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Exponential traveling-wave on a viscous fluid flowing down an inclined plane

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Abstract

An exponential traveling-wave solution of the free surface equation for a viscous film flowing down an inclined plane is presented. The free surface equation, valid for long waves of finite amplitude, is obtained from the general equation for waves of arbitrary amplitude, which is developed by the method of multiple-scales and considering a series solution expanded at two orders higher than the series used by Lin in 1974. Thus, confining ourselves to the case of weakly nonlinear wave motion, we use the Ince transformation method to find an exponential travelling-wave solution that depend of the physical parameters of the viscous fluid , the angle θ of the inclined plane and the wave parameters, which are subjected to three constraints between them. This solution show a free surface with a dynamic gap of thickness between the top and the bottom of the inclined plane. Also, we find a particular numerical example, for laminar flow, that may represent the evolution of the free surface h(x,t) of a viscous film flowing down an inclined plane, when a small quantity, of the same fluid, is added constantly to the primary fluid in the top of the inclined plane. An interesting result of this case show that the final state, when $t \to \infty$, is also stationary and has the same behavior as the unperturbed primary flow, however the final layer of fluid has a greater thickness.
Key words: Traveling waves, perturbation theory, viscous fluid.
AMS classification: 46.15.Ff, 47.10.ad, 05.45.-a

1 Introduction

One of the best-known solutions of Navier-Stokes equations for open channel flows is one that describes the flow due to gravity associated with a layer of viscous fluid, with thicknessh_0 , which is bounded above by a free surface and below by a fixed plane inclined at an angle θ respect to the horizontal. In this case, the velocity of the fluid is maximal on the free surface $y = h_0$ and decrease with the depth y, so that, the velocity distribution across the fluid is parabolic. Despite of the simplicity of this solution of the Navier-Stokes equations, generalizations via perturbation theory have been studied until today by several authors. Starting from a pioneering work by Kapitsa [1], various stability studies were performed during subsequent decades [3–5, 7]. Also there are several studies about the physical properties of the interphase and dynamical behavior of the free surface, such as, traveling waves along the surface [8, 9] and solitary waves [10]. In the geophysical context, this model is used to understand the evolving viscoplastic flows upon slopes [11]. A detailed review of several contributions, commonly obtained under the framework of lubrication approach, can be find in the monograph: "Falling Liquid Films" [2] and references therein. Originally, our motivation was oriented to find exact travelling wave solutions associated with the evolution equation for long waves of arbitrary amplitude proposed by Lin in 1974 (equation 11 in the original paper), which consider fluctuations on the motion associated with a viscous liquid flowing down by plane inclined and is

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presented as a power series expanded in terms of a small dimensionless parameter α that can be considered as the number of waves in a distance $2\pi h_0$. Apparently this series was truncated at order two, since the term $O(\alpha^3)$ appear in the series. However, the reader also can easily verify that the polynomial functions F,H,G and I in equation 11 [5] are proportional to α^2 , generating term of forth-order (α^4) in the series, causing an ambiguity that cannot be solved at a glance. In this point is important to remember that the paper of Lin (1974) is based on a previous work done by Benney (1966), which present a series solution truncated at order three, i.e.,one order major than the Lin's series, but according to Lin, with minor algebraic errors. The comments above presented show an inconsistency that simply could be a misprint, however is not trivial to find what is wrong. This issue motivate us to redo such a computation in terms of a solution series expanded up to $O(\alpha^4)$, thus we can to overcome the inconsistencies discussed above. Moreover, a series solution of this kind represent a better approximation than actual models and can eventually be used in future researches, wherever the requirements call for the highest precision. In this work, we follow closely the formulation of Lin (1974) to derive the evolution equation expanded to fourth-order (respect to α) that allow to describe long waves of arbitrary amplitude associated with a viscous fluid flowing down by plane inclined. The full calculations are presented in the appendix A. Furthermore, according with the expanded model, we find an exact solution for a particular case of long waves of finite amplitude, where the perturbation is very small ($\approx \alpha\alpha^4$). For this case we obtain a differential equation very similar to Benney-Lin equation, depending of three parameters, which can be written as functions of the Reynolds number R, the Weber number W, the inclination angle θ and α. Considering the Ince transformation method, we find an exact exponential traveling wave solution, which may represent a family of situations where the free surface evolves with time holding a constant gap of thickness between the layer of fluid on the top and on the bottom of the inclined plane. In this point is important to emphasize that the transition region moves faster than the primary fluid, showing an increase of the quantity of fluid in the system. Necessarily this additional fluid could be attributed to the disturbance. The physical interpretation of this solution show us the evolution of the free surface of a viscous film flowing down an inclined plane, when a small quantity, of the same fluid, is added constantly to the primary fluid. An interesting result of this solution shows that the final state, when $t \to \infty$, is stationary and has the same behavior as the unperturbed primary flow, but with a greater thickness. The article is organized as follows. In section III, we obtain the evolution equation valid for long waves of finite amplitude (9) from the fourth order equation valid for long waves of arbitrary amplitude. In section IV, we solve the equation like Benney-Lin proposed in III, obtaining an exact travelling wave solution for this case. Also, we find a numerical example, coherent with the constraint associated with our solution. We show graphically the dynamic evolution of the free surface . In section V we present the conclusions. In order to show the calculation and previous considerations that allows to obtain the motion equation at fourth order for the Lin model, we include an appendix A.

