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Forced oscillation of solutions of conformable hybrid elliptic partial differential equations

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Abstract

In this paper, we investigate the forced oscillation of solutions of conformable hybrid elliptic partial differential equations. We show that, the suitable condition for the infinite sequence of annular domains which gives every solution has a zero. Some examples are given to illustrate the effectiveness of our main result.

Key words: Oscillation, Conformable Elliptic differential equations, Hybrid differential equations.

AMS classification: 34A08, 34A34, 34K11, 35B05, 35R20.

1 Introduction

The problem of oscillation and nonoscillation solution of partial differential equations is well documented. For the fundamental theory and preliminary result one can refer the books and articles, see[2, 3, 4, 8, 9, 11, 13, 14, 17, 21, 26, 29, 31, 32, 33, 34, 35].

Recently, the study on conformable fractional derivative has been gaining more attention and it gives importance in numerous practical applications, see for example [1, 5, 10, 12, 18, 19, 20, 24, 28]

The past few years, the quadratic perturbation of nonlinear differential equations namely, hybrid differential equations have been extensively studied. The hybrid differential equation is an especially interesting type of nonlinear differential equations which has wider scope to research. The reason is that hybrid differential equations include several dynamic systems, as special cases. We refer the readers to the articles in [6, 15, 30, 37]. Applications with numerical solutions have been studied by several authors, see the example, [16, 25, 36].

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Dhage and Lakshmikantham [7] analyzed the existence of extremal solutions and a comparison result for first order hybrid differential equations with linear perturbations of the form

$$\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), a.e. t \in J, x(t_0) = x_0 \in \mathbb{R}$$

where $f \in$ C(J $\times$ $\mathbb{R}$,$\mathbb{R} - \{0\}$) and $g \in$ C(J $\times$ $\mathbb{R}$,$\mathbb{R}$).
On the other hand, Chatzarakis, Deepa, Nagajothi and Sadhasivam[6] studied, the second order hybrid differential equations and generalized the Riccati technique is of the form

$$\left(\frac{x(t)}{f(t,x(t))}\right)'' + q(t)x(t) = g(t,x(t)), \quad t \le t_0$$

By the motivation of the paper[27], discussed the fractional - order nonlinear impulsive hybrid partial differential equations of the form

$$\frac{\partial}{\partial t}\left(r(t)D^{\alpha}_{+,t}\left(\frac{u(x,t)}{h(t,u(x,t))}\right)\right) + q(x,t)g(D^{\alpha}_{+,t}u(x,t))$$
$$= a(t)\Delta u(x,t) - f(t, \int_0^t (t-s)^{-\alpha}u(x,s)ds) + F(x,t)$$

However, the best of our knowledge, we understand that there has been no previous research made on the forced oscillation of nonlinear conformable hybrid elliptic differential equations. Hence, we are the first forced oscillation of nonlinear conformable hybrid elliptic differential equations which had not been formerly studied. we propose the following model of the form

$$\Delta^{\alpha}_{\underline{x}}\left(\frac{u(x)}{g(u(x))}\right) + \varphi(x,u(x)) = f(x), \quad x \in E, \tag{1}$$

where $\alpha \in (0,1)$, $\Delta^{\alpha}_{\underline{x}}$ is the conformable nabla operator in Euclidean n-space $\mathbb{R}^n$ $(n \ge 2)$ and E is an exterior domain in $\mathbb{R}^n$. $g(u(x)) \in C(E, \mathbb{R} - \{0\})$ is convex function in R_+.
We assume throughout this paper:
(A_1) $\varphi(x,u(x))$ is a real-valued continuous function in E$\times$ $\mathbb{R}^1$.
(A_2) $\varphi(x,(ux)) \ge$ p(x)$u(x)$ for all $(x,u(x)) \in$ E$\times[0,\infty)$ and
$\varphi(x,u(x)) \le$ p(x)(x) for all $(x,u(x)) \in$ E$\times(-\infty,0)$.
(A_3) $f(x)$ is a real-valued continuous function in E.

