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# On the concept of solutions for fuzzy fractional initial value problem

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**Abstract**

It is evident that fuzzy arithmetic is different than that of normal arithmetic due to the different classes of functions and symbols. This article is concerned with an appropriate definition of the fractional derivative and integral in a fuzzy sense and a new concept of solutions for a fuzzy Caputo fractional initial value problem (IVP) is presented. Further, under some sufficient conditions on IVP, the existence, uniqueness, and stability results of the solution are established by applying the iterate methods. For the validation of established results, a particular fuzzy fractional Riccati differential equation is presented.

**Key words: Fractional derivatives and integrals, Initial value problems, Theory of fuzzy sets, Fuzzy ordinary differential equations**

**AMS classification: 26A33, 34A12, 03E72, 34A07**

## 1 Introduction

Fractional Calculus is one of the fastest-growing branches of mathematics. It has applications in many practical problems including problems in electrical and mechanical properties of materials, dynamics of turbulence, electro-chemistry, viscoelasticity, biology, and many more. Models involving fractional derivatives are capable of modifying our understanding of nature to an extent that was never achieved before. Bertram Ross [22] provided a brief history of fractional calculus. Books by A. A. Kilbas, H.M. Srivastava and J.J.Trujillo [10], Igor Podlubny [20]; K. Miller, B. Ross [17]; and K. B. Oldham, J. Spaneir [19] provide good literature to study

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fractional differential equations. Papers by V. Lakshmikantham and A. S. Vatsala [12, 13] and [8, 16] are also worth looking for further literature.

Fuzzy set theory was introduced by L. Zadeh in 1965. Fuzzy sets are used in modeling uncertainty, control theory, decision making, neural networks, and many other areas. Literature for fuzzy set theory can be found in the books by L. Zadeh [26] and H.J. Zimmermann [27], and that of fuzzy differential equations can be found in the books by V. Lakshmikantham et al. [11] and S. Chakraverty et al. [6]. For further readings the papers by O. Kaleva [9]; V. Lupulescu [15] and S. Seikkala [25] are referred in the references.

Only a decade ago, fractional calculus met with fuzzy set theory and a new branch of mathematics was born, named fuzzy fractional differential equations. Agarwal et al. [1] published a paper in which fuzzy fractional differential equations were introduced. This paper attracted researchers around the globe and rapid growth in this newborn topic was observed. Sadia Arshad, Lupulescu [4] and Salashour et al. [24] did some early works after the introduction of the topic. In 2018, Hao Van Ngo et al. [18] published a worth looking paper pointing out a mistake in previous works. A recent survey of the advancements in the theory and applications of the fuzzy fractional differential equations was done in 2018 by R. P. Agarwal et al. [2]. A more recent and broad survey can be found in the book by T. Allahviranloo [3]. Optimal control problems, diffusion equations are just to name a few areas of applications of fuzzy fractional differential equations. The topic can be greatly explored and has the potential to be used in many other practical problems.

Riccati differential equations are very well known and useful in many areas of science and pure mathematics [21]. Researchers have shown interest in this equation and have studied Fractional Riccati differential equations [23] and fuzzy Riccati differential equations [5]. Since Riccati differential equations are closely entangled with the Bessel functions, they arise in mathematical physics very often. The diverseness of their family makes it useful in areas of pure mathematics like calculus of variations and projective differential geometry of curves. Practically, they can be seen appearing in the areas like variational theory, optimal control, invariant embedding, dynamic programming, and many more. This variety of applicability motivated us to present an example of a particular fuzzy fractional Riccati differential equation.

Under fuzzy arithmetic and normal arithmetic, the class of functions are

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different, this idea encourages us to use different symbols for fuzzy Caputo fractional derivative operator and Riemann-Liouville fractional integral operator with predefined meaning. We found that this was missing in the aforesaid paper [18]. A small modification can be done in the paper by Hao Van Ngo et al. [18]. We noted that the theorem 3 in the paper [18] is stated on a continuous function, which may not be differentiable, is differentiated.

Motivated by the work initiated by Agarwal et al. [1], who laid the foundation of the theory and Hao Van Ngo et al. [18], who corrected a mistake by providing an appropriate condition on the non linear function $f$ to ensure the equivalence of fuzzy fractional integral and differential equations, the following fuzzy fractional IVP is taken into account:

$$\begin{gathered} {}_{t_0}^{C}D_t^{\alpha}x(t) = f(t, x(t)), t \in [t_0, T] \\ x(t_0) = x_0 \in \mathbb{F}_{\mathbb{R}} \end{gathered} \tag{1}$$

where, $\mathbb{F}_{\mathbb{R}}$ denotes the set of all fuzzy numbers with universe $\mathbb{R}$, $x(t)$ is the state of system (1) at time $t$, ${}_{t_0}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$ and $f : [t_0, T] \times B(x_0, \eta) \to \mathbb{F}_{\mathbb{R}}$ is a well defined given function. Where, $B(x_0, \eta)$ is a closed fuzzy ball with center $x_0$ and radius $\eta > 0$.