2 Viscous Film Flowing Steadily Down an Inclined Plane

The equation of motion of the free surface, obtained by S.P.Lin [5] is commonly used to describe the dynamic of long waves of arbitrary amplitude that could appear in the surface of a perturbed viscous film flowing steadily down an

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inclined plane. In this case, the Lin's equation is obtained from the kinematic boundary condition at free surface via perturbation theory with expansions in terms of a small parameter α, which is proportional to the number of waves in a distance equivalent to the film thickness. The condition of long waves implies that $\alpha < 1$. While Lin's equation is a truncated solution at $O(\alpha^2)$, in our paper, we find the full expression for $O(\alpha^4)$. In Addition, we point out that our procedure allows to obtain the correct series, in fourth-order, for both; the perturbed velocity and the perturbed pressure. Our results have been checked with the help of the MATHEMATICA, which are summarized in appendix A. The evolution equation for the free surface , in fourth-order perturbation theory, can be write in the following way

$$h_t + B_1 h_x + \alpha \frac{d}{dx}(B_2 h_x) + \alpha^2 \frac{d}{dx}(B_3 h_x^2 + B_4 h_{xx}) + \alpha^3 \frac{d}{dx}(B_5 h_x^3 + B_6 h_x h_{xx} + B_7 h_{xxx})$$
$$\alpha^4 \frac{d}{dx}(B_8 h_x^4 + B_9 h_x^2 h_{xx} + B_{10} h_{xx}^2 + B_{12} h_{xxxx}) = 0. \tag{1}$$

where the subscripts x and t represent partial derivatives with respect to x and time t, respectively. According to Appendix A, the variables x,t and h are dimensionless variables. The x-axis coincide with the plane bed inclined at the angle θ with respect to the horizontal, and the y-axis normal to the plane bed, so that, the function y = h(x,t) is the free surface associated with the fluid and B_i with i = 1,2,...,12 are polynomial functions of h(x,t) which are

$$B_1(h) = 2h^2, \qquad B_2(h) = \tfrac{2h^3}{15}(5\cot\theta - 4h^3 R)$$

$$B_3(h) = \tfrac{h^3}{630}(1192R^2h^6 - 819Rh^3\cot\theta + 2940)$$

$$B_4(h) = \tfrac{1}{14}h^3(4R^2h^6 - 5Rh^3\cot\theta + 28)$$

$$B_5(h) = h^3\left(\tfrac{24089R^3h^9}{1890} - \tfrac{144619R^2h^6\cot\theta}{22680} - \tfrac{3}{10}Rh^3\cot^2\theta + \tfrac{3751Rh^3}{90} - \tfrac{14\cot\theta}{3}\right)$$

$$B_6(h) = h^4\left(\tfrac{18093511R^3h^9}{4054050} - \tfrac{844073R^2h^6\cot\theta}{22680} - \tfrac{37}{210}Rh^3\cot^2\theta + \tfrac{1910Rh^3}{63} - 8\cot\theta\right)$$

$$B_7(h) = h^3\left(\tfrac{13241483R^3h^{11}}{8108100} - \tfrac{477523R^2h^8\cot\theta}{2494800} + \tfrac{331}{168}Rh^5 - \tfrac{6}{5}h^2\cot\theta + \tfrac{2W}{3}\right)$$

$$B_8(h) = h^3\left(\tfrac{9195404399R^4h^{12}}{70945875} - \tfrac{259902109R^3h^9\cot\theta}{4989600} + \tfrac{11890307}{22680}R^2h^6 - \tfrac{51733}{11340}R^2h^6\cot^2\theta\right)$$
$$-h^3\left(\tfrac{1571}{18}Rh^3\cot\theta - 30\right)$$

$$B_9(h) = \frac{h^{10}R^2}{100}\left(\tfrac{63126090901R^2h^6}{8513505} - \tfrac{4609042387Rh^3\cot\theta}{972972} - \tfrac{865589\cot^2\theta}{2268} - \tfrac{1450669}{28}\right)$$
$$+h^4\left(-\tfrac{3134}{21}Rh^3\cot\theta + \tfrac{32RWh}{5} + 130\right)$$

$$B_{10}(h) = \frac{h^{11}R^2}{5}\left(\tfrac{17773139299R^2h^6}{1157836680} - \tfrac{1076900219Rh^3\cot\theta}{77837760} + \tfrac{22262213}{124740}\right)$$
$$-\tfrac{h^5}{5}\left(\tfrac{31337R^2h^6\cot^2\theta}{33264} + \tfrac{43163}{504}Rh^3\cot\theta - 16RWh - 138\right)$$

$$B_{11}(h)=h^{11}\left(\frac{57362849R^4h^6}{13425750}-\frac{15415343R^3h^3\cot\theta}{3891888}+\frac{1031903R^2}{20790}-\frac{237697R^2\cot^2\theta}{1247400}\right)$$
$$-h^5\left(\frac{6289Rh^3\cot\theta}{252}-\frac{9RWh}{2}-\frac{652}{15}\right)$$
$$B_{12}(h)=h^6\left(\frac{33220639R^4h^{12}}{404838000}-\frac{39923447R^3h^9\cot\theta}{389188800}+\frac{7637717R^2h^6}{4989600}\right)$$
$$-h^6\left(\frac{2333Rh^3\cot\theta}{1890}-\frac{5RWh}{14}-\frac{11}{3}\right)$$