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A bounded domain G $\in E$ is said to be a nodal domain for (1) if there exists a nontrivial function u $\in$ Domain (G) $\equiv C^2(G;\mathbb{R}) \cap C(\overline{G};\mathbb{R})$ such that it has a solution of zero in G and u = 0 on ∂G.
Equation (1) is said to be nodally oscillatory in E if for any $r > 0$ equation (1) has a nodal domain contained in E_r. E $\subset \{x \in \mathbb{R}^n/|x| > r_0\}$ for some $r_0 > 0$, where $|x|$ denote the Euclidean norm of $x \in \mathbb{R}^n$.
The following notation will be used

$$S_r = \{x \in \mathbb{R}^n/|x| = r\}, \quad r > r_0$$
$$F(r) = \frac{1}{\sigma_n r^{n-1}} \int_{S_r} f(x) d\sigma, \quad r > r_0$$
$$U(r) = \frac{1}{\omega_n r^{n-1}} \int_{S_r} u(x) dS, \quad r > r_0.$$

where σ_n denotes the surface area of the unit sphere S_1.
This paper is organized as follows:
In Section 2, we present the relevant definitions, lemmas, corollary and results. In Section 3, we discuss the oscillation theorems of main result. In Section 4, we given few example is illustrate the effectiveness of our new result.
The results in this paper extend and improve numerous findings in the earlier publications. we believe that this research work would enable further researchers on the conformable elliptic partial differential equations.

2 Preliminaries

In this section, we give some fundamental definition of conformable derivatives and integrals. There are several basic definitions, lemmas, corollary and result which are useful throughout this paper

Definition 2.1 [19] Given $f : [0, \infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

For every $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t \to o^+} f^\alpha(t)$ exists, then we define

$$f^\alpha(0) = \lim_{t \to o^+} f^\alpha(t)$$

Definition 2.2 [19]

$$I_{\alpha}^{a}(f)(t) = I_{1}^{\alpha}(t^{\alpha-1})(f) = \int_{a}^{t} \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1)$.

Proposition 2.3 [19] Let $\alpha \in (0,1]$ and f, g be α-differentiable at some point $t > 0$. Then,

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g), \quad \forall a, b \in \mathbb{R}$
2. $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$
3. $T_{\alpha}(t^{p}) = pt^{p-\alpha}, \quad \forall p \in \mathbb{R}$
4. $T_{\alpha}(C) = 0,$ For all constant functions f(t)=C
5. $T_{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^{2}}$
6. If f is differential, then $T_{\alpha}(f(t)) = t^{1-\alpha}\frac{df(t)}{dt}$

Definition 2.4 [5] Let f be a function with m variable $x_1, ..., x_m$, and the conformable partial derivative of f of order $0 < \alpha \leq 1$ in x_i is defined as follows

$$\frac{\partial^{\alpha}}{\partial x_i^{\alpha}} f(x_1, ..., x_m) = \lim_{\epsilon \to 0} \frac{f(x_1, ..x_{i-1}, x_i + \epsilon x_i^{1-\alpha} ...., x_m) - f(x_1, ..., x_m)}{\epsilon}$$

Definition 2.5 [5] Consider the scalar field f(x) and the vector field F(x) that are assumed to possess partial conformable derivative of order α with respect to all the Cartesian coordinates $x_i, i = 1, 2, 3$. We define the conformable gradient order of α as begin the vector field

$$\Delta_{\underline{x}}^{\alpha} f = \Sigma_{i=1}^{3} (\partial_{x_i}^{\alpha} f) e_i$$

with, of course, e_i is the unit vector in the "i" direction.

The conformable of order α as the scalar field is defined as

$$\Delta_{\underline{x}}^{\alpha} f = \Sigma_{i=1}^{3} (\partial_{x_i}^{\alpha} F_i)$$

Definition 2.6 [5]The conformable divergence theorem
Let the vector field $\underline{v}$ have a continuous derivative on an open region space D containing then volume V and surface S of V positively outward orientated

$$\int\int\int \nabla.\underline{F} d_\alpha V = \int\int \underline{F}.\underline{n} d_\alpha S$$

Where, $d_\alpha V = d_\alpha x d_\alpha y d_\alpha z$ and $d_\alpha S = d_\alpha x d_\alpha y$

Lemma 2.7 [23, 27] Let u(x), g(u(x)), h(x) $\geq$ 0 be continuous functions on [a,b] such that

$$(u(x) - u(y)) \left(\frac{h(y)}{g(u(y))} - \frac{h(x)}{g(u(x))} \right) \geq 0 \tag{2}$$

Then the inequality

$$\frac{\int_a^b h(x)dx}{\int_a^b g(u(x))dx} \geq \frac{\int_a^b h(x)u(x)dx}{\int_a^b g(u(x))u(x)dx} \tag{3}$$

holds. If (2) reverses, then (3) reverses.