Further information about this work, has five sections. In section 2, we provided some necessary and useful notations, definitions, and lemmas along with some well-known results. Section 3 contains our main result in which we studied, under the assumptions of the theorem, the existence, uniqueness, and continuous dependence of the solution on some disturbed initial condition. Section 4 is devoted to an example, where we analyzed a fuzzy fractional Riccati differential equation. The conclusion is given in section 5 and useful references are listed at the end of the work.

## 2 Preliminary Supplements

In an attempt to make this paper self-sufficient we provide the following definitions and some well known results. For the general setting of fractional calculus and fuzzy arithmetic notations like Caputo's derivative ${}_{t_0}^{C}\mathcal{D}_t^{\alpha}$, Riemann-Liouville integral ${}_{t_0}\mathcal{J}_t^{-\alpha}$, Mittag Lefler function $E_{\alpha,\beta}$ addition $\oplus$, multiplication $\odot$, parametric form $r_a$, generalized Hukuhara difference $\ominus$ we refer [3, 7, 20]

**Definition 2.1** [3] The Hausdorff distance $D_H : \mathbb{F}_{\mathbb{R}} \times \mathbb{F}_{\mathbb{R}} \to \mathbb{R}$, between two fuzzy numbers $P$ and $Q$ is given by

$$D_H(P,Q) = \sup_{r\in[0,1]} \max\{|P_l(r) - Q_l(r)|, |P_u(r) - Q_u(r)|\}.$$

It is easy to see that $D_H$ is a metric in $\mathbb{F}_{\mathbb{R}}$ and here $P, Q, R, S \in \mathbb{F}_{\mathbb{R}}$. We have the following properties:

1. $D_H(P \ominus_g Q, R \ominus_g S) \le D_H(P,R) + D_H(Q,S)$.
2. $D_H(P \ominus_g Q, \mathbf{0}) = D_H(P,Q)$.

These properties are very useful as they relate the Hausdorff distance to the generalized Hukuhara difference.

**Definition 2.2** [3] The generalized Hukuhara derivative of a fuzzy number valued function $f : [0,T] \to \mathbb{F}_{\mathbb{R}}$ at $t_0 \in [0,T]$ is defined as

$$f'(t_0) = \lim_{h\to 0} \frac{f(t_0+h) \ominus_g f(t_0)}{h},$$

provided that the Hukuhara difference $f(t_0 + h) \ominus_g f(t_0)$ and the limit exists then the function $f$ is called $gH$-differentaible. The level-wise form of $gH$-differentaible function $f$ can be defined in following two cases:
***CASE I:*** $f'(t,r) = [f_l'(t,r), f_u'(t,r)]$, if $f$ is $i - gH$ differentiable at $t$.
***CASE II:*** $f'(t,r) = [f_u'(t,r), f_l'(t,r)]$, if $f$ is $ii - gH$ differentiable at $t$.

**Definition 2.3** [20] The Riemann-Liouville fractional integration with order $\alpha > 0$ of a function $f : [t_0, t] \to \mathbb{F}_{\mathbb{R}}$ such that $f \in L_{1,loc}([t_0,t], \mathbb{F}_{\mathbb{R}})$ in fuzzy sense is defined as

$${}_{t_0}\mathcal{J}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau) d\tau.$$

**Definition 2.4** [20] The Caputo fractional derivative with order $0 < \alpha < 1$ of a function $f : [t_0,t] \to \mathbb{F}_{\mathbb{R}}$ such that $f' \in L_{1,loc}([t_0,t], \mathbb{F}_{\mathbb{R}})$, and $gH$ differentiable in fuzzy sense is defined as

$${}_{t_0}^{C}\mathcal{D}_t^{\alpha} f(t) = {}_{t_0}\mathcal{J}_t^{-(1-\alpha)} \odot \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{-\alpha} \odot f'(\tau) d\tau,$$

where

$$f'(\tau) = \lim_{h\to 0} \frac{f(\tau+h) \ominus_g f(\tau)}{h},$$

is called $gH$-differentiable function of $f$ if limit exists.

We have the following remarks for the indorsee of above definition of fractional derivative and integral in fuzzy sense:

1. A function $f$ defined on any interval $I$ is absolutely continuous on $I$ if for every positive number $\epsilon$, there is a positive number $\delta$ such that whenever a finite sequence of point-wise disjoint subintervals $(x_k, y_k)$ of $I$ the function is continuous.
2. A function $f$ defined on a closed interval $[a, b]$ is absolutely continuous if and only if $f$ has a derivative almost everywhere, the derivative is Lebesgue integrable and

$$f(x) = f(a) + \int_a^x f'(t)dt, \forall x \in [a, b].$$

   A class of all absolutely continuous functions is denoted by $AC([a, b])$.
3. A function $f$ defined on a open interval $\Omega$ such that $\int_K |f|dx < +\infty$, that is, its Lebesgue integral is finite on all compact subsets $K$ of $\Omega$, then $f$ is called locally integrable. A class of all locally integrable functions on open interval $\Omega$ is denoted by $L_{1,loc}(\Omega)$.
4. A class of all absolutely continuous function $f : [a, b] \to \mathbb{R}$ such that its Lebesgue integral is finite on all compact subsets $K$ of $[a, b]$ is denoted by $f' \in L_{1,loc}([a, b], \mathbb{R})$.