where α is the number waves in a distance $2\pi h_0$ and h_0 is the non-perturbed film thickness. Equation (1), being an extension of Lin's equation, also describes long waves of arbitrary amplitude in the surface of the viscous film. Despite the complexity of this equation, our motivation is oriented to find exact solutions of (1). For convenience, we restrict ourselves to the case of weakly nonlinear wave motion, in this case we can consider that the motion wave perturbs the free surface only slightly. Thus, we write

$$h(x,t)=1+\varepsilon S(x,t) \tag{2}$$

where ε is a non-dimensional parameter, so that $\varepsilon<1$, whereas S(x,y) is an auxiliary function to determinate. Insert
(2) into (1) and expanding around $\varepsilon=0$, gives

$$S_t+2(2\varepsilon S+1)S_x+\frac{2\alpha}{3}\left(\frac{4R}{5}-\cot\theta\right)S_{xx}+\frac{\alpha^2}{14}(4(R^2+7)-5R\cot\theta)S_{xxx}$$
$$+\alpha^3\left(\frac{1241483R^3}{8108100}+\frac{331R}{168}+\frac{2W}{3}-\left(\frac{477523R^2}{2494800}+\frac{6}{5}\right)\cot\theta\right)S_{xxx}$$
$$+\alpha^4\left(\frac{33220639R^4}{404838000}+\frac{7637717R^2}{4989600}+\frac{5WR}{14}-\frac{(39923447R^2+480411360)\cot\theta R}{389188800}+\frac{11}{3}\right)S_{xxxxx}=0. \tag{3}$$

The equation (3) depends on three parameters; the Reynolds number R, the Weber number W and the inclination angle θ. Also the parameters α and ε appear in the equation. If we consider that $\varepsilon=\varepsilon(\alpha)$, then several cases can be investigated. In particular, we find that the case $\varepsilon=\alpha^4$ admit an exact solution, which is specially interesting because the dynamic of the free surface can be studied using the same methods that allow to solve the Benney-Lin equation. In fact, if we substitute

$$t=\frac{\tau}{A_0},\qquad S(x,\tau)=\frac{A_0u(x,\tau)-2}{4\varepsilon},\qquad A_0=\alpha^2\tilde{A_0} \tag{4}$$

Where

$$\tilde{A_0}=\frac{4R^2-5R\cot\theta_2 8}{14} \tag{5}$$

and defining three parameters γ,β and η so that

$$\gamma=\frac{28}{15R\alpha}\left(1-\frac{2}{\tilde{A_0}}\right) \tag{6}$$

$$\beta = \alpha\left(\frac{1}{25}\left(\frac{477523R}{35640}+\frac{84}{R}\right)+\frac{1}{\tilde{A_0}}\left(-\frac{32R^3}{3378375}-\frac{384467R}{6237000}+\frac{2W}{3}-\frac{168}{25R}\right)\right) \quad (7)$$

$$\beta = \alpha^2\left(\frac{1}{675}\left(\frac{39923447R^2}{205920}+2333\right)+\frac{1}{\tilde{A_0}}\left(-\frac{21176R^4}{3618239625}-\frac{4342309R^2}{138996000}+\frac{5WR}{14}-\frac{2191}{675}\right)\right) \quad (8)$$

then equation (3) becomes

$$u_\tau + \eta u_{xxxxx} + \beta u_{xxxx} + u_{xxx} + \gamma u_{xx} + u u_x = 0, \quad (9)$$

which have a similar form to Benney-Lin equation, in fact, if $\beta = \gamma$ then equation (9) has the same form than Benney-Lin equation. However, in our case u is associated with the height of the free surface, while the Benney-Lin equation is an equation commonly defined for velocities.

3 Exact Solution; Benney- Lin Equation Type

In this section, we present an exact solution of (9) and also provide a physical interpretation for this solution.

A. Exponential traveling wave solution. We now propose a travelling wave solution of (9). This class of solutions are sought by taking

$$u(x,\tau) = f(\xi) \quad where \qquad \xi = \xi(x,\tau) - kx - \omega\tau \quad (10)$$

Here k and ω are constants to be determined. Substituting (10) into (9) we have

$$-\omega f_\xi + \eta k^5 f_{\xi\xi\xi\xi\xi\xi} + \beta k^4 f_{\xi\xi\xi\xi\xi} + k^3 f_{\xi\xi\xi\xi} + \gamma k^2 f_{\xi\xi\xi} + k f f_\xi = 0. \quad (11)$$

Thus, integrating once with respect to ξ, we obtain

$$-\omega f + \eta k^5 f_{\xi\xi\xi\xi\xi} + \beta k^4 f_{\xi\xi\xi\xi} + k^3 f_{\xi\xi\xi} + \gamma k^2 f_{\xi\xi} + \frac{1}{2} k f^2 = C_1, \quad (12)$$

where C_1 is a constant of integration. Since equation (12) cannot be integrated directly, we use the same idea given by McIntosh [16]. Indeed, we consider the Ince transformation method, see [13]. If we assume that

$$f = f(z) = z^4 r(z) + f_0, \quad (13)$$

where $z = e^\xi$ and f_0 is a constant to determinate, then we can to reduce (12) to a directly integrable differential equation for r. In fact, substituting (13) into (12) and performing straightforward calculations we obtain that

