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On zero divisors graphs of direct product of finite fields

Subhash Mallinath Gaded[1] and Dr. Nithya Sai Narayana[2]

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Abstract

Zero divisor graphs are a fascinating area of current research. In this article, we look at some basic properties of zero divisor graphs of direct products of finite fields. We determine the girth, diameter, planarity, total domination number, connected domination number of the zero divisor graph as well as the complement graph of zero divisor graphs of direct products of finite fields.

Key words: zero divisor graphs, finite fields, diameter, planarity, domination number.

AMS classification: 13A70, 05C10, 05C12, 05C25, 05C69.

1 Introduction

Zero divisors graphs have been studied over general rings. In [7], I. Beck developed the idea of Zero Divisor graphs of a commutative ring R. A pair of vertices x and y are adjacent if and only if $x \cdot y = 0$, according to Beck's analysis of all the elements of the ring R as the vertices of the Zero divisor graph. Beck's definition of the zero divisor graph was modified by Anderson and Livingston ([6]), who considered the non-zero zero divisors of R to be the vertex set of the zero divisor graph of R, denoted by $\Gamma(R)$, and that two distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $x \cdot y = 0$. For more information on zero divisor graphs we refer to [1] [4], [9], [11], [12], [18], [19], [20]. Shane P. Redmond provided an approach in [16] that identifies (up to isomorphism) any commutative, reduced rings with 1 that give birth to a zero divisor graph on n vertices for any $n \geqslant 1$. Anderson and Livingston have demonstrated in [6],[Theorem 2.3] that if R is a commutative ring, then $\Gamma(R)$ is connected and diam$\Gamma(R) \leqslant 3$, and if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leqslant 7$. Additionally, Anderson and Livingston established [Theorem 2.4] that $gr(\Gamma(R)) \leqslant 4$ if R is a commutative Artinian ring and $\Gamma(R)$ contains a cycle. The authors D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston [5] also identified all n for

[1] Department of Mathematics, University of Mumbai, Mumbai, Maharashtra, India
Email:subhash.gaded@ssrkt.edu.in
[2] Department of Mathematics, University of Mumbai, Mumbai, Maharashtra, India.
Email:narayana_nithya@yahoo.com

which the planar zero divisor graph of $\mathbb{Z}_n$ exists and posed the problem of identifying the finite rings that do so. S. Akbari, H. R. Maimani, and S. Yassemi [3] were able to narrow down the question to local rings of cardinality at most thirty-two and provide a partial response. They demonstrated that $\Gamma(R)$ is not planar if R is a finite local ring that is not a field and has at least thirty-three members ([3]2003). The problem posed by Anderson, Frazier, Lauve, and Livingston, "For which finite commutative rings R is the zero divisor graph $\Gamma(R)$ planar?" is answered in [8]. Nader Jafari Rad, Sayyed Heidar Jafari, and Doost Ali Mojdeh have determined the domination number for the zero divisor graph of any commutative artinian ring in [15]. Pradeep Singh and Vijay Kumar Bhat provided a thorough overview of zero divisor graphs of finite commutative rings in [17].
We use the definitions of [21] and [10] for fundamental graph theoretical terms. The girth of a graph G refers to the shortest cycle length that graph G can contain. In the absence of any cycles, the graph's girth is assumed to be infinite. The shortest $x-y$ path in a graph G is measured by two vertices' distances, $d_G(x,y)$. We set $d_G(x,y)=\infty$ if there is no such path. The diameter of a graph G, indicated by $diam(G)$, is the largest distance between any two vertices in the graph. The subgraph of G that is caused by vertices of degree $\Delta(G)$, where $\Delta(G)$ specifies the greatest degree of G, is the core of G, indicated by G_{Δ}.[2]
Let $F_1,F_2,\cdots,F_n$, $(n\geqslant 2)$ be finite fields. Consider the reduced ring $R=F_1\times F_2\times\cdots\times F_n$, $(n\geqslant 2)$ of direct product of finite fields. In this paper we determine the girth, diameter, planarity, total domination number, connected domination number of the zero divisor graph as well as the complement graph of zero divisor graphs of direct products of finite fields. We also determine the domination number of the complement graph of zero divisor graphs of direct products of finite fields. Let $Z^*(R)$ be the set of non-zero zero divisors of the reduced ring $R=F_1\times F_2\times\cdots\times F_n$, $(n\geqslant 2)$ and $\Gamma(R)$ denote the graph with vertex set as $Z^*(R)$ and edge set as $\{rs : r\cdot s=0, r,s\in Z^*(R)\}$.
Since $Z(R)$ is closed under multiplication, the complement graph $\overline{\Gamma(R)}$ of the zero divisor graph $\Gamma(R)$ satisfies the property: $rs\in E(\overline{\Gamma(R)})$ if and only if $r\cdot s\in Z^*(R)$.