Corollary 2.8 In particular, replacing h(x), h(y)=1 and g(u(x)) by $\frac{1}{g(u(x))}$ in the above lemma under the condition

$$(u(x) - u(y)) \left(g(u(y)) - g(u(x)) \right) \geq 0$$

then

$$\int_\Omega \frac{u(x)}{g(u(x))} dx \geq \int_\Omega u(x)dx \int_\Omega \frac{1}{g(u(x))} dx$$

holds. If

$$(u(x) - u(y)) \left(g(u(y)) - g(u(x)) \right) \leq 0$$

then

$$\int_\Omega \frac{u(x)}{g(u(x))} dx \leq \int_\Omega u(x)dx \int_\Omega \frac{1}{g(u(x))} dx$$

Remark 2.9 [27] It is well known that if f(x) is convex and 2f - $f^2 \geq 0$ then $\frac{1}{f(x)}$ is convex.

Lemma 2.10 If $u \in C^{2\alpha}(A(r_0, \infty), \mathbb{R})$ and $g(u(x)) \in C(E \times [0, \infty)/[0, \infty])$ is convex function in R_+ then

$$\frac{1}{\omega_n r^{n-1}}\int_{S_r}\Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S\leq r^{1-n}\left(T_\alpha\left(r^{(\alpha+n-2)}\left(T_\alpha\left(\frac{U(r)}{g(U(r))}\right)\right)\right)\right)$$

Proof: Let u(x) = $\tilde{u}$(r,θ) at x = (r,θ)
The function

$$U(r)=\frac{1}{\omega_n}\int_{S_1}\tilde{u}(r)d\omega,$$

The divergence theorem implies

$$\begin{aligned}\int_{A(a,r)}\Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha x&=\int_{A(a,r)}div_\alpha\left(grad_\alpha\left(\frac{u(x)}{g(u(x))}\right)\right)d_\alpha x\\&=\int_{\partial A(a,r)}grad_\alpha\left(\frac{u(x)}{g(u(x))}\right).\widehat{n}d_\alpha S\\&=\int_{\partial A(a,r)}\Sigma_{i=1}^{n}\frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u(x)}{g(u(x))}\right)\widehat{n_i}d_\alpha S\\&\leq\int_{\partial A(a,r)}\Sigma_{i=1}^{n}\frac{\partial^\alpha u(x)}{\partial x_i^\alpha}\left(\frac{1}{g(u(x))}\right)\widehat{n_i}d_\alpha S\\&\leq\int_{\partial A(a,r)}\frac{\partial^\alpha u(x)}{\partial n^\alpha}\left(\frac{1}{g(u(x))}\right)d_\alpha S\end{aligned}$$

$$\int_a^r\int_{S_r}\Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S d_\alpha r\leq\int_{S_r}\frac{\partial^\alpha u(x)}{\partial n^\alpha}\left(\frac{1}{g(u(x))}\right)d_\alpha S-\int_{S_a}\frac{\partial^\alpha u(x)}{\partial n^\alpha}\left(\frac{1}{g(u(x))}\right)d_\alpha S$$

Taking α derivative with respect to r

$$\int_{S_r}\Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S\leq T_\alpha\left(\int_{S_r}\frac{\partial^\alpha u(x)}{\partial n^\alpha}\left(\frac{1}{g(u(x))}\right)d_\alpha S\right)$$

$$\int_{S_1}\Delta_{\underline{x}}^{\alpha}\left(\frac{\tilde{u}(r,\theta)}{g(\tilde{u}(r,\theta))}\right)r^{n-1}d_\alpha\omega\leq T_\alpha\left(\int_{S_r}\frac{\partial^\alpha u(x)}{\partial n^\alpha}\left(\frac{1}{g(u(x))}\right)d_\alpha S\right)$$

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Since,

$$\frac{\partial^\alpha u(x)}{\partial n^\alpha} = \frac{\partial^\alpha \tilde{u}(r)}{\partial r^\alpha} \quad on \quad S_r$$

$$\begin{aligned}
\int_{S_r} \Delta^\alpha_{\underline{x}} \left(\frac{u(x)}{g(u(x))} \right) d_\alpha S &= \int_{S_1} \Delta^\alpha_{\underline{x}} \left(\frac{\tilde{u}(r,\theta)}{g(\tilde{u}(r,\theta))} \right) r^{n-1} d_\alpha \omega \\
&\le T_\alpha \left(\int_{S_r} \frac{\partial^\alpha u(x)}{\partial n^\alpha} \left(\frac{1}{g(u(x))} \right) d_\alpha S \right) \\
&\le T_\alpha \left(\int_{S_1} \frac{\partial^\alpha \tilde{u}(r)}{\partial r^\alpha} \left(\frac{1}{g(\tilde{u}(r))} \right) r^{n-1} d_\alpha \omega \right) \\
&\le T_\alpha \left(r^{(\alpha+n-2)} \int_{S_1} \frac{\partial^\alpha \tilde{u}(r)}{\partial r^\alpha} \left(\frac{1}{g(\tilde{u}(r))} \right) d\omega \right) \\
&\le \omega_n T_\alpha \left(r^{(\alpha+n-2)} T_\alpha \left(\frac{1}{\omega_n} \int_{S_1} \left(\frac{\tilde{u}(r)}{g(\tilde{u}(r))} \right) d\omega \right) \right)
\end{aligned}$$

