family higher order differential equations and its application to solve multipoint-integral problems

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Abstract

The family multipoint-integral problems of higher order differential equations is considered. An effective method for solving to family multipoint-integral problems for higher order differential equations is offered. A domain is divided into m parts, the values of a solution at the beginning lines of the subdomains are considered as functional parameters, and the family higher order differential equations are reduced to the family Cauchy problems on the subdomains for system of differential equations with functional parameters. Using the solutions to these family problems, new general solutions to family higher order differential equations are introduced and their properties are established. Based on the general solution, family multipoint-integral problems, and continuity conditions of a solution at the interior lines of the partition, the linear system of functional equations with respect to parameters is composed. Algorithms for finding solutions to families of multipoint-integral problems for higher order differential equations are constructed and conditions for unique solvability are established in the terms of initial data.

Key words: Family multipoint-integral problems, family higher order differential equations, functional parameters, algorithm, solvability.

AMS classification: 34A12; 34A30; 34A99; 34B10; 45D05.

1 Introduction

On the domain $\Omega = [0, T] \times [0, \omega]$ consider the family multipoint-integral problems for the higher order differential equations

$$\begin{aligned} \frac{\partial^n u(t, y)}{\partial t^n} = & a_1(t, y) \frac{\partial^{n-1} u(t, y)}{\partial t^{n-1}} + a_2(t, y) \frac{\partial^{n-2} u(t, y)}{\partial t^{n-2}} + a_3(t, y) \frac{\partial^{n-3} u(t, y)}{\partial t^{n-3}} + \dots + \\ & + a_{n-1}(t, y) \frac{\partial u(t, y)}{\partial t} + a_n(t, y) u(t, y) + f(t, y), \end{aligned} \quad (1.1)$$

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$$\sum_{j=0}^{n-1} \sum_{s=0}^m b_{j,s}^k(y) \frac{\partial^j u(t, y)}{dt^j} \Big|_{t=t_s} + \int_0^T \sum_{j=0}^{n-1} c_j^k(\tau, y) \frac{\partial^j u(\tau, y)}{\partial \tau^j} \partial \tau = d_k(y), \quad y \in [0, \omega], \quad k = 1, 2, \dots, n, \quad (1.2)$$

where $u(t, y)$ is unknown function, the functions $a_i(t, y)$, $i = \overline{1, n}$, and $f(t, y)$ are continuous on Ω ; $b_{j,s}^k(y)$ are continuous on $[0, \omega]$, the functions $c_j^k(t, y)$ are continuous on Ω , $d_k(y)$ are continuous on $[0, \omega]$, $k = \overline{1, n}$, $j = \overline{0, n-1}$, $s = \overline{0, m}$; $0 = t_0 < t_1 < \dots < t_{m-1} < t_m = T$.

Let $C(\Omega, \mathbb{R})$ be the space of continuous on Ω functions $u(t, y)$ with norm $\|u\|_0 = \max_{(t,y) \in \Omega} |u(t, y)|$.

A solution to family multipoint-integral problems (1.1), (1.2) is called the function $u(t, y) \in C(\Omega, \mathbb{R})$, having derivatives $\frac{\partial^p u(t, y)}{\partial t^p} \in C(\Omega, \mathbb{R})$, $p = 1, 2, \dots, n$, satisfies to the family of differential equations (1.1) for all $(t, y) \in \Omega$ and multipoint-integral conditions (1.2).

Various processes of the theory of oscillations, the theory of impulse systems, and the theory of multi-support beams are considered as a family multipoint-integral problems for higher order differential equations [1-13, 16-18, 22-31]. The development of effective and constructive methods for solving the family multipoint-integral problems for higher order differential equations is important and interesting.

The goal of the present paper is to develop an effective method for solving the family multipoint-integral problems for higher order differential equations and to construct an algorithm for finding solution to the family multipoint-integral problems (1.1), (1.2).

We apply a modification of new concept of general solution and Dzhumabaev's parametrization method [19-21].

The paper is organized as follows.

In Section 2, the family multipoint-integral problems for higher order differential equations (1.1), (1.2) is considered. Introducing a new unknown functions the family multipoint-integral problems for higher order differential equations (1.1), (1.2) is transferred to an equivalent family multipoint-integral problems for system of differential equations. Further, the Dzhumabaev's parametrization method [21] is applied for solving the equivalent family problems.

Using the lines $t = t_j$, $j = 0, 1, \dots, m$, we make a partition Δ_m of domain Ω :
$$\Omega = \bigcup_{r=1}^m \Omega_r, \Omega_r = [t_{r-1}, t_r) \times [0, \omega], r = 1, 2, \dots, m, t_0 = 0 < t_1 < t_2 < \dots < t_m = T.$$

The values of a solution at the lines $t = t_{r-1}$ of the subdomains $[t_{r-1}, t_r) \times [0, \omega]$ are considered as functional parameters $\lambda_r(y)$, $r = 1, 2, \dots, m$. The family systems of differential equations is reduced to the family Cauchy problems on the subdomains for the system of differential equations with functional parameters. Using the solutions to these problems, Δ_m general solutions to the family systems of differential equations and initial the family higher order differential equations are introduced and their properties are established. The Δ_m general solution, denoted by $x(t, \Delta_m, \lambda)$, contains an arbitrary vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^{nm}$.