**Lemma 2.1** Let $x(t)$ be a $i - gH$ differentiable or $ii - gH$ differentiable fuzzy number valued function. Then $x(t)$ is the solution of (1) if and only if, $x(t)$ is the solution of the following integral equation:

$$x(t) \ominus_g x_0 = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x(\tau))d\tau.$$

## 3 Main Results

This section contains existence, uniqueness, and stability results for the solution of a fuzzy Caputo fractional initial value problem with generalized Hukuhara derivative.

**Theorem 3.1** Let $f$ be a continuous function in IVP (1) such that there exists $M \geq 0$ satisfying,

$$D_H(f(t, x(t)), \boldsymbol{0}) \leq M, \ \forall (t, x(t)) \in [t_0, T] \times B(x_0, \eta), \tag{2}$$

and, for any $z, w \in C([t_0, T], B(x_0, \eta))$ and $L > 0$,

$$D_H(f(t, z(t)), f(t, w(t))) \leq L D_H(z(t), w(t)), \tag{3}$$

then the IVP (1) has a unique solution on $[t_0, T^*]$. Where, $T^* = min\{t^*, T\}$ and,

$$t_0 \leq t^* \leq t_0 + \left(\frac{\eta \Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}. \tag{4}$$

Proof: Let us consider the following sequence, we define the sequence with increasing length of $x_n(t) \ominus_g x_0$ as,

$$x_n(t) \ominus_g x_0 = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x_{n-1}(\tau)) d\tau, \ x_0(t) = x_0, \quad n = 1, 2, 3, \ldots \tag{5}$$

First, we'll prove that $\{x_n\}$ is well defined and $x_n \in C([t_0, T], B(x_0, \eta)), \forall n$.
We have,

$$D_H(x_n(t) \ominus_g x_0, \ \mathbf{0}) \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(f(\tau, x_{n-1}(\tau)), \ \mathbf{0}) d\tau,$$

using equation (2), we get,

$$\begin{aligned} &\leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} d\tau \\ &= \frac{M}{\Gamma(\alpha+1)} (t-t_0)^{\alpha} \end{aligned}$$

using equation (4), we get,

$$\leq \frac{M}{\Gamma(\alpha+1)} (T^* - t_0)^{\alpha} < \eta.$$

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This implies that $x_n(t) \in B(x_0, \eta)$. Which means that $\{x_n\}$ is well defined for all $n$. Now we'll show that $\{x_n\}$ is continuous on $[t_0, T^*]$ for all $n$. For this, let $t_1, t_2 \in [t_0, T^*]$ with $t_1 < t_2$ and consider,

$$\begin{aligned}
&D_H(x_n(t_1) \ominus_g x_0, x_n(t_2) \ominus_g x_0) = D_H(x_n(t_1), x_n(t_2))\\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} [(t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}] D_H(f(\tau, x_{n-1}(\tau)),\ \mathbf{0}) d\tau\\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} D_H(f(\tau, x_{n-1}(\tau)),\ \mathbf{0}) d\tau\\
&\leq \frac{M}{\Gamma(\alpha+1)} [(t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}] + \frac{M}{\Gamma(\alpha+1)} (t_2-t_1)^{\alpha}\\
\implies &\lim_{t_2 \to t_1} D_H(x_n(t_1), x_n(t_2)) \to 0.
\end{aligned}$$

That is, $x_n$ is continuous on $[t_0, T^*]$ and $x_n(t) \in B(x_0, \eta)$.

Now we'll show that $x_n$ satisfies the following inequality for all $n$,

$$D_H(x_n(t), x_{n-1}(t)) \leq \frac{ML^{n-1}(t-t_0)^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{6}$$

We'll use the Principle of mathematical induction to prove this. Clearly for $n = 1$, using equation (6), we have,

$$D_H(x_1(t), x_0(t)) \leq \frac{M(t-t_0)^{\alpha}}{\Gamma(1+\alpha)}, \tag{7}$$

for $n = 2$, consider,

$$\begin{aligned}
D_H(x_2(t), x_1(t)) &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(f(\tau, x_1(\tau)), f(\tau, x_0(\tau))) d\tau\\
&\leq \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(x_1(\tau), x_0) d\tau,
\end{aligned}$$

using equation (7), we get,

$$\begin{aligned}
&\leq \frac{L}{\Gamma(\alpha)}\frac{M}{\Gamma(1+\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}(\tau-t_0)^{\alpha}d\tau\\
&= \frac{ML}{\Gamma(\alpha)\Gamma(1+\alpha)}\times\frac{\Gamma(\alpha)\Gamma(1+\alpha)}{\Gamma(\alpha+1+\alpha)}(t-t_0)^{\alpha+\alpha}\\
&= \frac{ML(t-t_0)^{2\alpha}}{\Gamma(1+2\alpha)}.
\end{aligned}$$