Using the Corollary (2.7), we get

$$\int_{S_r} \Delta^\alpha_{\underline{x}} \left(\frac{u(x)}{g(u(x))} \right) d_\alpha S \le \omega_n T_\alpha \left(r^{(\alpha+n-2)} \left(T_\alpha \left(\frac{U(r)}{g(U(r))} \right) \right) \right)$$

Divided by $\omega_n r^{n-1}$, then

$$\frac{1}{\omega_n r^{n-1}} \int_{S_r} \Delta^\alpha_{\underline{x}} \left(\frac{u(x)}{g(u(x))} \right) d_\alpha S \le r^{1-n} \left(T_\alpha \left(r^{(\alpha+n-2)} \left(T_\alpha \left(\frac{U(r)}{g(U(r))} \right) \right) \right) \right)$$

Hence the lemma is proved.

Lemma 2.11 (see Lebedev[22], p.127]) The function $J_n(z)$ (n = 0,1,2..) has no complex zeros, and has an infinite number of real zeros symmetrically located with respect to the point z = 0, which is itself a zero if $n > 0$. All the zeros of $J_n(z)$ are simple, expect the point z = 0, which is a zero of order n if $n > o$.

The following several properties of α - conformable Bessel functions are given

Proposition 2.12 [12] Let n is a nonnegative integer. Then

1. $T_\alpha(x^{n\alpha}(J_\alpha)_n(x)) = \alpha x^{n\alpha}(J_\alpha)_{n-1}(x)$
2. $T_\alpha(x^{-n\alpha}(J_\alpha)_n(x)) = -\alpha x^{n\alpha}(J_\alpha)_{n+1}(x)$
3. $T_\alpha((J_\alpha)_n(x)) = \alpha(J_\alpha)_{n-1}(x) - \frac{\alpha n}{x^\alpha}(J_\alpha)_{n-1}(x)$
4. $T_\alpha((J_\alpha)_n(x)) = \frac{\alpha n}{x^\alpha}(J_\alpha)_n(x) - \alpha(J_\alpha)_{n+1}(x)$
5. $(J_\alpha)_{n+1}(x) = \frac{2n}{x^\alpha}(J_\alpha)_n(x) - \alpha(J_\alpha)_{n-1}(x)$
6. $(J_\alpha)_{-n}(x) = (-1^n)(J_\alpha)_n(x)$

3 Main Results

In this section, we prove our main results :

Theorem 3.1 Let G $\subset \mathbb{R}$ be a bounded domain with piecewise smooth boundary ∂G=$B_1 \cup B_2$. Let $\Psi(x)$ be a solution of the differential inequality

$$\Delta_{\underline{x}}^\alpha\left(\frac{\Psi(x)}{g(\Psi(x))}\right) + p(x)\Psi(x) \geq 0 \quad in \quad G \tag{4}$$

which satisfies

$$\begin{aligned}
&\Psi(x) \geq 0 \quad in \quad G\\
&\Psi(x) = 0 \quad in \quad \partial G\\
&\frac{\partial^\alpha\Psi(x)}{\partial n^\alpha} < 0 \quad on \quad B_1 \qquad \frac{\partial^\alpha\Psi(x)}{\partial n^\alpha} \leq 0 \quad in \quad B_2.
\end{aligned}$$

where n is the unit Exterior normal vector to ∂G.
If $\int_G f(x)\Psi(x)d_\alpha x = 0$
Then every solution $u \in D(G) \equiv C^{2\alpha}(G) \cap C^\alpha(\bar{G})$ of (1) has a zero in G $\cup B_1$.
Proof: Suppose to the contrary that there is a solution u of (1) which has no zero in G $\cup B_1$. Let u > 0 in G $\cup B_1$.
By assumption(A_1) we get $\varphi(x,u) \geq p(x)u(x)$
Hence we have

$$\Delta_{\underline{x}}^\alpha\left(\frac{u(x)}{g(u(x))}\right) + p(x)u(x) \leq f(x) \quad in \quad G \tag{5}$$

multiply (5) by $\Psi(x)$ and integrating on over G

$$\int_G \left(\Delta_{\underline{x}}^\alpha\left(\frac{u(x)}{g(u(x))}\right) + p(x)u(x)\right)\Psi(x)d_\alpha x \leq \int_G f(x)\Psi(x)d_\alpha x \tag{6}$$