Based on the Δ_m general solution, multipoint-integral conditions, and continuity conditions of a solution at the interior lines of the partition is composed the system of functional equations with respect to parameters in Section 3.

Using $u(t, y, \Delta_m, \lambda)$, we set solvability criteria of considered problem and propose an algorithm for finding its solution. Applying the new general solution reduces the solvability of family multipoint-integral problems for higher order differential equations (1.1), (1.2) to the solvability of the system of functional equations for parameters. An effective method for solving the family multipoint-integral problems for higher order differential equations (1.1), (1.2) is offered. This method includes to solve the system of functional equations and the family Cauchy problems for the system of differential equations.

The principal differences results of the paper from other existing analogues are included the following assertions:

the development of algorithms for the parametrization method for solving family multipoint-integral problems for higher order differential equations;

the construction a new general solution for family higher order differential equations and the establishment its properties;

the creation an effective method for solving family multipoint-integral problems for higher order differential equations based on the composition and solution of systems of linear functional equations for arbitrary vectors of new general solutions;

the establishment unique solvability conditions for family multipoint-integral problems for higher order differential equations.

2 The Δ_m general solution to a family higher order differential equations and its properties

Problem (1.1), (1.2) by introducing new functions

$$x_1(t, y) = u(t, y), \quad x_2(t, y) = \frac{\partial u(t, y)}{\partial t}, \quad \dots, \quad x_{n-1}(t, y) = \frac{\partial^{n-2} u(t, y)}{\partial t^{n-1}}, \quad x_n(t, y) = \frac{\partial^{n-1} u(t, y)}{\partial t^{n-1}},$$

is reduced to a family multipoint-integral problems for a system of differential equations

$$\frac{\partial x(t, y)}{\partial t} = A(t, y)x(t, y) + g(t, y), \quad (t, y) \in \Omega, \quad x \in \mathbb{R}^n, \quad (2.1)$$

$$\sum_{s=0}^m B_s(y)x(t_s, y) + \int_0^T C(\tau, y)x(\tau, y)d\tau = d(y), \quad y \in [0, \omega], \quad d \in \mathbb{R}^n, \quad (2.2)$$

where $x(t, y) = \text{col}(x_1(t, y), x_2(t, y), \dots, x_n(t, y))$ is unknown vector function, the $n \times n$ matrices $A(t, y)$, $C(t, y)$ and n - vector function $g(t, y)$ have the next form

$$A(t, y) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_n(t, y) & a_{n-1}(t, y) & a_{n-2}(t, y) & a_{n-3}(t, y) & \dots & a_2(t, y) & a_1(t, y) \end{pmatrix},$$

$$g(t, y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ f(t, y) \end{pmatrix},$$

$$C(t, y) = \begin{pmatrix} c_0^1(t, y) & c_1^1(t, y) & c_2^1(t, y) & c_3^1(t, y) & \dots & c_{n-2}^1(t, y) & c_{n-1}^1(t, y) \\ c_0^2(t, y) & c_1^2(t, y) & c_2^2(t, y) & c_3^2(t, y) & \dots & c_{n-2}^2(t, y) & c_{n-1}^2(t, y) \\ c_0^3(t, y) & c_1^3(t, y) & c_2^3(t, y) & c_3^3(t, y) & \dots & c_{n-2}^3(t, y) & c_{n-1}^3(t, y) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_0^n(t, y) & c_1^n(t, y) & c_2^n(t, y) & c_3^n(t, y) & \dots & c_{n-2}^n(t, y) & c_{n-1}^n(t, y) \end{pmatrix}, \text{ and are}$$

continuous on Ω ; the $n \times n$ matrices $B_s(y)$ and n - vector function $d(y)$ have the form

$$B_s(y) = \begin{pmatrix} b_{0,s}^1(y) & b_{1,s}^1(y) & b_{2,s}^1(y) & b_{3,s}^1(y) & \dots & b_{n-2,s}^1(y) & b_{n-1,s}^1(y) \\ b_{0,s}^2(y) & b_{1,s}^2(y) & b_{2,s}^2(y) & b_{3,s}^2(y) & \dots & b_{n-2,s}^2(y) & b_{n-1,s}^2(y) \\ b_{0,s}^3(y) & b_{1,s}^3(y) & b_{2,s}^3(y) & b_{3,s}^3(y) & \dots & b_{n-2,s}^3(y) & b_{n-1,s}^3(y) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{0,s}^n(y) & b_{1,s}^n(y) & b_{2,s}^n(y) & b_{3,s}^n(y) & \dots & b_{n-2,s}^n(y) & b_{n-1,s}^n(y) \end{pmatrix}, \quad d(y) = \begin{pmatrix} d_1(y) \\ d_2(y) \\ d_3(y) \\ \dots \\ d_{n-1}(y) \\ d_n(y) \end{pmatrix},$$

$$y \in [0, \omega], \quad s = 0, 1, 2, \dots, m.$$

Family multipoint-integral problems (2.1), (2.2) is played important role for investigating a nonlocal problems for partial differential equations of hyperbolic type [14-16].