Now, assume that the following inequality holds for $n-1$,

$$D_H(x_{n-1}(t), x_{n-2}(t)) \leq \frac{ML^{n-2}(t-t_0)^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}. \tag{8}$$

We now prove that equation (6) holds for $n$,

$$\begin{aligned}
D_H(x_n(t), x_{n-1}(t)) &\leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(f(\tau, x_{n-1}(\tau)), f(\tau, x_{n-2}(\tau)))d\tau\\
&\leq \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(x_{n-1}(\tau), x_{n-2}(\tau))d\tau,
\end{aligned}$$

using equation (8), we get,

$$\begin{aligned}
&\leq \frac{L}{\Gamma(\alpha)}\frac{ML^{n-2}}{\Gamma(1+(n-1)\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}(\tau-t_0)^{(n-1)\alpha}d\tau\\
&= \frac{ML^{n-1}}{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)}\times\frac{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)}{\Gamma(\alpha+1+(n-1)\alpha)}(t-t_0)^{(n-1)\alpha+\alpha}\\
&= \frac{ML^{n-1}(t-t_0)^{n\alpha}}{\Gamma(1+n\alpha)}.
\end{aligned}$$

Which proves the inequality (6). Now using this inequality (6) we'll show that $\{x_n\}$ is uniformly convergent. For this, observe that,

$$\frac{ML^{n-1}(t-t_0)^{n\alpha}}{\Gamma(1+n\alpha)} \leq \frac{ML^{n-1}(T^*-t_0)^{n\alpha}}{\Gamma(1+n\alpha)},$$

which is a sequence of numbers. We have corresponding series,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{ML^{n-1}(T^{*}-t_0)^{n\alpha}}{\Gamma(1+n\alpha)} &= \frac{M}{L}\sum_{n=1}^{\infty} \frac{[L(T^{*}-t_0)^{\alpha}]^{n}}{\Gamma(1+n\alpha)}\\ &= \frac{M}{L}E_{\alpha,1}(L(T^{*}-t_0)^{\alpha}-1).\end{aligned}$$

Which is just a number. This suggests that by Weirstrass' test $\{x_n\}$ is a uniformly convergent sequence of fuzzy functions. Let $x(t)$ be the limit function as the ball $B(x_0, \eta)$ is complete. Now, we can allow $n$ to go $\infty$ in (5) and get,

$$x(t) \ominus_g x_0 = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x(\tau))d\tau.$$

This implies that $x(t)$ is the solution of system (1) and hence proves the existence.

**Uniqueness:** For this, suppose there is another solution $\tilde{x}(t)$ satisfying system (1). Consider, $z(t) = x(t) \ominus_g \tilde{x}(t)$. Then clearly $z$ is continuous as $x$ and $\tilde{x}$ are continuous on $[t_0, T^{*}]$, which is compact. This implies that $z$ is bounded $\forall\ t \in [t_0, T^{*}]$. That is,

$$D_H(z(t), \mathbf{0}) \leq A, \text{where } A \text{ is a constant.} \tag{9}$$

Now, computing the following,

$$D_H(z(t), \mathbf{0}) = D_H(x(t) \ominus_g \tilde{x}(t), \mathbf{0})$$

Using properties 1 and 2 from definition 2.1,

$$
\begin{aligned}
&= D_H(x(t) \ominus_g x_0, \tilde{x}(t) \ominus_g x_0) \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(f(\tau, x(\tau)), f(\tau, \tilde{x}(\tau)))d\tau \\
&\leq \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(x(\tau), \tilde{x}(\tau))d\tau \\
&= \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(x(t) \ominus_g \tilde{x}(t),\ \mathbf{0})d\tau \\
&= \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(z(t),\ \mathbf{0})d\tau \\
&\leq \frac{LA}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} d\tau = \frac{LA}{\Gamma(\alpha)} \frac{(t-t_0)^{\alpha}}{\alpha} \leq \frac{LA(T^{*}-t_0)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Which is a new estimate. Applying this technique again with this estimate, we get,

$$
\begin{aligned}
D_H(z(t), \mathbf{0}) = D_H(x(t) \ominus_g \tilde{x}(t), \mathbf{0}) &= D_H(x(t) \ominus_g x_0, \tilde{x}(t) \ominus_g x_0) \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(f(\tau, x(\tau)), f(\tau, \tilde{x}(\tau)))d\tau \\
&\leq \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(x(\tau), \tilde{x}(\tau))d\tau \\
&= \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(x(t) \ominus_g \tilde{x}(t),\ \mathbf{0})d\tau \\
&= \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(z(t),\ \mathbf{0})d\tau \\
&\leq \frac{L^2 A}{\Gamma(\alpha)\Gamma(1+\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} (\tau - t_0)^{\alpha} d\tau \\
&= \frac{L^2 A}{\Gamma(\alpha)\Gamma(1+\alpha)} \frac{\Gamma(\alpha)\Gamma(1+\alpha)}{\Gamma(\alpha+1+\alpha)} (t-t_0)^{\alpha+\alpha} \\
&\leq \frac{L^2 A (T^{*}-t_0)^{2\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Using the same technique again $n$ times, we get the following,

$$
D_H(z(t), \mathbf{0}) \leq \frac{L^n A (T^{*}-t_0)^{n\alpha}}{\Gamma(n\alpha+1)}.
$$

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Which goes to 0 as $n$ goes to $\infty$. Which means that $x \equiv \tilde{x}$. Hence Uniqueness is proved. This completes the proof of Theorem.