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From green's formula and the hypothesis, we get

$$\int_G \Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)\Psi(x)d_\alpha x \geq \int_{\partial G}\frac{\partial_\alpha}{\partial n^\alpha}\left(\frac{u(x)}{g(u(x))}\right)\Psi(x)d_\alpha S - \int_{\partial G}\frac{\partial_\alpha \Psi(x)}{\partial n^\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S + \int_G \left(\frac{u(x)}{g(u(x))}\right)\Delta_{\underline{x}}^{\alpha}\Psi(x)d_\alpha x$$

$$\int_G \Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)\Psi(x)d_\alpha x \geq -\int_{B1}\frac{\partial_\alpha \Psi(x)}{\partial n^\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S + \int_G \left(\frac{u(x)}{g(u(x))}\right)\Delta_{\underline{x}}^{\alpha}\Psi(x)d_\alpha x \tag{7}$$

combining the equation (5) and (6), we get

$$\int_G \Delta_{\underline{x}}^{\alpha}\left(\frac{u(x)}{g(u(x))}\right)\Psi(x)d_\alpha x + \int_G p(x)u(x)\Psi(x)d_\alpha x \geq -\int_{B1}\frac{\partial_\alpha \Psi(x)}{\partial n^\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S + \int_G \left(\frac{u(x)}{g(u(x))}\right)\Delta_{\underline{x}}^{\alpha}\Psi(x)d_\alpha x + \int_G p(x)u(x)\Psi(x)d_\alpha x$$

$$-\int_{B1}\frac{\partial_\alpha \Psi(x)}{\partial n^\alpha}\left(\frac{u(x)}{g(u(x))}\right)d_\alpha S + \int_G \left(\frac{u(x)}{g(u(x))}\right)\Delta_{\underline{x}}^{\alpha}\Psi(x)d_\alpha x + \int_G p(x)u(x)\Psi(x)d_\alpha x \leq \int_G f(x)\Psi(x)d_\alpha x \tag{8}$$

which is contradiction, it is easy to see that the right-hand side of (8) is zero.
If $u < 0$ in $G \cup B_1$, $v \equiv -u$, applying the first part for v, we get

$$\Delta_{\underline{x}}^{\alpha}\left(\frac{v(x)}{g(v(x))}\right) + p(x)v(x) \leq -f(x) \quad in \quad G$$

we lead to be contradiction, hence the proof.

We consider the lemma (2.11), where G is an annular domain
$A(r_1, r_2) = \{x \in \mathbb{R}^n / r_1 < |x| < r_2\}$

Corollary 3.2 Assume that p(x)=p(r). Let $\Psi(r)$ be a solution of the inequality

$$r^{1-n}\left(T_\alpha\left(r^{(\alpha+n-2)}\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right)\right)\right) + p(r)\Psi(r) \geq 0 \quad in \quad (r_1, r_2)$$

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which satisfies

$$\begin{aligned}&\Psi(r) \geq 0 \quad in \quad (r_1, r_2)\\&\Psi(r_1) = \Psi(r_2) = 0\\&\frac{\partial^\alpha \Psi(r_1)}{\partial n^\alpha} \geq 0, \qquad \frac{\partial^\alpha \Psi(r_2)}{\partial n^\alpha} < 0\end{aligned}$$

If $\int_{r_1}^{r_2} F(r)\Psi(r)r^{(\alpha+n-2)}d_\alpha r = 0$
then every solution $u \in D(A(r_1, r_2))$ of (1) has a zero in $A(r_1, r_2)$
where $A(r_1, r_2) = \{x \in \mathbb{R}^n / r_1 < |x| < r_2\}$
Proof: Let we take $B_1 = S_{r_2}$ and $B_2 = S_{r_1}$

$$\begin{aligned}\int_{A(r_1,r_2)} f(x)\Psi(x)d_\alpha x &= \int_{r_1}^{r_2}\int_{S_r} f(x)\Psi(x)d_\alpha S d_\alpha r\\&= \sigma_n \int_{r_1}^{r_2} F(r)\Psi(r)r^{(\alpha+n-2)}d_\alpha r\end{aligned}$$

The conclusion follows from Theorem (3.1) and the hypothesis. Hence the proof.