For solving family multipoint-integral problems (2.1), (2.2) we apply Dzhumabaev's parametrization method [21].

Using the lines $t = t_j$, $j = 0, 1, \dots, m$, we make a partition Δ_m of domain Ω :

$$\Omega = \bigcup_{r=1}^m \Omega_r, \quad \Omega_r = [t_{r-1}, t_r] \times [0, \omega], \quad r = 1, 2, \dots, m, \quad t_0 = 0 < t_1 < t_2 < \dots < t_m = T.$$

By Δ_1 denote the case of no partitioning of the domain Ω .

Let $C(\Omega, \Delta_m, \mathbb{R}^{nm})$ be the space of systems functions

$x([t], y) = (x_{(1)}(t, y), x_{(2)}(t, y), \dots, x_{(m)}(t, y))$, where the functions $x_{(r)} : \Omega_r \rightarrow \mathbb{R}^n$ are continuous and have a finite left-hand side limits $\lim_{t \rightarrow t_r-0} x_{(r)}(t, y)$ uniformly with respect to $y \in [0, \omega]$ for all $r = 1, 2, \dots, m$ with norm

$$\|x([\cdot], y)\|_1 = \max_{r=\overline{1,m}} \sup_{t \in [t_{r-1}, t_r)} \|x_{(r)}(t, y)\|.$$

Let vector function $x(t, y)$ be the solution to family systems (2.1) and $x_{(r)}(t, y)$ is its restriction to sub-domain Ω_r , $r = \overline{1, m}$. Then the system functions

$x([t], y) = (x_{(1)}(t, y), x_{(2)}(t, y), \dots, x_{(m)}(t, y))$ belongs to $C(\Omega, \Delta_m, \mathbb{R}^{nm})$, and its elements $x_{(r)}(t, y)$, $r = 1, \dots, m$, satisfy to family systems of differential equations in the following form

$$\frac{\partial x_{(r)}(t, y)}{\partial t} = A(t, y)x_{(r)}(t, y) + g(t, y), \quad (t, y) \in \Omega_r, \quad r = \overline{1, m}. \quad (2.3)$$

Now, we introduce a functional parameters $\lambda_r(y) = x_{(r)}(t_{r-1}, y)$, $r = 1, \dots, m$.

Making a replacement $v_{(r)}(t, y) = x_{(r)}(t, y) - \lambda_r(y)$ on each sub-domain Ω_r , we obtain the following family systems of differential equations with functional parameters

$$\frac{\partial v_{(r)}(t, y)}{\partial t} = A(t, y)[v_{(r)}(t, y) + \lambda_r(y)] + g(t, y), \quad (t, y) \in \Omega_r, \quad r = \overline{1, m}, \quad (2.4)$$

and initial conditions

$$v_{(r)}(t_{r-1}, y) = 0_n, \quad r = \overline{1, m}, \quad (2.5)$$

0_n is an n -dimensional zero vector.

For any fixed parameter $\lambda_r(y) \in C([0, \omega], \mathbb{R}^n)$ and r , the family Cauchy problems (2.4), (2.5) has a unique solution $v_{(r)}(t, y, \lambda_r)$, and the system functions $v([t], y, \lambda) = (v_{(1)}(t, y, \lambda_1), v_{(2)}(t, y, \lambda_2), \dots, v_{(m)}(t, y, \lambda_m))$ belongs to $C(\Omega, \Delta_m, \mathbb{R}^{nm})$.

The system functions $v([t], y, \lambda)$ is called a solution to the family Cauchy problems with functional parameters (2.4), (2.5).