**Theorem 3.2** Considering the same system (1) and the following IVP with disturbed initial condition,

$$ {}_{t_0}^{C}D_t^{\alpha}\tilde{x}(t) = f(t, \tilde{x}(t)), \qquad \tilde{x}(t_0) = x_0 \oplus \boldsymbol{\delta} \tag{10} $$

in the same environment as in the theorem 3.1. Where $\boldsymbol{\delta}$ is a fuzzy singleton, then the solutions are continuously dependent on the initial conditions, for $t \in [t_0, T^*]$.

**Proof** We have the following sequences from the same setting as in Theorem 3.1,

$$ x_n(t) = x_0 \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x_{n-1}(\tau))d\tau, \quad n = 1, 2, 3, \dots \tag{11} $$

and

$$ \tilde{x}_n(t) = (x_0 \oplus \boldsymbol{\delta}) \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, \tilde{x}_{n-1}(\tau))d\tau, \quad n = 1, 2, 3, \dots \tag{12} $$

As we know, $\tilde{x}_n(t) \ominus_g x_0$ has increasing length (i.e, It is $i - gH$ differentiable) then by definition we have,

$$ \tilde{x}_n(t) \ominus_g (x_0 \oplus \boldsymbol{\delta}) = \tilde{x}_n(t) \ominus_g x_0 \ominus_g \boldsymbol{\delta}. \tag{13} $$

Now consider,

$$ D_H(\tilde{x}_1(t), x_1(t)) = D_H\left((x_0 \oplus \boldsymbol{\delta}) \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, \tilde{x}_0(\tau))d\tau, x_0 \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x_0(\tau))d\tau\right). $$

Using equation (13) and properties (1) and (2) from definition 2.1

$$ \begin{aligned} &= D_H(\tilde{x}_1(t) \ominus_g \boldsymbol{\delta} \ominus_g x_0, x_1(t) \ominus_g x_0) \\ &= D_H(\tilde{x}_1(t) \ominus_g \boldsymbol{\delta}, x_1(t)) \\ &\leq D_H(\boldsymbol{\delta}, \mathbf{0}) + D_H(\tilde{x}_1(t), x_1(t)) \\ &\leq D_H(\boldsymbol{\delta}, \mathbf{0}) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(f(\tau, \tilde{x}_0(\tau)), f(\tau, x_0(\tau)))d\tau \end{aligned} $$

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using equation (3), we get,

$$\begin{aligned}
&\leq D_H(\boldsymbol{\delta},\mathbf{0}) + \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(x_0\oplus\boldsymbol{\delta},x_0)d\tau\\
&= D_H(\boldsymbol{\delta},\mathbf{0}) + \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(\boldsymbol{\delta},\mathbf{0})d\tau\\
&= D_H(\boldsymbol{\delta},\mathbf{0}) + D_H(\boldsymbol{\delta},\mathbf{0})\frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}d\tau\\
&= D_H(\boldsymbol{\delta},\mathbf{0})\left[1+\frac{L}{\Gamma(\alpha)}\frac{(t-t_0)^{\alpha}}{\alpha}\right] = D_H(\boldsymbol{\delta},\mathbf{0})\sum_{i=0}^{1}\frac{L^i(t-t_0)^{i\alpha}}{\Gamma(1+i\alpha)},
\end{aligned}$$