The Bessel function $(J_\alpha)_{(\frac{n}{2})-1}(r)$ and $(J_\alpha)_{(\frac{n}{6})-1}(r)$ has a oscillatory character.
Let a_k, b_k $(k \geq 1)$ be number such that $\lambda r_0 < a_k < b_k$,
$(J_\alpha)_{(\frac{n}{2})-1}(a_k) = (J_\alpha)_{(\frac{n}{2})-1}(b_k) = 0$,
$(J_\alpha)_{(\frac{n}{6})-1}(a_k) = (J_\alpha)_{(\frac{n}{6})-1}(b_k) = 0$ and
$(J_\alpha)_{(\frac{n}{2})-1}(r) > 0$, $(J_\alpha)_{(\frac{n}{6})-1}(r) > 0$ in (a_k, b_k).

Theorem 3.3 Assume that (A_1) and (A_3) hold and let (A_2) be satisfied for $p(x) = \frac{\lambda^2}{r^\mu}$. If

$$\int_{a_k/\lambda}^{b_k/\lambda} F(r)r^{((\frac{\alpha(4-n)}{2})+1)}(J_\alpha)_{(\frac{n}{2})-1}(\lambda r)d_\alpha r = 0 \quad for \quad some \quad k \geq 1$$

then every solution $u \in C^{2\alpha}(E)$ of (1) has a zero in $A(a_k/\lambda, b_k/\lambda]$.
Proof: Let define the function $\Psi(r) = r^{((1-(\frac{n}{2}))\alpha+\mu)}(J_\alpha)_{(\frac{n}{2})-1}(\lambda r)$ is a solution of

$$r^{1-n}\left(T_\alpha\left(r^{(\alpha+n-2)}\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right)\right)\right) + p(r)\Psi(r) = 0$$

$$r^{\alpha-1}\left(T_\alpha\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right)\right) + \frac{(\alpha+n-2)}{r}\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right) + p(r)\Psi(r) = 0$$

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where $g(\Psi(r)) = r^{\mu}$ for some $\mu > 1$

$$(J_\alpha)_{\frac{1}{2}}(\lambda r)\left(\left(\frac{3\alpha^2}{4}\right)r^{\left(\frac{-3\alpha-2}{2}\right)} - \frac{\alpha(\alpha+1)}{2}r^{\left(\frac{-3\alpha-2}{2}\right)} + \lambda^2 r^{\left(\frac{-\alpha}{2}\right)} - \frac{\alpha^2\lambda^2}{2}r^{\left(\frac{\alpha-2}{2}\right)}\right) + (J_\alpha)_{\frac{-1}{2}}(\lambda r)\left(\frac{\alpha\lambda}{2}r^{\left(\frac{-\alpha-2}{2}\right)}\right) - (J_\alpha)_{\frac{3}{2}}(\lambda r)\left(\frac{\alpha\lambda}{2}r^{\left(\frac{-\alpha-2}{2}\right)}\right) + (J_\alpha)_{\frac{-3}{2}}(\lambda r)\left(\frac{\alpha^2\lambda^2}{4}r^{\left(\frac{\alpha-2}{2}\right)}\right) + (J_\alpha)_{\frac{5}{2}}(\lambda r)\left(\frac{\alpha^2\lambda^2}{4}r^{\left(\frac{\alpha-2}{2}\right)}\right) = 0$$

(see,[34]Theorem 2)
show that

$$\Psi(r) \geq 0 \quad in \quad (a_k/\lambda, b_k/\lambda)$$
$$\Psi(a_k/\lambda) = \Psi(b_k/\lambda) = 0$$
$$\frac{\partial^\alpha \Psi(a_k/\lambda)}{\partial n^\alpha} \geq 0, \qquad \frac{\partial^\alpha \Psi(b_k/\lambda)}{\partial n^\alpha} < 0$$

Follows from corollary (3.2), we get

$$\int_{a_k/\lambda}^{b_k/\lambda} F(r) r^{\left(\frac{\alpha(4-n)}{2}+1\right)} (J_\alpha)_{\left(\frac{n}{2}-1\right)}(\lambda r) d_\alpha r = 0.$$

Hence the proof is complete.

Remark 3.4 The following remark in the paper see[12]. If

$$(J_\alpha)_{\left(\frac{1}{2}\right)}(x) = \Sigma_{n=0}^{\infty}\left(\frac{(-1)^n}{n!\Gamma(\frac{3}{2}+n)}\left(\frac{x^\alpha}{2}\right)^{\left(2n+\frac{1}{2}\right)}\right)$$

then

$$(J_\alpha)_{\left(\frac{1}{2}\right)}(x) = \left(\frac{2}{\pi x^\alpha}\right)^{\frac{1}{2}} sin(x^\alpha)$$