If a system functions $\tilde{x}([t], y) = (\tilde{x}_{(1)}(t, y), \tilde{x}_{(2)}(t, y), \dots, \tilde{x}_{(m)}(t, y))$ belongs to $C(\Omega, \Delta_m, \mathbb{R}^{nm})$, and the functions $\tilde{x}_{(r)}(t, y)$, $r = \overline{1, m}$, satisfy family systems of differential equations (2.3), then the system functions $v([t], y, \tilde{\lambda}) = (v_{(1)}(t, y, \tilde{\lambda}_1), v_{(2)}(t, y, \tilde{\lambda}_2), \dots, v_{(m)}(t, y, \tilde{\lambda}_m))$ with the elements $v_{(r)}(t, y, \tilde{\lambda}_r) = \tilde{x}_{(r)}(t, y) - \tilde{\lambda}_r(y)$, $\tilde{\lambda}_r(y) = \tilde{x}_{(r)}(t_{r-1}, y)$, $r = 1, \dots, m$, is a solution to the family Cauchy problems with functional parameters (2.4), (2.5) for $\lambda_r(y) = \tilde{\lambda}_r(y)$, $r = 1, \dots, m$. Conversely, if a system functions $v([t], y, \lambda^*) = (v_{(1)}(t, y, \lambda_1^*), v_{(2)}(t, y, \lambda_2^*), \dots, v_{(m)}(t, y, \lambda_m^*))$ is a solution to the family Cauchy problems (2.4), (2.5) for $\lambda_r(y) = \lambda_r^*(y)$, $r = 1, \dots, m$, then the system functions $x^*([t], y) = (x_{(1)}^*(t, y), x_{(2)}^*(t, y), \dots, x_{(m)}^*(t, y))$ with $x_{(r)}^*(t, y) = \lambda_r^*(y) + v_{(r)}(t, y, \lambda_r^*)$, $r = 1, \dots, m$, belongs to $C(\Omega, \Delta_m, \mathbb{R}^{nm})$, and the functions $x_{(r)}^*(t, y)$, $r = 1, \dots, m$, satisfy the family systems (2.3).

Further, we introduce a new definition of general solutions to the family systems of differential equations (2.1) and the original higher order differential equations (1.1).

Definition 2.1 Let $v([t], y, \lambda) = (v_{(1)}(t, y, \lambda_1), v_{(2)}(t, y, \lambda_2), \dots, v_{(m)}(t, y, \lambda_m))$ be the solution to the family Cauchy problems (2.4), (2.5) for a functional parameter $\lambda(y) = (\lambda_1(y), \lambda_2(y), \dots, \lambda_m(y)) \in C([0, \omega], \mathbb{R}^{nm})$. Then a function $u(t, y, \Delta_m, \lambda)$ defining by the equalities

$$x(t, y, \Delta_m, \lambda) = \lambda_r(y) + v_{(r)}(t, y, \lambda_r) \text{ for } (t, y) \in \Omega_r, r = 1, \dots, m, \text{ and}$$

$$x(T, y, \Delta_m, \lambda) = \lambda_m(y) + \lim_{t \rightarrow T-0} v_{(m)}(t, y, \lambda_m),$$

is called the Δ_m general solution to family systems of differential equations (2.1).

Definition 2.2 Let $v([t], y, \lambda) = (v_{(1)}(t, y, \lambda_1), v_{(2)}(t, y, \lambda_2), \dots, v_{(m)}(t, y, \lambda_m))$ be a solution to the family Cauchy problems (2.4), (2.5) for a functional parameter $\lambda(y) = (\lambda_1(y), \lambda_2(y), \dots, \lambda_m(y)) \in C([0, \omega], \mathbb{R}^{nm})$ with elements $v_{(r)}(t, y, \lambda_r) = (v_{1,(r)}(t, y, \lambda_r), v_{2,(r)}(t, y, \lambda_r), \dots, v_{n,(r)}(t, y, \lambda_r))$, and $\lambda_r(y) = (\lambda_{1,r}(y), \lambda_{2,r}(y), \dots, \lambda_{n,r}(y)) \in C([0, \omega], \mathbb{R}^n)$, respectively, for $r = 1, \dots, m$. Then the function $u(t, y, \Delta_m, \lambda)$ and its derivatives $\frac{\partial^i u(t, y, \Delta_m, \lambda)}{\partial t^i}$, $i = 1, \dots, n - 1$, determining by the equalities

$$u(t, y, \Delta_m, \lambda) = \lambda_{1,r}(y) + v_{1,(r)}(t, y, \lambda_r), \quad (t, y) \in \Omega_r, \quad r = 1, \dots, m,$$

$$\frac{\partial^i u(t, y, \Delta_m, \lambda)}{\partial t^i} = \lambda_{i+1,r}(y) + v_{i+1,(r)}(t, y, \lambda_r), \quad (t, y) \in \Omega_r, \quad r = 1, \dots, m, \quad i = 1, \dots, n-1,$$

$$u(t, y, \Delta_m, \lambda)|_{t=T} = \lambda_{1,m}(y) + \lim_{t \rightarrow T-0} v_{1,(m)}(t, y, \lambda_m), \quad y \in [0, \omega],$$

$$\frac{\partial^i u(t, y, \Delta_m, \lambda)}{\partial t^i} |_{t=T} = \lambda_{i+1,m}(y) + \lim_{t \rightarrow T-0} v_{i+1,(m)}(t, y, \lambda_m), \quad y \in [0, \omega], \quad i = 1, \dots, n-1,$$

is called the Δ_m general solution to the family higher order differential equations (1.1).