$$r^2 + 2k^4 \eta r_z^{''''} = 0. \quad (14)$$

and also the following constraint relations must be satisfied

$$\eta = \frac{1}{179k^2}, \quad (15)$$

$$\beta = -\frac{22}{179k}, \tag{16}$$

$$\gamma = -\frac{638}{179k}, \tag{17}$$

$$f_0 = \frac{840k^3 + (-1)^p\sqrt{2}\sqrt{352800k^6 - 32041C_1k}}{179k}, \tag{18}$$

$$\omega = \frac{(-1)^p\sqrt{2}}{179}\sqrt{352800k^6 - 32041C_1k}, \tag{19}$$

where p takes values 0 or 1. The differential equation (14) has a solution of the form

$$r(z) = A(Bz + C)^{-4}, \tag{20}$$

Where

$$A = -1980k^4\eta B^4, \tag{21}$$

and B and C are arbitrary constants [17].
Using (21), solution (20) can be written as follows:

$$r(z) = -\frac{1680k^2}{179(C_3 + z)^4}, \tag{22}$$

where C_3 is another constant defined by $C = BC_3$. Finally, according to (13) and (4), the free surface h(x,t) of the fluid can be written

$$h(x,t) = \frac{1}{2} + \frac{1}{4}\tilde{A_0}f_0\alpha^2 - \frac{420\tilde{A_0}k^2\alpha^2}{179(e^{-\xi(x,t)}C_3 + 1)^4} \tag{23}$$

where $\tilde{A_0}$ and f_0 are defined in (5) and (18) respectively, and

$$\xi(x,t) = kx - \omega\alpha^2\tilde{A_0}t, \tag{24}$$

where ω is defined in (19). Moreover, in order to guarantee a solution without discontinuities in the free surface, we can to choose $C_3 > 0$ as a sufficiency condition. From (23) and performing straightforward calculations, for the case $k > 0$, we can to rewrite this solution in term of your asymptotic values, so that

$$h(x,t) = h_{-\infty} - \frac{h_{-\infty} - h_{\infty}}{(e^{-\xi(x,t)}C_3 + 1)^4}, \tag{25}$$

Where

$$h_{-\infty} = \lim_{x\to-\infty} h(x,t) = \frac{1}{2} + \frac{\alpha^2\tilde{A_0}f_0}{4}, \tag{26}$$

and

$$h_{-\infty} = \lim_{x\to\infty} h(x,t) = \frac{1}{2} + \frac{\alpha^2\tilde{A_0}f_0}{4} - \frac{420\tilde{A_0}k^2\alpha^2}{179}, \tag{27}$$

This solution shows that there are a dynamic gap of thickness between the top and the bottom of the inclined plane. At time t = 0 the system is perturbed increasing the original thickness h_0 of the fluid layer to $h_0+\Delta h$, later the system evolves to the original steady regime at $t\to\infty$, in fact, $h_\infty = h_0$ and $h_{-\infty} = h_0+\Delta h$.

B. Particular Solution and physical interpretation.

The relationships (6) to (8) shows that the parameters γ,β and η in the equation (9) depend on the physical parameters α,R,W and θ. On the other hand, the solution of (12) impose three constraining relations (15) to (17), where γ,β and η depend only on k. Consequently, the Reynolds number, the Weber number and the inclination angle are determinate by

$$W=\frac{16R^3}{1126125}-\left(\frac{60204416101R^2}{1865170692000}+\frac{9}{25}\right)G+\frac{631620}{32014G}-\frac{9625963R}{89714800}, \tag{28}$$

$$0=\frac{29699149844375(23RG-203)}{91765424}+17R\left(\frac{2059912136270951R^2}{258411433984}+\frac{16023735}{128}\right)G^3$$
$$+14\left(3315\left(\frac{38329731519749R^2}{258411433984}+\frac{363825}{256}\right)-R^4\right)G^2, \tag{29}$$

where $G=G(R,\theta)$ is an auxiliary function defined by

$$G(R,\theta)=4R-5\cot\theta \tag{30}$$

The equation (29) show that the Reynolds number R, for this fluid, depends on the inclination angle θ. This relation is numerically solved using an iterative routine based on the Newton-Raphson method, so that those results are presented in the figure 1.
If R and θ are known then Weber number W can be determined using the equation (28). In particular, we find that, if R = 40 then

$$\theta = 0.03111163068911944 \quad rad,$$
$$W = 1.0481979630110703$$

and

$$\alpha=\frac{0.21458672674674478}{k}. \tag{31}$$

Defining $\alpha=h_0k$, then for $h_0=1$ we have $\alpha=k=0.463235$. According with this numerical example, the figure (2) show a particular case of the exponential travelling wave solution obtained in the last section. This solution represent the flow due to gravity of a layer of viscous fluid with thickness h_0 falling down through a fixed inclined plane , which has been subject to a small perturbation that may be understood as the action of adding, constantly, a small amount of the same fluid, in a position considered at $x=-\infty$ (top of plane). This additional fluid travel on the primary flux forming a gap of thickness Δh respect to free surface of the primary fluid. This gap evolves with the time covering all the primary fluid at $t\to\infty$. In this case, the final state, when $t\to\infty$, is also stationary and has the same behavior as the unperturbed primary flow, however the final layer of fluid has a greater thickness.
FIG. 1: Graphical solution of the equations (6) to (8) in terms of R and θ. Given a fixe angle, we obtain a sixth order polynomial equation in R, which is numerically solved using an iterative routine based on the Newton-Raphson method.