similarly,

$$(J_\alpha)_{(\frac{-1}{2})}(x) = \left(\frac{2}{\pi x^\alpha}\right)^{\frac{1}{2}} cos(x^\alpha)$$
$$(J_\alpha)_{(\frac{3}{2})}(x) = \left(\frac{2}{\pi x^\alpha}\right)^{\frac{1}{2}} \left(\frac{sin(x^\alpha)}{x^\alpha} - cos(x^\alpha)\right)$$
$$(J_\alpha)_{(\frac{-3}{2})}(x) = \left(\frac{2}{\pi x^\alpha}\right)^{\frac{1}{2}} \left(\frac{-cos(x^\alpha)}{x^\alpha} - sin(x^\alpha)\right)$$
$$(J_\alpha)_{(\frac{5}{2})}(x) = \left(\frac{2}{\pi x^\alpha}\right)^{\frac{1}{2}} \left(\left(\frac{3-x^{2\alpha}}{x^{2\alpha}}\right) sin(x^\alpha) - \frac{3}{x^\alpha}cos(x^\alpha)\right)$$
$$(J_\alpha)_{(\frac{-5}{2})}(x) = \left(\frac{2}{\pi x^\alpha}\right)^{\frac{1}{2}} \left(\left(\frac{3-x^{2\alpha}}{x^{2\alpha}}\right) cos(x^\alpha) + \frac{3}{x^\alpha}sin(x^\alpha)\right)$$

Corollary 3.5 Assume that (A_1) - (A_3) hold and that p(x) = $\frac{\lambda^2}{r^\mu}$ $(\lambda > 0)$. Let n=3 and M be a positive integer with $2M\pi/\lambda > r_0$. If

$$\int_{2M\pi/\lambda}^{(2M+1)\pi/\lambda} F(r) r sin(\lambda r)^\alpha d_\alpha r = 0 \tag{9}$$

Then every solution $u \in C^{2\alpha}$(E) of (1) has a zero in $A(2M\pi/\lambda, (2M+1)\pi/\lambda]$.
Proof: From the remark, we get

$$(J_\alpha)_{(\frac{1}{2})}(\lambda r) = \left(\frac{2}{\pi(\lambda r)^\alpha}\right)^{\frac{1}{2}} sin(\lambda r)^\alpha$$

$$r^{(\frac{3\alpha}{2})}(J_\alpha)_{(\frac{1}{2})}(\lambda r) = \left(\frac{2}{\pi(\lambda)^\alpha}\right)^{\frac{1}{2}} r^\alpha sin(\lambda r)^\alpha$$

It is clear that $(J_\alpha)_{\frac{1}{2}}(2M\pi) = (J_\alpha)_{\frac{1}{2}}((2M+1)\pi) = 0$ and $(J_\alpha)_{\frac{1}{2}}(r) > 0$ in $(2M\pi, (2M+1)\pi)$. Follows from Theorem (3.3), we get

$$\int_{2M\pi/\lambda}^{(2M+1)\pi/\lambda} F(r) r sin(\lambda r)^\alpha d_\alpha r = 0$$

Hence the proof.

Theorem 3.6 Assume that (A_1) and (A_3) hold and let (A_2) be satisfied for

$p(x) = \frac{\lambda^2}{r^\mu}$. If

$$\int_{a_k/\lambda}^{b_k/\lambda} F(r) r^{(\frac{\alpha(4-n)}{2}+1)} (J_\alpha)_{(\frac{n}{6})-1}(\lambda r) d_\alpha r = 0 \quad for \quad some \quad k \geq 1$$

then every solution $u \in C^{2\alpha}(E)$ of (1) has a zero in $A(a_k/\lambda, b_k/\lambda]$.
Proof: Let define the function $\Psi(r) = r^{((\frac{n}{6})-1)\alpha+\mu}(J_\alpha)_{(\frac{n}{6})-1}(\lambda r)$ is a solution of

$$r^{1-n}\left(T_\alpha\left(r^{(\alpha+n-2)}\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right)\right)\right) + p(r)\Psi(r) = 0$$

$$r^{\alpha-1}\left(T_\alpha\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right)\right) + \frac{(\alpha+n-2)}{r}\left(T_\alpha\left(\frac{\Psi(r)}{g(\Psi(r))}\right)\right) + p(r)\Psi(r) = 0$$