2 Zero divisor graphs of direct product of finite fields

If F is a finite field of order p, we label the elements of F as $\{0,1,2,\cdots,p-1\}$.

Theorem 2.1 If $\Gamma(R)$ is the zero divisor graph of the reduced ring,

$R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 2)$, of direct product of n finite fields, then,
$|Z^*(R)| = |V(\Gamma(R))| = \prod_{i=1}^{n} |F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1$.

Proof: $|\ |R| = \prod_{i=1}^{n} |F_i|$. The number of elements of R with non-zero entries in all the n co-ordinate positions is $\prod_{i=1}^{n}(|F_i| - 1)$ and clearly these elements are not zero divisors of R. Also, the additive identity of R will not be included in the vertex set of $\Gamma(R)$ as we are considering the zero divisor graph by Anderson and Livingston. Thus, the number of non-zero zero divisors of R is
$|Z^*(R)| = |V(\Gamma(R))| = \prod_{i=1}^{n} |F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1$.

Theorem 2.2 In a zero divisor graph $\Gamma(R)$, of the reduced ring $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$, where F_i are finite fields with $|F_1| = |F_2| = \cdots = |F_n| = p$, the number of vertices of degree $p^r - 1$ is $(p-1)^{n-r}C(n,r)$ for $1 \leq r \leq n-1$.

Proof: Consider a non-zero zero divisor of the Semi-local ring $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$, with $|F_1| = |F_2| = \cdots = |F_n|$. This element can have zeros in $1, 2, \cdots, n$ coordinate positions. Consider a zero divisor $(0,0,\cdots,0,*,*,\cdots,*)$ with exactly r zeros in the first r positions. It will be adjacent with a vertex with zeros in positions $r+1$ to n and we count the vertices adjacent to $(0,0,\cdots,0,*,*,\cdots,*)$. We observe that number of vertices with non-zero elements in l positions out of r positions is $C(r,l)(p-1)^l$ and hence the total number of vertices which are adjacent with a vertex with r zeros in first r position is
${}^rC_1(p-1) + {}^rC_2(p-1)^2 + \cdots + {}^rC_r(p-1)^r$
$= {}^rC_0(p-1)^0 + {}^rC_1(p-1) + {}^rC_2(p-1)^2 + \cdots + {}^rC_r(p-1)^r - {}^rC_0(p-1)^0$
$= ((p-1)+1)^r - 1 = p^r - 1 \Rightarrow$ Degree of such vertex is $p^r - 1$.
The choices for non-zero entries is $(p-1)^{n-r}$ and the choices for zero positions is $C(n,r)$. Therefore, total number of vertices of degree $p^r - 1$ is $(p-1)^{n-r}C(n,r)$, for $1 \leqslant r \leqslant n-1$.

Corollary 2.3 If $|F_1| = |F_2| = \cdots = |F_n| = p$, then the number of vertices of degree $p^n - (p-1)^n - p^r - 1$, $(1 \leqslant r \leqslant n-1)$ in $\overline{\Gamma(R)}$ is $C(n,r)(p-1)^r$.

Corollary 2.4 The zero divisor graph $\Gamma(R)$ of the reduced ring $R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 2)$ contains induced subgraph complete graph K_n and $\overline{\Gamma(R)}$ contains induced subgraph, complete graph K_n if $n \geqslant 3$.

Proof: The subgraph spanned by $n-$tuple vertices containing $'1'$ in the i^{th} co-ordinate position $(1 \leqslant i \leqslant n)$ and $'0'$ in the remaining co-ordinate positions is complete graph K_n in $\Gamma(R)$. The subgraph spanned by $n-$tuple vertices $(n \geqslant 3)$ containing $'0'$ in the i^{th} co-ordinate position $(1 \leqslant i \leqslant n)$ and $'1'$ in the remaining co-ordinate positions is complete graph K_n in $\overline{\Gamma(R)}$.

Corollary 2.5 Let $R = F_1 \times F_2$ with $|F_1| = m_1$, $|F_2| = m_2$. Then

(i) $\Gamma(R)$ is Complete bipartite graph K_{m_1-1,m_2-1}.

(ii) $\overline{\Gamma(R)}$ is disconnected graph with two copies of complete graph K_{m_1-1} and K_{m_2-1}.