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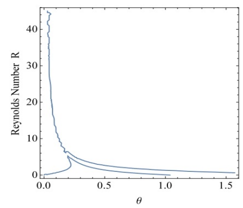
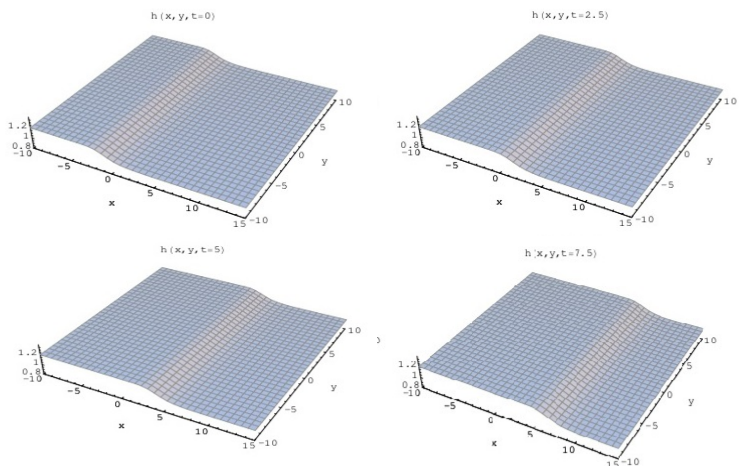


FIG. 2: Temporal evolution of an exponential traveling wave on the free surface of a viscous film flowing steadily down an inclined plane in the x direction. Figures are shown for values t=0, t=2.5, t=5 and t=7.5 respectively.



4 Discussion

In this work, we derive an order four evolution equation for long waves of arbitrary amplitude associated with a viscous liquid film flowing steadily down an inclined plane. Our result is an extension to the commonly used [5] and it can be applied when highest precision is required. The motivation in undertaking this rather laborious computation is supported by the difficulty when explaining an inconsistency detected in equation (11) in [5]. The equation of motion up to an order four approximation is derived and then simplified under the finite-amplitude approximation (here small deviations of order fourth are considered (α^4)). For this case, the motion equation is reduced to a Benney-Lin type equation. And, by considering the Ince transformation method, we find an exact exponential traveling wave solution for this model. Note that, the free surface evolves with time holding a constant gap of thickness between the layer of fluid on the top and on the bottom of the inclined plane.

Physically, this solution may be interpreted as the evolution of the free surface of a viscous film flowing down an inclined plane when a small quantity of the same fluid is added constantly to the primary fluid. In this fashion, the additional fluid form a small thin layer that evolves with the time covering the primary flux at $t \to \infty$.

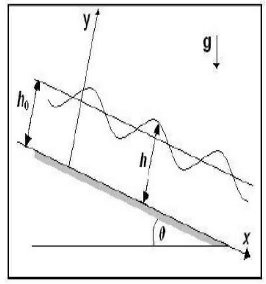


FIG. 3: Viscous film steadily flowing down an inclined plane along the x direction.

It is worth noticing that a result of this solution shows that the final state, when $t \to \infty$, is stationary and it has the same behavior as the unperturbed primary flow (but with a greater thickness). We also show that for each inclination angle there is finite number possible values for the Reynolds number R which are solutions of a polynomial equation of degree six in R. Consequently, the Weber number also depends on the inclination angle and it can be calculated using the values obtained for R and θ, see figure (1). We expect to motivate experimental researches to verify this wave solution and the possible constraints between R,W and θ.

5 ACKNOWLEDGMENT

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APPENDIX A: Fourth-order perturbative model for a viscous film flowing steadily down an inclined plane

The motion of a viscous fluid is described from first principles by

$$\vec{\nabla}\vec{V} = 0. \tag{32}$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}.\vec{\nabla})\vec{\nabla} = -\frac{1}{\rho}\vec{\nabla}P + \nu(\vec{\nabla}.\vec{\nabla})\vec{V} + \vec{g} \tag{33}$$

where $\vec{V}$ is the velocity vector, t the time, ν the kinematic viscosity, P the pressure, ρ the density and $\vec{g}$ the gravitational acceleration vector.

If we consider a perturbation stream function for a viscous film steadily flowing down an inclined plane, then the velocity takes the following form

$$\vec{V} = [\bar{u}(y) + u^{'}(x,y,t), v^{'}(x,y,t)] \tag{34}$$

where $\bar{u}(y) = y(2h_0 - y^{'})\frac{g}{2\nu}\sin\theta$ will be regarded as the unperturbed part of $\vec{v}$ and $u^{'}(x,y,t)$ and $v^{'}(x,y,t)$ as the parallel (x) and normal (y) components of the perturbed velocity. The angle θ is the inclination of the plane respect to horizontal and h_0 is the film thickness associated with the unperturbed fluid. The pressure also is perturbed, so that

$$P = \bar{p}(y) + p^{'}(x,y,t) \tag{35}$$

where $\bar{p} = p_0 + \rho g(h_0 - y)\cos\theta$ is the unperturbed part, $p'(x,y,t)$ is the perturbation and $p_0$is the atmospherical pressure.
The equation (32) is trivialy satisfied if we assume $u' = \phi_y$ and $v' = -\phi_x$, where $\phi(x,y,t)$ needs to be determined. The indices x,y and t represent the usual notation for partial derivatives in the spatial directions and time, respectively. Equation (33) can be writen in the form