So, the Δ_m general solution depends on m arbitrary functional vectors $\lambda_r(y) \in C([0, \omega], \mathbb{R}^n)$ and satisfies the family systems (2.1) for all $t \in (0, T) \setminus \{t_p, p = \overline{1, N-1}\}$, $y \in [0, \omega]$. Analogously, the Δ_m general solution depends on m arbitrary vectors $\lambda_{1,r}(y) \in C([0, \omega], \mathbb{R})$ and satisfies the family higher order differential equations (1.1) for all $t \in (0, T) \setminus \{t_p, p = \overline{1, N-1}\}$, $y \in [0, \omega]$.

Let $\Phi_r(t, y)$ be a fundamental matrix of the family systems of differential equations

$$\frac{\partial \Phi_r(t, y)}{\partial t} = A(t, y)\Phi_r(t, y), \quad \Phi_r(t_{r-1}, y) = 0_{n \times n}, \quad (t, y) \in \Omega_r, \quad r = 1, \dots, m,$$

where $0_{n \times n}$ is n - dimensional square identity matrix.

Solution to the family Cauchy problems with parameters (2.4), (2.5) rewrite in the next form:

$$v_{(r)}(t, y, \lambda_r) = \Phi_r(t, y) \int_{t_{r-1}}^t \Phi_r^{-1}(\tau, y)A(\tau, y)d\tau \lambda_r(y) + \Phi_r(t, y) \int_{t_{r-1}}^t \Phi_r^{-1}(\tau, y)g(\tau, y)d\tau,$$

$$(t, y) \in \Omega_r, \quad r = 1, \dots, m.$$

Consider the families Cauchy problems on the subdomains

$$\frac{\partial z(t, y)}{\partial t} = A(t, y)z(t, y) + F(t, y), \quad u(t_{r-1}) = 0, \quad (t, y) \in \Omega_r, \quad r = 1, \dots, m. \tag{2.6}$$

where $F(t, y)$ is a square matrix or a vector of dimension n , continuous on Ω .

Denote by $A_r(t, y, F)$ a unique solution to the family Cauchy problems (2.6) on each r th domain. The uniqueness of the solution to the family Cauchy problems for linear system of differential equations give

$$D_r(t, y, F) = \Phi_r(t, y) \int_{t_{r-1}}^t \Phi_r^{-1}(\tau, y)F(\tau, y)d\tau, \quad (t, y) \in \Omega_r, \quad r = 1, \dots, m.$$

Then, we can write the Δ_m general solution to family systems (2.1) in the following form:

$$x(t, y, \Delta_m, \lambda) = \lambda_p(y) + D_p(t, y, A)\lambda_p(y) + D_p(t, y, g), \quad (t, y) \in \Omega_p, \quad r = 1, \dots, m-1, \tag{2.7}$$

$$x(t, y, \Delta_m, \lambda) = \lambda_m(y) + D_m(t, y, A)\lambda_m(y) + D_m(t, y, g), \quad (t, y) \in [t_{m-1}, t_m] \times [0, \omega]. \tag{2.8}$$

The following statements characterize the properties of the functions $x(t, y, \Delta_m, \lambda)$ and $u(t, y, \Delta_m, \lambda)$ as a general solution.

Theorem 2.3 Let a piecewise continuous on Ω function $\tilde{x}(t, y)$ with the possible

discontinuity lines $t = t_p$, $p = 1, \dots, m - 1$, be given, and $x(t, y, \Delta_m, \lambda)$ be the Δ_m general solution to the family systems (2.1). Suppose that the function $\tilde{x}(t, y)$ has a continuous partial derivative $\frac{\partial \tilde{x}(t, y)}{\partial t}$ and satisfies the family systems (2.1) for all $t \in (0, T) \setminus \{t_p, p = 1, \dots, m - 1\}$, $y \in [0, \omega]$. Then there exists a unique $\tilde{\lambda}(y) = (\tilde{\lambda}_1(y), \tilde{\lambda}_2(y), \dots, \tilde{\lambda}_m(y)) \in C([0, \omega], \mathbb{R}^{nm})$ such that the equality $x(t, y, \Delta_m, \lambda) = \tilde{x}(t, y)$ holds for all $(t, y) \in \Omega$.