Proof:

(i) The vertices of $\Gamma(R)$ are 2-tuple zero divisors
$\{(1,0),(2,0),\cdots,(m_1-1,0),(0,1),(0,2),\cdots,(0,m_2-1)\}$.
Let $V_1 = \{(1,0),(2,0),\cdots,(m_1-1,0)\}$
and $V_2 = \{(0,1),(0,2),\cdots,(0,m_2-1)\}$.
Then, $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \emptyset$.
Moreover, every vertex in V_1 is adjacent with each and every vertex in V_2 and vice-versa. Also, no two vertices in $V_i,(i=1,2)$ are adjacent. Thus, $\Gamma(R)$ is Complete bipartite graph K_{m_1-1,m_2-1} if $R = F_1 \times F_2$ with $|F_1| = m_1$, $|F_2| = m_2$.

(ii) In $\overline{\Gamma(R)}$ the vertices in V_i, $i = 1,2$ are all mutually adjacent with each other. Since, $(x,0)\cdot(0,y) \notin Z(R)^*, x \in F_1, y \in F_2$, the vertices in V_1 are not adjacent with vertices in V_2. Thus, $\overline{\Gamma(R)}$ is disconnected graph with two copies of Complete graph K_{m_1-1} and K_{m_2-1}.

2.1 Girth, Diameter, Planarity

Theorem 2.6 Let $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$, then

(i) if $n = 2$ with $|F_1| \geqslant 3$, $|F_2| \geqslant 3$ then girth of $\Gamma(R)$ is 4, diameter of $\Gamma(R)$ is 2

(ii) girth and diameter of $\overline{\Gamma(R)}$ is ∞.

(iii) if $n \geqslant 3$ girth of $\Gamma(R)$ and $\overline{\Gamma(R)}$ is 3.

(iv) if $n \geqslant 3$ diameter of $\Gamma(R)$ is 3.

(v) $\overline{\Gamma(R)}$ is connected and diameter of $\overline{\Gamma(R)}$ is 2.

Proof:

(i) If $n=2$, then by theorem, 2.5, $\Gamma(R)$ is complete bipartite graph. Then the cycle $(1,0),(0,1),(2,0),(0,2),(1,0)$ is of minimum length 4 in $\Gamma(R)$. Therefore, girth of $\Gamma(R)=4$ and diameter of $\Gamma(R)$ is 2.

(ii) $\overline{\Gamma(R)}$ is disconnected graph with two copies of Complete graph K_{m_1-1} and K_{m_2-1}. Therefore, girth and diameter of $\overline{\Gamma(R)}$ is ∞.

(iii) If $n\geqslant 3$, then the cycle $(1,0,\cdots,0),(0,1,\cdots,0),(0,0,\cdots,1),(1,0,\cdots,0)$ in $\Gamma(R)$ is of length 3. Therefore, girth of $\Gamma(R)$ is 3. And the cycle $(1,0,\cdots,0),(1,1,\cdots,0),(1,0,\cdots,1),(1,0,\cdots,0)$ in $\overline{\Gamma(R)}$ is of length 3. Therefore, girth of $\overline{\Gamma(R)}$ is 3.

(iv) By [6](Theorem 2.3), any zero divisor graph G is connected and $diam(G)\leq 3$. Consider $R=F_1\times F_2\times\cdots\times F_n$, $n\geqslant 3$.
Let $v_1=(1,1,\cdots,1,0), v_2=(0,1,\cdots,1,1)$. Then v_1 is adjacent to $x=(0,0,\cdots,0,1)$ and $y=(1,0,\cdots,0,0)$ is adjacent to v_2. Also x is adjacent to y. This indicates that $d(v_1,v_2)=3\Rightarrow diam(G)=3$.

(v) Let $n\geqslant 3$. Let u, $v\in V(\overline{\Gamma(R)})$. If $(u,v)\in E(\overline{\Gamma(R)})$, then $d(u,v)=1$. Suppose, $(u,v)\notin E(\overline{\Gamma(R)})$. Then u and v do not contain non-zero entries in the same position. Let u contains $t_i\in\{1,2,\cdots,m_i-1\}$ in the i^{th} position and v contains $t_j\in\{1,2,\cdots,m_j-1\}$ in the j^{th} position. Clearly, $i\neq j$. Let $z\in V(\overline{\Gamma(R)})\setminus\{u,v\}$ contain non-zero entry in the i^{th} and j^{th} position and $'0'$ elsewhere. Then, clearly, $(u,z),(v,z)\in E(\overline{\Gamma(R)})$. Therefore, $u-z-v$ is a path connecting u and v. Thus, $d(u,v)=2$. Hence, $\overline{\Gamma(R)}$ is connected and diameter of $\overline{\Gamma(R)}$ is 2.