$$-\frac{1}{\rho}p'(x) = \phi_{y_t} + (\bar{u} + \phi_y)\phi_{yx} - \phi_x\bar{u}_y - \phi_x\phi_{yy} - \nu\phi_{yxx} - \nu\phi_{yyy} \tag{36}$$

$$\begin{aligned} \nu\phi_{yyyy} &= \phi_{yyt} + \bar{u}\phi_{yyx} + \phi_{yyx}\phi_y - \phi_x\bar{u}_{yy} - \phi_x\phi_{yyy} - 2\nu\phi_{yyxx} \\ &+ \phi_{xxt} + \bar{u}\phi_{xxx} + \phi_{xxx}\phi_y + \phi_{xx}\bar{u}_x - \phi_x\phi_{xxy} - \nu\phi_{xxxx}. \end{aligned} \tag{37}$$

Additionally, the model satisfies the following boundary conditions:

1. On the surface of the inclined plane, no slipping is allowed,

$$u' = \phi_y = 0, v' = -\phi_x = 0 \quad at \quad y = 0. \tag{38}$$

2. If we assume that, at any point (x,y) of the free surface y = h(x,t), the fluid velocity must be tangential to the moving surface at all times, then is easy to show that

$$h_t = -\phi_x - h_x(\bar{u} + \phi_y) \quad at \quad y = h, \tag{39}$$

which is known as the kinematic boundary condition.

3. Also, at each point of the free surface we identify two characteristic pressures; the tangential pressure $p_{\|}$ and the normal pressure $p_{\perp}$. The tangential pressure $p_{\|}$ is null on the free surface,

$$p_{||} = 0 \quad at \quad y = h. \tag{40}$$

This boundary condition expresses the fact that, upon neglecting the influence of air, there are no shear stress on the free surface.

4. The normal pressure $p_{\perp}$ allows to define the balance between the total normal force per unit area with the curvature of the surface, according to Laplace's formula

$$p_{\perp} + p_0 = \frac{T h_{xx}}{(1 + h_x^2)^{\frac{3}{2}}} \quad at \quad y = h, \tag{41}$$

where T defines the surface-tension coefficient.

The normal pressure $p_{\perp}$ is calculated by rotating of the cartesian stress tensor

$$\sigma_{ij} = -P\delta_{ij} + \rho\nu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{42}$$

In our case the components of the cartesian stress tensor are

$$\sigma_{11} = -P + 2\rho v\phi_{yx} \tag{43}$$

$$\sigma_{22} = -P - 2\rho v\phi_{yx} \tag{44}$$

$$\sigma_{12} = \rho\nu(\bar{u}_y + \phi_{yy} - \phi_{xx}) \tag{45}$$

hence, if we rotate the x and y axes in an angle ν, so that $\tan(\nu) = h_x$, then is easy to prove that

$$p_{\perp} = \sigma_{22}\cos^2(\nu) + \sigma_{11}\sin^2(\nu) - \sigma_{12}\sin(2\nu) \tag{46}$$

and

$$p_{||} = 0 = \sigma_{12}\cos(2\nu) + \frac{1}{2}(\sigma_{22} - \sigma_{11})\sin(2\nu) \tag{47}$$

substituting (43)- (45) into (46) and (47) we have

$$p_{\perp} = -P - \rho\nu\left(2\phi_{yx}\frac{1-h_x^2}{h_x^2+1} + (\bar{u}_y + \phi_{yy} - \phi_{xx})\frac{2h_x}{h_x^2+1}\right) \tag{48}$$

$$(\bar{u}_y + \phi_{yy} - \phi_{xx}) - \phi_{yx}\frac{4h_x}{1-h_x^2} = 0 \tag{49}$$

Finally, using the fact that $P = p^{'} + p_0 + \rho g(h_0 - y)\cos\theta$, we combine the equations (41) and(48), obtaining

$$T\left(\frac{h_{xx}}{(1+h_x^2)^{\frac{3}{2}}}\right) = -p^{'} - \rho g(h_0 - y)\cos\theta - \rho\nu\left(2\phi_{yx}\frac{1-h_x^2}{h_x^2+1} + (\bar{u}_y + \phi_{yy} - \phi_{xx})\frac{2h_x}{h_x^2+1}\right). \tag{50}$$

In this point is recommendable to rewrite the Navier-Stokes's equations (36)-(37) and the boundary conditions ((38),(39), (49),(50)) in dimensionless form. For this purpose we re-scale the independent variables (x,y,t) by

$$x \to L\hat{x} \quad y \to h_0\hat{y} \quad t = \frac{L}{\bar{u}_{max}}\hat{t}, \tag{51}$$

where $(\hat{x},\hat{y},\hat{t})$ are the new dimensionless variables, $\bar{u}_{max} = \frac{gh_0^2\sin\theta}{2\nu}$ is the unperturbed maximum velocity of the fluid, and $L = \frac{\lambda}{2\pi}$ defines a characteristic length, representing the inverse of the wave number. Also, we use the following substitutions

$$h(x,t) \to h(\hat{x},\hat{t}) = h_0\hat{h}(\hat{x},\hat{t}) \tag{52}$$

$$\bar{u}(y) \to \bar{u}(\hat{y}) - \bar{u}_{max}\hat{u}(\hat{y}) \quad where \quad \hat{u}(\hat{y}) = \hat{y}(2-\hat{y}) \tag{53}$$