where $g(\Psi(r)) = r^\mu$ for some $\mu > 1$

$$\begin{aligned}(J_\alpha)_{\frac{-1}{2}}(\lambda r)&\left(\left(\frac{3\alpha^2}{4}\right)r^{(\frac{-\alpha-4}{2})} - \frac{\alpha(\alpha+1)}{2}r^{(\frac{-3\alpha-2}{2})} + \lambda^2 r^{(\frac{-\alpha}{2})} - \frac{\lambda^2}{2}r^{(\frac{\alpha-2}{2})}\right)\\ &+ (J_\alpha)_{\frac{1}{2}}(\lambda r)\left(\frac{\alpha^2\lambda}{4}r^{(\frac{-\alpha-2}{2})} - \frac{\lambda\alpha^2}{4}r^{(\frac{\alpha-4}{2})} = \frac{\alpha(\alpha+1)\lambda}{2}r^{(\frac{-\alpha-2}{2})}\right)\\ &- (J_\alpha)_{\frac{-3}{2}}(\lambda r)\left(\frac{\alpha^2\lambda}{4}r^{(\frac{-\alpha-2}{2})} - \frac{\lambda\alpha^2}{4}r^{(\frac{\alpha-4}{2})} = \frac{\alpha(\alpha+1)\lambda}{2}r^{(\frac{-\alpha-2}{2})}\right)\\ &+ (J_\alpha)_{\frac{3}{2}}(\lambda r)\left(\frac{\lambda^2}{4}r^{(\frac{\alpha-2}{2})}\right) + (J_\alpha)_{\frac{-5}{2}}(\lambda r)\left(\frac{\lambda^2}{4}r^{(\frac{\alpha-2}{2})}\right) = 0\end{aligned}$$

(see[34],Theorem 2)
show that

$$\begin{aligned}&\Psi(r) \geq 0 \quad in \quad (a_k/\lambda, b_k/\lambda)\\ &\Psi(a_k/\lambda) = \Psi(b_k/\lambda) = 0\\ &\frac{\partial^\alpha \Psi(a_k/\lambda)}{\partial n^\alpha} \geq 0, \qquad \frac{\partial^\alpha \Psi(b_k/\lambda)}{\partial n^\alpha} < 0\end{aligned}$$

Follows from corollary (3.3), we get

$$\int_{a_k/\lambda}^{b_k/\lambda} F(r) r^{(\frac{\alpha(4-n)}{2}+1)} (J_\alpha)_{(\frac{n}{6}-1)}(\lambda r) d_\alpha r = 0.$$

Hence the proof is complete.

Corollary 3.7 Assume that (A_1) - (A_3) hold and that p(x) = $\frac{\lambda^2}{r^\mu}$ $(\lambda > 0)$. Let n=3 and M be a positive integer with $(4M+3)\pi/2\lambda > r_0$. If

$$\int_{(4M+3)\pi/2\lambda}^{(4M+5)\pi/2\lambda} F(r)rcos(\lambda r)^\alpha d_\alpha r = 0 \tag{10}$$

Then every solution $u \in C^{2\alpha}$(E) of (1) has a zero in $A((4M+3)\pi/2\lambda, (4M+5)\pi/2\lambda]$.
Proof:From the remark, we get

$$(J_\alpha)_{(\frac{-1}{2})}(\lambda r) = \left(\frac{2}{\pi(\lambda r)^\alpha}\right)^{\frac{1}{2}} cos(\lambda r)^\alpha$$

$$r^{(\frac{3\alpha}{2})}(J_\alpha)_{(\frac{-1}{2})}(\lambda r) = \left(\frac{2}{\pi(\lambda)^\alpha}\right)^{\frac{1}{2}} r^\alpha cos(\lambda r)^\alpha$$

It is clear that $(J_\alpha)_{\frac{-1}{2}}((4M+3)\pi/2) = (J_\alpha)_{\frac{-1}{2}}((4M+5)\pi/2) = 0$ and $(J_\alpha)_{\frac{-1}{2}}(r) > 0$ in $((4M+3)\pi/2, (4M+5)\pi/2)$ Follows from Theorem (3.3), we get

$$\int_{(4M+3)\pi/2\lambda}^{(4M+5)\pi/2\lambda} F(r)rcos(\lambda r)^\alpha d_\alpha r = 0$$

Hence the corollary is proved.

Corollary 3.8 Assume that (A_1) - (A_3) hold and that p(x) = $\frac{\lambda^2}{r^\mu}$ $(\lambda > 0)$. Every solution $u \in C^{2\alpha}(E)$ of

$$\Delta_{\underline{x}}^\alpha \left(\frac{u(x)}{g(u(x))}\right) + \varphi(x, u(x)) = 0$$

has a zero in $A(a_k/\lambda, b_k/\lambda]$ for some k ≥ 1.

4 Conclusion

In this article, we have derived some new sufficient condition for oscillation of solutions of conformable hybrid elliptic partial differential equations. We have given some examples to illustrate the effectiveness of our new results. Our newly- derived oscillation results extend and improve numerous findings in the recent publications on classical literature to conformable hybrid elliptic partial differential equations. We belived that this research work would lead to further work on the conformable hybrid elliptic partial differential equations.

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