Theorem 2.4 Let a piecewise continuous on Ω function $\tilde{u}(t, y)$ with the possible discontinuity lines $t = t_p$, $p = 1, \dots, m - 1$, be given, and $u(\Delta_m, t, y, \lambda)$ be the Δ_m general solution to the family higher order differential equations (1.1). Suppose that the function $\tilde{u}(t, y)$ has a continuous partial derivatives up to the n th order by t and satisfies the family higher order differential equations (1.1) for all $t \in (0, T) \setminus \{t_p, p = \overline{1, m-1}\}$, $y \in [0, \omega]$. Then there exists a unique functional parameter $\tilde{\lambda}(y) = (\tilde{\lambda}_1(y), \tilde{\lambda}_2(y), \dots, \tilde{\lambda}_m(y)) \in C([0, \omega], \mathbb{R}^{nm})$ with elements $\tilde{\lambda}_r(y) = (\tilde{\lambda}_{1,r}(y), \tilde{\lambda}_{2,r}(y), \dots, \tilde{\lambda}_{n,r}(y))$, $r = 1, \dots, m$, such that the equality $u(t, y, \Delta_m, \lambda) = \tilde{u}(t, y)$ holds for all $(t, y) \in \Omega$.

Corollary 2.5 Let $x^*(t, y)$ be a solution to family systems (2.1) and $u(t, y, \Delta_m, \lambda)$ be the Δ_m general solution to the family systems (2.1). Then there exists a unique functional parameter $\lambda^*(y) = (\lambda_1^*(y), \lambda_2^*(y), \dots, \lambda_m^*(y)) \in C([0, \omega], \mathbb{R}^{nm})$ such that the equality $u(t, y, \Delta_m, \lambda^*) = x^*(t, y)$ holds for all $(t, y) \in \Omega$.

Corollary 2.6 Let $u^*(t, y)$ be a solution to the family higher order differential equations (1.1) and $u(t, y, \Delta_m, \lambda)$ be the Δ_m general solution to the family higher order differential equations (1.1). Then there exists a unique functional parameter $\lambda^*(y) = (\lambda_1^*(y), \lambda_2^*(y), \dots, \lambda_m^*(y)) \in C([0, \omega], \mathbb{R}^{nm})$ with elements $\lambda_r^*(y) = (\lambda_{1,r}^*(y), \lambda_{2,r}^*(y), \dots, \lambda_{n,r}^*(y))$, $r = 1, \dots, m$, such that the equality $u(t, y, \Delta_m, \lambda^*) = u^*(t, y)$ holds for all $(t, y) \in \Omega$.

If $x(t, y)$ is a solution to the family systems (2.1), and $x([t], y) = (x_{(1)}(t, y), x_{(2)}(t, y), \dots, x_{(m)}(t, y))$ is a system functions composed of its restrictions to the subdomains Ω_r , $r = 1, \dots, m$, then the equations

$$\lim_{t \rightarrow t_p - 0} x_{(p)}(t, y) = x_{(p+1)}(t_p, y), \quad y \in [0, \omega], \quad p = 1, \dots, m - 1, \quad (2.9)$$

hold. These equations are the continuity conditions for the solution to the family systems (2.1) at the interior lines $t = t_p$, $p = 1, \dots, m - 1$, of the partition Δ_m .

Theorem 2.7 Let a system functions $x([t], y) = (x_{(1)}(t, y), x_{(2)}(t, y), \dots, x_{(m)}(t, y))$ belong to $C(\Omega, \Delta_m, \mathbb{R}^{nm})$. Assume that the functions $x_{(r)}(t, y)$, $r = 1, \dots, m$, satisfy the family systems (2.3) and continuity conditions (2.9). Then the function $x^*(t, y)$, given by the equalities

$x^*(t, y) = x_{(r)}(t, y)$ for $(t, y) \in \Omega_r$, $r = 1, \dots, m$, and $x^*(T, y) = \lim_{t \rightarrow T-0} x_{(m)}(t, y)$, is continuous on Ω , continuously differentiable by t on Ω , and satisfies the family system (2.1).

Proof: Equations (2.9), the equality $x^*(T, y) = \lim_{t \rightarrow T-0} x_{(m)}(t, y)$, and belonging of $x([t], y) = (x_{(1)}(t, y), x_{(2)}(t, y), \dots, x_{(m)}(t, y))$ to $C(\Omega, \Delta_m, \mathbb{R}^{nm})$ provide continuity of the function $x^*(t, y)$ on the domain Ω . Since the functions $x_{(r)}(t, y)$, $r = 1, \dots, m$, satisfy the family systems (2.3), the function $x^*(t, y)$ has continuous partial derivative by t and satisfies the family systems (2.1) for all $t \in [0, T] \setminus \{t_p, p = 1, \dots, m - 1\}$, $y \in [0, \omega]$. The existence and continuity of the partial derivative of the function $x^*(t, y)$ by t at the lines $t = t_p$, $p = 1, \dots, m - 1$, follow from the relations:

$$\lim_{t \rightarrow t_p-0} \frac{\partial x^*(t, y)}{\partial t} = A(t_p, y)x^*(t_p, y) + g(t_p, y) = \lim_{t \rightarrow t_p+0} \frac{\partial x^*(t, y)}{\partial t}, \quad p = 1, \dots, m - 1.$$

Hence the function $x^*(t, y)$ satisfies the family systems (2.1) at the interior lines $t = t_p$, $p = 1, \dots, m - 1$, of the partition Δ_m as well. Theorem 2.3 is proved.