$$\phi(x,y,t) \to \phi(\hat{x},\hat{y},\hat{t}) = h_0\bar{u}_{max}\Psi(\hat{x},\hat{y},\hat{t}) \tag{54}$$

$$\bar{p}(x,y,t) \to \bar{p}(\hat{x},\hat{y},\hat{t}) = \rho g h_0 \sin\theta\,\hat{p}(\hat{x},\hat{y},\hat{t}) \tag{55}$$

$$p^{'}(x,y,t) \to p^{'}(\hat{x},\hat{y},\hat{t}) = \rho g h_0 \sin\theta\,\hat{p}^{'}(\hat{x},\hat{y},\hat{t}) \tag{56}$$

$$p_0 = \rho g h_0 \sin\theta\,\hat{p}_0 \tag{57}$$

and we define the physical parameters

$$R = \frac{h_0\bar{u}_{max}}{\nu} \qquad W = \frac{T}{\rho g h_0^2 \sin\beta} \qquad \alpha = \frac{h_0}{L}, \tag{58}$$

where R is the Reynolds number, W the Weber number and α is the wave number in a distance $2\pi h_0$. After performing all the substitutions above mentioned, the equations (36),(37)and the boundary conditions(38), (39), (49)and (50) can be rewrite in dimensionless form, however, for simplicity of the notation, henceforth we use $\hat{x}\rightarrow x, \hat{y}\rightarrow y, \hat{z}\rightarrow z$

$$\Psi_{yyyy} = \alpha R(\Psi_{yyt} + \hat{u}\Psi_{yyx} + \Psi_{yyx}\Psi_y - \Psi_x\hat{u}_{yy} - \Psi_x\Psi_{yyy}) - 2\alpha^2\Psi_{xxyy}$$

$$+\alpha^3 R(\Psi_{xxt} + \hat{u}\Psi_{xxx} + \Psi_{xxx}\Psi_x + \Psi_{xx}\hat{u}_x - \Psi_x\Psi_{xxy}) - \alpha^4\Psi_{xxxx}, \quad (59)$$

$$\hat{p}'_x = \frac{1}{2\alpha}\Psi_{yyy} - \frac{R}{2}(\Psi_{yt} + (\hat{u}+\Psi_y)\Psi_{xy} - \Psi_x\hat{u}_y - \Psi_x\Psi_{yy}) + \frac{\alpha}{2}\Psi_{yxx}, \quad (60)$$

$$\Psi_x = 0 \quad and \quad \Psi_y = 0 \quad at \quad y = 0, \quad (61)$$

$$h_t = -\Psi_x - h_x(\hat{u}+\Psi_y) \quad at \quad y = h, \quad (62)$$

$$0 = (\hat{u} + \Psi_{yy} - \alpha^2\Psi_{xx})(1-\alpha^2 h_x^2) - 4\alpha^2\Psi_{yx}h_x \quad at \quad y = h, \quad (63)$$

$$\hat{p}' = -\frac{\alpha^2 W h_{xx}}{(1+\alpha^2 h_x^2)^{\frac{3}{2}}} - (1-h)\cot\theta - \alpha\frac{\Psi_{yx}(\alpha^2 h_x^2+1)}{(1-\alpha^2 h_x^2)} \quad at \quad y = h. \quad (64)$$

The case long waves of arbitrary amplitude can be investigated assuming that the wavelength is greater than the film thickness, this implies that α is small. For this reason, we assume a series solution to the above system will be expanded in terms of α as follows

$$\Psi_{x,y,t} = \sum_{n=0}\alpha^n\hat{\Psi}_{(n)} \quad (65)$$

$$\hat{p}'x,y,t = \sum_{n=0}\alpha^n\hat{p}'_{(n)} \quad (66)$$

with

$$\hat{\Psi}_{(n)} = \sum_{m=0}\hat{\Psi}_{(nm)}y^m \quad (67)$$

and

$$\hat{p}'_{(n)} = \sum_{m=0}\hat{p}'_{(nm)}y^m \quad (68)$$

where the coefficient $\hat{\Psi}_{(nm)}$ and $\hat{P}'_{(nm)}$ are determined by demanding that the expansion of $\Psi(x,y,t)$ satisfied the equations ((59)-(64)) order by order. In this work, we calculate $\Psi(s,z,\tau)$ to $O(\alpha^4)$, so that, the first three terms are

$$\hat{\Psi}_{(0)} = (h(x,t)-1)y^2 \quad (69)$$

$$\hat{\Psi}_{(1)} = \frac{y^2}{30}(10(y-3h)\cot\theta + hR(y^3 - 5hy^2 + 20h^3))h_x, \quad (70)$$

$$\hat{\Psi}_{(2)} = \frac{y^2}{5040}(W_{20}h_{xx} + W_{21}h_x^2) \quad (71)$$

where
$W_{20} = 1856R^2h^8 - 2352Rh^5\cot(\theta) + 12600h^2 - 1680hy$

$$+(-560R^2h^6+840R\cot(\theta)h^3-840)y^2+112R(h^5R-3h^2\cot(\theta))y^3 \\ +56R(Rh^4+\cot(\theta)h)y^4+8R(-4Rh^3-\cot(\theta))y^5+9R^2h^2y^6-R^2hy^7 \tag{72}$$

$$W_{21}=12160R^2h^7-8400R\cot(\theta)h^4+25200h-1680y+(2520h^2R\cot(\theta)-3360h^5R^2)y^2 \\ +(560h^4R^2-672hR\cot(\theta))y^3+224R^2h^3y^4-56R^2h^2y^5+9R^2hy^6-R^2y^7 \tag{73}$$