Corollary 2.7

(i) If $R=F_1\times F_2$ with $|F_1|=2, |F_2|=p\geqslant 2$, then $\Gamma(R)$ is a star graph and hence is planar.

(ii) If $R=F_1\times F_2$ with $|F_1|=3, |F_2|\geqslant 3$, then $\Gamma(R)$ is a planar bipartite graph.

(iii) If $n=3$, then $\Gamma(R)$ is planar if $|F_1|=|F_2|=|F_3|=2$.

(iv) $\overline{\Gamma(R)}$ is planar graph if $n = 2$, $|F_1|$, $|F_2| =$ 2 *or* 3 *or* 5.

(v) By [8], $\Gamma(R)$ is not planar if
(a) $n = 2$, 3, $|F_1| \neq 2, 3$
(b) if $n \geqslant 4$.

Corollary 2.8 Let $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$. Then $\overline{\Gamma(R)}$ is not planar

(i) if $n = 2$ and $|F_1| = 2, 3, 5$, $|F_2| > 5$ or $|F_1| > 5$.

(ii) if $n \geq 3$ and order of at least two fields are distinct.
Proof:

(i) If $n = 2$ and $|F_1| = 2$, 3, 5, $|F_2| > 5$ or $|F_1| > 5 \implies |F_2| \geqslant 7$. Therefore, the subgraph induced by $S = \{(0,1), (0,2), (0,3), (0,4), (0,5)\}$ is complete graph K_5 contained in $\overline{\Gamma(R)}$. Hence $\overline{\Gamma(R)}$ is not planar in this case.

(ii) Let $n \geqslant 3$. Let $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$. Since, order of at least two fields are distinct $\implies |F_3| \geqslant 3$. Then the subgraph induced by
$S = \{(1,1,0,\cdots,0), (1,0,1,\cdots,0), (1,0,2,\cdots,0), (0,1,1,\cdots,0), (0,1,2,\cdots,0)\}$
contains K_5 and hence $\overline{\Gamma(R)}$ is not planar if $n \geqslant 3$.

3 Total domination Number, Connected domination number

"A dominating set for the graph $G = (V, E)$ is a subset D of V such that each vertex outside of D is adjacent to at least one member of D. The set of all vertices adjacent to a vertex $v \in V$ is the open neighborhood of v denoted by $N(v)$. The closed neighborhood of v is the set of all vertices adjacent to v along with v. For a set $S \subset V, N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The set S is a dominating set of G if $N[S] = V$, and a total dominating set of G (or just TDS) if $N(S) = V$. The domination number, represented by $\gamma(G)$, is the minimum cardinality of a dominating set of G, and the least cardinality of a TDS of G is the total domination number, denoted by $\gamma_{t(G)}$. The subgraph induced by $S \subseteq V$ is denoted by $G[S]$. If S is a dominating set and $G[S]$ is connected, then S is referred to as a connected dominating set. The lowest cardinality of a connected dominating set is represented by the connected domination number of G, denoted by $c(G)$." [13], [14] [15]

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Remark 3.1

(i) If the ring $R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 2)$, is direct product of n finite fields, then by the corollary 2.11, [15], the domination number of the zero divisor graph is $\gamma(\Gamma(R)) = n$.

(ii) If R is an integral domain, then the connected domination number $c(\Gamma(\mathbb{Z}_2 \times R)) = 1$ and the total domination number $\gamma_t(\mathbb{Z}_2 \times R)) = 2$. (Proposition 2.2, [15])

Theorem 3.2 If $R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 2)$, then the total domination number $\gamma_t(\Gamma(R)) = n$ and connected domination number $c(\Gamma(R)) = n$.
Proof: Using the corollary 2.11, [15], the domination number of the zero divisor graph is $\gamma(\Gamma(R)) = n$. Let $n \geqslant 3$. Let $D = \{x_1, x_2, \cdots, x_n\}$ where x_i contains $'1'$ in the i^{th} co-ordinate position and $'0'$ in remaining co-ordinate positions. Let $z \in V(\Gamma(R)) \setminus D$. Since, z contains at least one $'0'$, suppose z contains $'0'$ in i^{th} co-ordinate position, then z is adjacent to $x_i \in D$ containing $'1'$ in the i^{th} co-ordinate position and $'0'$ in remaining co-ordinate positions. Thus, every vertex not in D is adjacent to at least one member of D. Therefore, D is dominating set and $\gamma(\Gamma(R)) = n = |D|$.
Also, $N(D) = V(\Gamma(R))$. Therefore, D is total dominating set, and thus $\gamma_t(\Gamma(R)) = n$. $G(D)$ is complete graph K_n(Corollary 2.4). Hence, the connected domination number, $c(\Gamma(R)) = n$.