From the Theorem 2.3 it follows that the function $u^*(t, y)$, given by the equalities $u^*(t, y) = x_{1,(r)}(t, y)$ for $(t, y) \in \Omega_r$, $r = 1, \dots, m$, and $u^*(T, y) = \lim_{t \rightarrow T-0} x_{1,(m)}(t, y)$, has a partial derivatives up to the $(n - 1)$ st order in t on Ω , defining by equalities

$$\frac{\partial^i u^*(t, y)}{\partial t^i} = x_{i+1,(r)}(t, y), \quad (t, y) \in \Omega_r, \quad r = 1, \dots, m,$$

and

$$\frac{\partial^i u^*(t, y)}{\partial t^i} \Big|_{t=T} = \lim_{t \rightarrow T-0} x_{i+1,(m)}(t, y), \quad y \in [0, \omega], \quad i = 1, \dots, n - 1,$$

has a partial derivatives of n th order in t on Ω , and satisfies the family higher order differential equations (1.1).

3 Algorithms for finding of solution to the family multipoint-integral problems for higher order differential equations (1.1), (1.2) and conditions for its unique solvability

The Δ_m general solution allows us to reduce the solvability of a family multipoint-integral problems to the solvability of a system of linear functional equations for arbitrary vectors $\lambda_r(y) \in C([0, \omega], \mathbb{R}^n)$, $r = 1, \dots, m$. Substituting the suitable expressions of Δ_m general solution (2.7), (2.8) into the multipoint-integral conditions (2.2) and continuity conditions (2.9), we have the system of linear functional equations

$$\sum_{s=0}^{m-1} B_s(y)\lambda_{s+1}(y) + B_m(y)[I + D_m(T, y, A)]\lambda_m(y) + \sum_{r=1}^m \int_{t_{r-1}}^{t_r} C(\tau, y)[I + D_r(\tau, y, A)]d\tau \lambda_r(y) =$$

$$= d(y) - B_m(y)D_m(T, y, g) - \sum_{r=1}^m \int_{t_{r-1}}^{t_r} C(\tau, y)D_r(\tau, y, g)d\tau, \quad (3.1)$$

$$\lambda_p(y) + D_p(t_p, y, A)\lambda_p(y) - \lambda_{p+1}(y) = -D_p(t_p, y, g), \quad y \in [0, \omega], \quad p = 1, \dots, m-1. \quad (3.2)$$

Denote by $Q_*(\Delta_m, y)$ the $nm \times nm$ matrix corresponding to the left-hand side of functional system (3.1), (3.2) and rewrite the system in the next form

$$Q_*(\Delta_m, y)\lambda(y) = -G_*(\Delta_m, y), \quad y \in [0, \omega], \quad \lambda(y) \in C([0, \omega], \mathbb{R}^{nm}), \quad (3.3)$$

where $G_*(\Delta_m, y) = \left(B_m(y)D_m(T, y, g) - d(y) \right.$

$$\left. + \sum_{r=1}^m \int_{t_{r-1}}^{t_r} C(\tau, y)D_r(\tau, y, g)d\tau, D_1(t_1, y, g), D_2(t_2, y, g), \dots, D_{m-1}(t_{m-1}, y, g) \right) \in C([0, \omega], \mathbb{R}^{nm}).$$

For any partition Δ_m , Theorems 2.1 and 2.3 provide the validity of the following statement.

Lemma 3.1 If $u^*(t)$ is a solution to problem with non-separated multipoint-integral condition (2.1), (2.2) and $\lambda_r^* = u^*(t_{r-1})$, $r = \overline{1, m}$, then the vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*) \in \mathbb{R}^{nm}$ is a solution to system (2.12). Conversely, if $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_m) \in \mathbb{R}^{nm}$ is a solution to system (2.12) and

$v[t, \tilde{\lambda}] = (v_{(1)}(t, \tilde{\lambda}_1), v_{(2)}(t, \tilde{\lambda}_2), \dots, v_{(m)}(t, \tilde{\lambda}_m))$ is the solution to the Cauchy problems (2.4), (2.5) for the parameter $\tilde{\lambda} \in \mathbb{R}^{nm}$, then the function $\tilde{u}(t)$ given by the equalities $\tilde{u}(t) = \tilde{\lambda}_r + v_{(r)}(t, \tilde{\lambda}_r)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, m}$, and $\tilde{u}(T) = \tilde{\lambda}_m + \lim_{t \rightarrow T-0} v_{(m)}(t, \tilde{\lambda}_m)$, is a solution to problem with non-separated multipoint-integral condition (2.1), (2.2).