similarly,

$$\begin{aligned}
&D_H(\tilde{x}_2(t)\ominus_g(x_0\oplus\boldsymbol{\delta}),x_2(t)\ominus_g x_0) = D_H(\tilde{x}_2(t)\ominus_g x_0\ominus_g\boldsymbol{\delta},x_2(t)\ominus_g x_0)\\
&\quad= D_H(\tilde{x}_2(t)\ominus_g\boldsymbol{\delta},x_2(t))\\
&\quad\leq D_H(\boldsymbol{\delta},\mathbf{0}) + D_H(\tilde{x}_2(t),x_2(t))\\
&\quad\leq D_H(\boldsymbol{\delta},\mathbf{0}) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(f(\tau,\tilde{x}_1(\tau)),f(\tau,x_1(\tau)))d\tau\\
&\quad\leq D_H(\boldsymbol{\delta},\mathbf{0}) + \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(\tilde{x}_1(\tau),x_1(\tau))d\tau\\
&\quad\leq D_H(\boldsymbol{\delta},\mathbf{0}) + \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}D_H(\delta,\mathbf{0})\left[1+\frac{L(\tau-t_0)^{\alpha}}{\Gamma(1+\alpha)}\right]d\tau\\
&\quad= D_H(\boldsymbol{\delta},\mathbf{0})\left[1+\frac{L}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}d\tau+\frac{L^2}{\Gamma(\alpha)\Gamma(1+\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}(\tau-t_0)^{\alpha}d\tau\right]\\
&\quad= D_H(\boldsymbol{\delta},\mathbf{0})\left[1+\frac{L}{\Gamma(\alpha)}\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha)}+\frac{L^2}{\Gamma(\alpha)\Gamma(1+\alpha)}\frac{\Gamma(\alpha)\Gamma(1+\alpha)}{\Gamma(\alpha+1+\alpha)}(t-t_0)^{\alpha+\alpha}\right]\\
&\quad= D_H(\boldsymbol{\delta},\mathbf{0})\left[1+\frac{L(t-t_0)^{\alpha}}{\Gamma(1+\alpha)}+\frac{L^2}{\Gamma(1+2\alpha)}(t-t_0)^{2\alpha}\right]
\end{aligned}$$

$$= D_H(\boldsymbol{\delta},\mathbf{0})\sum_{i=0}^{2}\frac{L^i(t-t_0)^{i\alpha}}{\Gamma(1+i\alpha)}.$$

Hence, by induction, we have the following inequality,

$$D_H(\tilde{x}_n(t),x_n(t)) \leq D_H(\boldsymbol{\delta},\mathbf{0})\sum_{i=1}^{n}\frac{L^i(t-t_0)^{i\alpha}}{\Gamma(1+i\alpha)}.$$

Making $n \to \infty$, we get,

$$\begin{aligned} D_H(\tilde{x}(t), x(t)) &\leq D_H(\boldsymbol{\delta}, \mathbf{0}) E_{\alpha,1}(L(t-t_0)^\alpha - 1) \\ &\leq D_H(\boldsymbol{\delta}, \mathbf{0}) E_{\alpha,1}(L(T^*-t_0)^\alpha - 1). \end{aligned}$$

This implies that $D_H(\tilde{x}(t), x(t))$ is controlled by the disturbance $\boldsymbol{\delta}$, which can be made arbitrarily small. Hence, under the stated assumptions, the solution is continuously dependent on the initial conditions. This completes the proof of Theorem.

**Theorem 3.3** Let $f$ be a continuous function in system (1) such that there exists $M \geq 0$ satisfying,

$$D_H(f(t, x(t)), \boldsymbol{0}) \leq M, \ \forall (t, x(t)) \in [t_0, T] \times B(x_0, \eta), \tag{14}$$

then the system (1) has a solution on $[t_0, T^*]$. Where, $T^* = min\,\{t^*, T\}$ and,

$$t_0 \leq t^* \leq t_0 + \left(\frac{\eta \Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}. \tag{15}$$

**Proof** Let us consider the following family of sequence, we define the family with increasing length of $x_n(t) \ominus_g x_0$ as,

$$x_n(t) \ominus_g x_0 = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x_{n-1}(\tau)) d\tau, \ x_0(t) = x_0 \quad n = 1, 2, 3, ... \tag{16}$$

We have,

$$D_H(x_n(t) \ominus_g x_0, \ \mathbf{0}) \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} D_H(f(\tau, x_{n-1}(\tau)), \ \mathbf{0}) d\tau,$$

using equation (14), we get,

$$\begin{aligned} &\leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} d\tau \\ &= \frac{M}{\Gamma(\alpha+1)} (t-t_0)^\alpha, \end{aligned}$$

using equation (15), we get,

$$\leq \frac{M}{\Gamma(\alpha+1)}(T^* - t_0)^{\alpha} < \eta.$$

Now, observe that,

$$D_H(x_n(t),\ \mathbf{0}) = D_H(x_n(t) \oplus x_0, x_0) \leq D_H(x_n(t) \ominus_g x_0,\ \mathbf{0}) + D_H(x_0,\ \mathbf{0})$$

using above information, we get,

$$D_H(x_n(t) \ominus_g x_0,\ \mathbf{0}) + D_H(x_0,\ \mathbf{0}) \leq \eta + D_H(x_0,\ \mathbf{0}).$$

Which proves that the family (16) is uniformly bounded family of functions.
Now, let $t_1, t_2 \in [t_0, T^*]$ with $t_1 < t_2$ and consider,

$$\begin{aligned}
&D_H(x_n(t_1) \ominus_g x_0, x_n(t_2) \ominus_g x_0) = D_H(x_n(t_1), x_n(t_2)) \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} [(t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}] D_H(f(\tau, x_{n-1}(\tau)),\ \mathbf{0}) d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} D_H(f(\tau, x_{n-1}(\tau)),\ \mathbf{0}) d\tau \\
&\leq \frac{M}{\Gamma(\alpha+1)}[(t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}] + \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha},
\end{aligned}$$

having in mind that $t_1 < t_2$, above equation gives,

$$D_H(x_n(t_1), x_n(t_2)) \leq \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} < \epsilon \ \ \text{(let)},$$

which implies,

$$t_2 - t_1 < \frac{\epsilon^{\frac{1}{\alpha}}\Gamma(\alpha+1)}{M} = \delta. \ \ \text{(let)}$$