In third order we have

$$\hat{\Psi}_{(3)}=y^2(W_{30}h_{xxx}+W_{31}h_x^3+W_{32}h_xh_{xx}) \tag{74}$$

Where

$$W_{30}=Wh+\frac{6179R^3h^{12}}{31185}-\frac{2813R^2h^9}{11340}\cot(\theta)+\frac{221Rh^6}{90}-\frac{3h^3}{2}\cot(\theta) \\ -\left(\frac{7Rh^5}{90}-\frac{h^2}{6}\cot(\theta)+\frac{W}{3}\right)y-\left(\frac{58R^3h^{10}}{945}-\frac{7R^2h^7}{90}\cot(\theta)+\frac{19Rh^4}{36}-\frac{h}{6}\cot(\theta)\right)y^2 \\ +\left(\frac{58R^3h^9}{4725}-\frac{7R^2h^6}{450}\cot(\theta)+\frac{7Rh^3}{60}-\frac{\cot(\theta)}{30}\right)y^3+\left(\frac{R^3h^8}{135}-\frac{R^2h^5}{90}\cot(\theta)+\frac{Rh^2}{180}\right)y^4 \\ +\left(-\frac{4}{945}R^3h^7+\frac{R^2h^4}{126}\cot(\theta)-\frac{Rh}{420}\right)y^5+\left(\frac{R^3h^6}{1260}-\frac{R^2h^3}{336}\cot(\theta)+\frac{R}{2520}\right)y^6 \\ +\left(\frac{R^3h^5}{3780}+\frac{R^2h^2}{1512}\cot(\theta)\right)y^7+\left(-\frac{47R^3h^4}{226800}-\frac{R^2h}{11340}\cot(\theta)\right)y^8 \\ +\left(\frac{19h^3R^3}{311850}+\frac{R^2}{124740}\cot(\theta)\right)y^9-\frac{R^3h^2}{95040}y^{10}+\frac{R^3h}{1235520}y^{11}, \tag{75}$$

$$W_{31}=\frac{114R^3h^{10}}{7}-\frac{41}{5}R^2h^7\cot(\theta)-\frac{1}{3}Rh^4\cot^2(\theta)+49Rh^4-5h\cot(\theta) \\ +\frac{1}{9}\left(3\cot(\theta)-14h^3R\right)y-\frac{1}{315}Rh^2\left(1432R^2h^6-749Rh^3\cot(\theta)+1995\right)y^2 \\ +\frac{Rh}{1890}\left(1328R^2h^6-735R\cot(\theta)h^3+63\cot^2(\theta)+1008\right)y^3 \\ +\frac{1}{270}R\left(112R^2h^6-60R\cot(\theta)h^3+9\right)y^4-\frac{2}{945}R^2h^2\left(59h^3R-30\cot(\theta)\right)y^5 \\ +\frac{R^2h}{1680}\left(22h^3R-13\cot(\theta)\right)y^6+\frac{R^2}{45360}\left(48Rh^3+49\cot(\theta)\right)y^7 \\ -\frac{11R^3h^2}{10800}y^8+\frac{R^3h}{10800}y^9, \tag{76}$$

$$W_{32}=h^2\left(\frac{11908R^3h^9}{2079}-\frac{1}{5}R\cot^2(\theta)h^3+\frac{553Rh^3}{15}-\left(\frac{6031R^2h^6}{1260}+\frac{19}{2}\right)\cot(\theta)\right) \\ +h\left(\cot(\theta)-\frac{7h^3R}{6}\right)y+\left(-\frac{524}{315}R^3h^9-\frac{19Rh^3}{3}+\frac{1}{2}\left(\frac{127R^2h^6}{45}+1\right)\cot(\theta)\right)y^2$$

$$+\frac{h^2R}{5}\left(\frac{1352R^2h^6}{945}-\frac{56}{45}R\cot(\theta)h^3+\frac{cot^2(\theta)}{6}+\frac{53}{12}\right)y^3$$
$$+\frac{1}{15}hR\left(\frac{8R^2h^6}{3}-\frac{43}{18}R\cot(\theta)h^3-\frac{\cot^2(\theta)}{6}+1\right)y^4$$
$$+\frac{1}{63}R\left(-\frac{74}{15}R^2h^6+5R^2\cot(\theta)h^3+\frac{\cot^2(\theta)}{10}+\frac{1}{20}\right)y^5$$
$$+\frac{1}{84}h^2R^2\left(h^3R-\frac{101\cot(\theta)}{60}\right)y^6+\frac{hR^2}{11340}\left(\frac{61Rh^3}{2}+29\cot(\theta)\right)y^7$$
$$-\frac{31h^3R^3}{18900}y^8+\frac{229h^2R^3}{831600}y^9-\frac{hR^3}{31680}y^{10}+\frac{R^3}{411840}y^{11}. \quad (77)$$

Similarly, fourth order we can obtain W_{44}.

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