Definition 3.2 The problem with non-separated multipoint-integral condition (2.1), (2.2) is called uniquely solvable if for any pair $(g(t), d)$, with $g(t) \in C([0, T], \mathbb{R}^n)$ and $d \in \mathbb{R}^n$, it has a unique solution.

Definition 3.3 The problem with non-separated multipoint-integral conditions for high-order differential equations (1.1), (1.2) is called uniquely solvable if for any $f(t) \in C([0, T], \mathbb{R})$ and $d_k \in \mathbb{R}$, $k = \overline{1, n}$, it has a unique solution.

Lemma 3.1 and well known theorems of linear algebra imply the following two assertions.

Theorem 3.4 The problem with non-separated multipoint-integral condition (2.1), (2.2) is solvable if and only if the vector $G_*(\Delta_m)$ is orthogonal to the kernel of the transposed matrix $(Q_*(\Delta_m))'$, i.e. iff the equality

$$(G_*(\Delta_m), \zeta) = 0$$

is valid for all $\zeta \in Ker(Q_*(\Delta_m))'$, where (\cdot, \cdot) is the inner product in \mathbb{R}^{nm} .

Theorem 3.5 The problem with non-separated multipoint-integral condition (2.1), (2.2) is uniquely solvable if and only if the $nm \times nm$ matrix $Q_*(\Delta_m)$ is invertible.

Theorem 3.6 The problem with non-separated multipoint-integral conditions for high-order differential equations (1.1), (1.2) is solvable if and only if the vector $G_*(\Delta_m)$ is orthogonal to the kernel of the transposed matrix $(Q_*(\Delta_m))'$, i.e. iff the equality

$$(G_*(\Delta_m), \zeta) = 0$$

is valid for all $\zeta \in Ker(Q_*(\Delta_m))'$, where (\cdot, \cdot) is the inner product in \mathbb{R}^{nm} .

Theorem 3.7 The problem with non-separated multipoint-integral conditions for high-order differential equations (1.1), (1.2) is uniquely solvable if and only if the $nm \times nm$ matrix $Q_*(\Delta_m)$ is invertible.

Based on these results, we propose the following algorithm for finding a solution to the problem with non-separated multipoint-integral condition (2.1), (2.2).

Step 1. Solve the Cauchy problems on the subintervals

$$\frac{du(t)}{dt} = A(t)u(t) + A(t), \quad u(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$
$$\frac{du(t)}{dt} = A(t)u(t) + g(t), \quad u(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

and find $D_r(A, t_r)$ and $D_r(g, t_r)$, $r = \overline{1, m}$.

Step 2. Using found matrices and vectors compose the system of linear algebraic equations (2.12).

Step 3. Solve the composed system and find $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*) \in \mathbb{R}^{nm}$. Note that the elements of λ^* are the values of the solution to problem (2.1), (2.2) at the left-end points of the subintervals: $\lambda_r^* = u^*(t_{r-1})$, $r = \overline{1, m}$.

Step 4. Solve the Cauchy problems

$$\frac{du(t)}{dt} = A(t)u(t) + g(t), \quad u(t_{r-1}) = \lambda_r^*, \quad t \in [t_{r-1}, t_r],$$

and define the values of the solution $u^*(t)$ at the remaining points of the subintervals.

Step 5. Using the founded values of the solution $u^*(t)$, we determine the values of the solution $x^*(t)$ to the problem with non-separated multipoint-integral conditions for high-order differential equations (1.1), (1.2).

As it follows from Lemma 3.1, any solution to system (2.12) determines the values of the solution to problem (2.1), (2.2) at the beginning points of the subintervals. The accuracy of the algorithm offered depends on the accuracy of computing the coefficients and right-hand sides of the system (2.12). The Cauchy problem for system of differential equations is the main auxiliary problem in the algorithm proposed. By choosing an approximate method for solving that problem, we get an approximate method for solving the problem (2.1), (2.2). Solving the Cauchy problems by numerical methods leads to the numerical methods for solving problem with non-separated multipoint-integral condition (2.1), (2.2). These methods allow us to find numerical solutions to problem with non-separated multipoint-integral conditions for high-order differential equations (1.1), (1.2).

4 Conclusion

Dzhumabaev's parametrization method is applied to solving problems with non-separated multipoint-integral conditions for high-order differential equations. A new general solution of high-order differential equations is constructed and its properties is clarified. Using a new general solution, criteria for the unique solvability are established for the problems with non-separated multipoint-integral conditions for high-order differential equations. The basic idea behind the proposed method is to construct and solve system of algebraic equations with respect to arbitrary parameters of new general solutions. The Cauchy problems for system of differential equations on the subintervals are the main tool of composing this system. Methods and results will be apply to a various problems for differential equations of higher order [1-11, 17-18, 22-31].

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