Which means that the family (16) is an equi-continuous family of functions. Thus, by Arzela-Ascoli theorem, there exists a subsequence of family (16) which is uniformly convergent to the limit function $x(t)$ given by,

$$x(t) \ominus_g x_0 = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x_{n-1}(\tau)) d\tau$$

which gives,

$$x(t) \ominus_g x_0 = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, x(\tau)) d\tau.$$

This implies that $x(t)$ is a solution of the system (1) and hence, existence is proved. This completes the proof of the Theorem.

## 4 Example

We are presenting an analysis of a particular fuzzy fractional Riccati differential equation. Along with the applications stated in section 1 some of its specific applications are in the real world models which can be converted into Riccati differential equations, which include models for linear regulator problem, linear filtering, invariant embedding, the Mycielski-Paszkowski diffusion problem and more. A problem of multiple non-uniform transmission lines in steady state to determine when the transmission is lossless also gets converted into a Riccati differential equation. So we are considering an example which satisfies the conditions of our Theorem and works as an illustration of the Theorem. For $t \in [0, T]$, we have:

$$ {}_{0}^{C}D_t^{\alpha} x(t) = \mathbf{2} \odot x(t) \ominus_g x^2(t) \oplus \mathbf{1}, \qquad x(0) = [1 \ \ 2 \ \ 3] \in \mathbb{F}_{\mathbb{R}}, \tag{17}$$

here, $\mathbf{1}$ and $\mathbf{2}$ are singleton fuzzy numbers. Since

$$f(t, x(t)) = \mathbf{2} \odot x(t) \ominus_g x^2(t) \oplus \mathbf{1}, \quad x(t) \in B([1 \ \ 2 \ \ 3], \eta)$$

is a polynomial in $x$, it is continuous. We wish to compute the bound $M$ for $f$ the number $T^*$ such that the solution exists in the interval $[0, T^*]$ and the Lipschitz constant $L$. Consider the following:

$$\begin{aligned} D_H(f(t, x(t)), \mathbf{0}) &= D_H(\mathbf{2} \odot x(t) \ominus_g x^2(t) \oplus \mathbf{1}, \mathbf{0}) \\ &\leq D_H(\mathbf{2} \odot x(t), \mathbf{0}) + D_H(x^2(t), \mathbf{0}) + D_H(\mathbf{1}, \mathbf{0}). \end{aligned} \tag{18}$$

Since $x(t)$ is a fuzzy number, it can be represented in parametric form as follows:

$$r_{x(t)} = [p(r), q(r)], \qquad 0 \leq r \leq 1,$$

and the fuzzy number $[1\ \ 2\ \ 3]$ in parametric form is given by,

$$r_{[1\ \ 2\ \ 3]} = [r+1, 3-r].$$

Now, by definition of Hausdorff distance, we have:

$$D_H(x(t), [1\ \ 2\ \ 3]) = \sup_{r\in[0,1]} \max\{|p(r)-r-1|, |q(r)+r-3|\} \leq \eta.$$

Now, either

$$\begin{aligned}
&\sup_{r\in[0,1]} |p(r)-r-1| \leq \eta \\
&\implies |p(r)-r-1| \leq \eta \\
&\implies -\eta \leq p(r)-r-1 \leq \eta \\
&\implies -\eta+r+1 \leq p(r) \leq r+1+\eta
\end{aligned}$$

or,

$$\begin{aligned}
&\sup_{r\in[0,1]} |q(r)+r-3| \leq \eta \\
&\implies |q(r)+r-3| \leq \eta \\
&\implies -\eta \leq q(r)+r-3 \leq \eta \\
&\implies -\eta-r+3 \leq q(r) \leq -r+3+\eta.
\end{aligned}$$

Conclusively, we get,

$$-\eta+r+1 \leq p(r) \leq q(r) \leq -r+3+\eta.$$

Now, consider a trapezoidal fuzzy number $B$ with parametric form,

$$r_B = [-\eta+r+1, -r+3+\eta] \text{ i.e, } B = [1-\eta\ \ 2-\eta\ \ 2+\eta\ \ 3+\eta]. \tag{19}$$

An alternate way of finding the bound $B$ is presented below:
Consider the interval, $N = [-\eta, \eta]$ as a fuzzy number with membership function, $\mu_N = \begin{cases} 1 \text{ if } y \in [-\eta,\eta] \\ 0 \text{ otherwise} \end{cases}$, which has the parametric form $r_N = [-\eta, \eta]$.
Now, the parametric form of bound $B$ is given by $r_B = r_N + r_{[1\ \ 2\ \ 3]}$, which turns

out to be,

$$r_B = [-\eta + r + 1, -r + 3 + \eta] \text{ i.e, } B = [1 - \eta \;\; 2 - \eta \;\; 2 + \eta \;\; 3 + \eta].$$

Which, evidently, is the same as computed in (19).