Remark 3.3 If $R = F_1 \times F_2$, then clearly the set $D = \{(1,0),(0,1)\}$ is dominating set of $\overline{\Gamma(R)}$. Since, $\overline{\Gamma(R)}$ is disconnected set, any singleton subset of $V(\overline{\Gamma(R)})$ can not be a dominating set. Thus, the domination number $\gamma(\overline{\Gamma(R)}) = 2$ and the total dominating set $\gamma_t(\overline{\Gamma(R)}) = 2$ and the connected dominating set $c(\overline{\Gamma(R)})$ does not exist.

Theorem 3.4 If $R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 3)$, then the domination number $\gamma(\overline{\Gamma(R)}) = n$, the total domination number $\gamma_t(\overline{\Gamma(R)}) = n$ and connected domination number $c(\overline{\Gamma(R)}) = n$.
Proof: Let $D' = \{y_1, y_2, \cdots, y_n\}$ where y_i contains $'0'$ in the i^{th} co-ordinate position and $'1'$ in remaining co-ordinate positions. Let $z \in V(\overline{\Gamma(R)}) \setminus D'$. Since, z contains at least one $'0'$, suppose z contains $0's$ in positions $i_1, \cdots, i_r$ and non-zero entries in

positions $j_1, \cdots, j_{n-r}$, where $i_1, \cdots, i_r \neq j_1, \cdots, j_{n-r}$. Then z is adjacent to $y_{j_1}, \cdots, y_{j_{n-r}}$ in D'. Thus, every vertex not in D' is adjacent to at least one member of D'. Therefore, D' is dominating set $\implies \gamma(\overline{\Gamma(R)}) \leqslant n = |D|$.

Let $S \subset V(\overline{\Gamma(R)})$ with $|S| = n-1$. We claim that S is not a dominating set in $\overline{\Gamma(R)}$.

Case (i) $S \cap D' \neq \emptyset$. If $S \subset D'$ then $D' \setminus S = \{y_j\}$ for some $1 \leqslant j \leqslant n$. Consider a vertex x with a $'1'$ in the j^{th} co-ordinate position and 0 entries in the remaining co-ordinate position. Clearly, x is not adjacent with any vertex in S. Therefore S is not a dominating set in this case.

Case(ii) $S \cap D' \neq \emptyset$ but $S \not\subset D'$. We have $|S| = n-1$ and $|D' \cap S| = k$ where $1 \leqslant k \leqslant n-2$. Let $y_{i_1}, y_{i_2}, \cdots, y_{i_k} \in D' \cap S$ with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Let $y_{k+1}, \cdots, y_{n-1}$ be the remaining vertices in S. The vertices $y_{k+1}, \cdots, y_{n-1}$ may all contain exactly one 0 entry or some or none of them may contain exactly one 0 entry. Let j_s be the number of vertices with exactly one 0 entry where $0 \leqslant s \leqslant (n-1)-k$. Since, $|S| = n-1$, there exists a set of vertices with exactly one 0 entry in the t^{th} co-ordinate position where $t \neq i_1, i_2, \cdots, i_k, j_1, \cdots, j_s$ and non-zero entries in remaining co-ordinate positions. Consider a vertex $x \notin S$ with a non-zero entry in the t^{th} co-ordinate position and 0 entries in the remaining co-ordinate position. Clearly, x is not adjacent with any vertex in S. Therefore S is not a dominating set in this case.

Case (iii) Suppose $S \cap D' = \emptyset$. The vertices of S may all contain exactly one 0 entry or some or none of them may contain exactly one 0 entry, and since $|S| = n-1$, there exists a set of vertices with exactly one 0 entry in j^{th} position and non-zero in remaining co-ordinate positions and this set of vertices does not belong to S. Then there exists a vertex $x \notin S$ with a non-zero entry in the j^{th} co-ordinate position and 0 entries in the remaining co-ordinate positions. Clearly, x is not adjacent with any vertex in S. Therefore, S is not a dominating set in this case. Thus, $\gamma(\overline{\Gamma(R)}) \geqslant n$.

Hence, $\gamma(\overline{\Gamma(R)}) = n$.

Also, $N(D') = V(\overline{\Gamma(R)})$. Therefore, D' is total dominating set, and thus $\gamma_t(\overline{\Gamma(R)}) = n$.

$G(D')$ is complete graph K_n(Corollary 2.4). Hence, the connected domination number, $c(\overline{\Gamma(R)}) = n$.

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