This $B$ acts as an envelop (bound) for all $x(t)$ and can be considered as the largest (in a vague sense) $x(t)$ there can be, for any $t$. Using $B$, the bounds for $\mathbf{2} \odot x(t)$ and $x^2(t)$ can easily be calculated as follows:

$$r_{2B} = r_B + r_B = [-2\eta + 2r + 2, -2r + 6 + 2\eta],$$

and

$$\begin{aligned} r_{B^2} = r_B \times r_B =& [\min\{(-\eta + r + 1)^2, (-\eta + r + 1)(-r + 3 + \eta), \\ & (-r + 3 + \eta)^2\}, (-r + 3 + \eta)^2]. \end{aligned}$$

Now, let us calculate the size of $2B$ and $B^2$.

$$D_H(2B, \mathbf{0}) = \sup_{r\in[0,1]} \max\{|-2\eta + 2r + 2|, |-2r + 6 + 2\eta|\} = 2\eta + 6, \tag{20}$$

$$\begin{aligned} D_H(B^2, \mathbf{0}) = & \sup_{r\in[0,1]} \max\{|\min\{(-\eta + r + 1)^2, (-\eta + r + 1)(-r + 3 + \eta), \\ & (-r + 3 + \eta)^2\}|, |(-r + 3 + \eta)^2|\}, \\ = & \sup_{r\in[0,1]} (-r + 3 + \eta)^2 = (\eta + 3)^2. \end{aligned} \tag{21}$$

Now, let us calculate $D_H(\mathbf{1}, \mathbf{0})$. By definition, we have, $r_{\mathbf{1}} = [1, 1]$ and $r_{\mathbf{0}} = [0, 0]$. This implies,

$$D_H(\mathbf{1}, \mathbf{0}) = \sup_{r\in[0,1]} \max\{|1 - 0|, |1 - 0|\} = 1. \tag{22}$$

Using equations (20), (21) and (22) in equation (18); we get,

$$D_H(f(t, x(t)), \mathbf{0}) \leq 2\eta + 6 + (\eta + 3)^2 + 1 = (\eta + 4)^2.$$

This suggests that $f$ is bounded by $M$ where, $M = (\eta + 4)^2$.
Now, we'll compute $T^*$ using the value of $M$ in (15), which tells us the interval on

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which the solution exists. We have,

$$T^* = \left( \frac{\eta\Gamma(\alpha+1)}{(\eta+4)^2} \right)^{\frac{1}{\alpha}}.$$

Now, let us compute a lower bound for the Lipschitz constant $L$. We have the following,

$$L = \sup_{t\in[0,T^*]} D_H \left( \frac{\partial}{\partial x} f(t,x(t)), \mathbf{0} \right) = \sup_{t\in[0,T^*]} D_H \left( \frac{d}{dx} f(t,x(t)), \mathbf{0} \right), \tag{23}$$

where, derivatives are in Hukuhara sense. We get the following,

$$\begin{aligned} \frac{d}{dx} f(t,x(t)) &= \frac{d}{dx}(\mathbf{2}\odot x(t)) \ominus_g \frac{d}{dx}(x^2(t)) \oplus \frac{d}{dx}(\mathbf{1}) \\ &= \mathbf{2} \ominus_g (\mathbf{2}\odot x(t)) \oplus \mathbf{0} = \mathbf{2} \ominus_g \mathbf{2}\odot x(t). \end{aligned} \tag{24}$$

Using equation (24) in equation (23), we get,

$$\begin{aligned} \sup_{t\in[0,T^*]} D_H \left( \frac{d}{dx} f(t,x(t)), \mathbf{0} \right) &= \sup_{t\in[0,T^*]} D_H \left( \mathbf{2} \ominus_g \mathbf{2}\odot x(t), \mathbf{0} \right) \\ &\leq \sup_{t\in[0,T^*]} \left[ D_H(\mathbf{2},\mathbf{0}) + D_H(\mathbf{2}\odot x(t), \mathbf{0}) \right] \leq 2 + 2\eta + 6, \end{aligned}$$

that is, we can set $L = 2(4+\eta)$ and existence of this bound suggests that $f$ is Lipschitz continuous. This shows that the problem (17) considered in our example is consistent with the established theory.

## 5 Conclusion

In our work, we have provided the new concept and notation of fractional derivative and integral in fuzzy sense and then considered a general fuzzy fractional initial value problem with an initial condition as a fuzzy number. We discussed the existence, uniqueness, and stability of the solution of the IVP. The method of the proof uses Picard's type iterates. In the end, an analysis of the fuzzy fractional Riccati differential equation is presented as a practical example under the stated assumptions because Riccati differential equations appear in many areas of science and technology.

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