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Statistical $\triangle^{\mathfrak{M}}$ - Convergence in Neutrosophic Normed Space

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Abstract

Characterizing Statistical $\Delta^{\mathfrak{M}}$-Convergence in the setup of Neutrosophic Normed Space for Double sequences is the main objective of our study in the current paper. We have provided a few instances that demonstrate how generalized this approach of convergence is. Additionally, we described the Statistical $\Delta^{\mathfrak{M}}$- Cauchy sequence and Cauchy convergence criterion that exist in the Neutrosophic Normed space.

Key words: Statistical $\Delta^{\mathfrak{M}}$- Convergence, Statistical $\Delta^{\mathfrak{M}}$ - Cauchy sequence, Double sequence, Neutrosophic Normed Space.

AMS classification: 40A35; 26E50.

1 Introduction

Fast [7] introduced a new, more expansive idea of convergence in 1951 under the name Statistical Convergence ($\mathcal{SC}$). Numerous mathematical disciplines, including topology, fourier analysis, measure theory, approximation theory, etc., have fascinating applications for it. After the first investigation of Zadeh [19], an immense of research works have appeared on fuzzy theory and its applications. Fuzzy Sets ($\mathcal{FS}$) have found widespread use across a wide range of fields and technology. The theory of Intuitionistic Fuzzy Sets ($\mathcal{IFS}$) was presented by Atanassov [3]. A substantial amount of research has been done using the $\mathcal{FS}$ and $\mathcal{IFS}$, notably in the decision-making domain, on a wide range of challenging problems related to many disciplines. Intuitionistic Fuzzy Metric Space ($\mathcal{IFMS}$) was investigated by Park [16]. In [12], motivated by Park's definition of an $\mathcal{IFMS}$, Lael and Nourouzi first defined an $\mathcal{IFNS}$. In $\mathcal{IFS}$, membership degrees are described with a pair of

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a membership degree and a non-membership degree. Karakus [11] and Mursaleen [14], studied the notion of Statistical Convergence ($\mathscr{SC}$) and lacunary statistical convergence in intuitionistic fuzzy normed space respectively. Smarandache [17] proposed Neutrosophic Sets ($\mathscr{NS}$) as a development of the $\mathscr{IFS}$. For the situation when the aggregate of the components is 1, in the wake of satisfying the condition by applying the neutrosophic set operators, various results can be acquired by applying the intuitionistic fuzzy operators, whereas the neutrosophic operators are taken into the cognizance of the indeterminacy at a degree akin to truth-membership and falsehood-non membership, the operators disregard the indeterminacy. Jeyaraman et al. [9] developed approximate fixed point theorems for weak contractions on neutrosophic normed spaces ($\mathscr{NNS}$) in 2022. In the present paper, our aim is to discuss Statistical $\Delta^{\mathfrak{M}}$- Convergence in $\mathscr{NNS}$. In this connection, we put forward the notion of Statistical $\Delta^{\mathfrak{M}}$- Cauchy sequence and Cauchy convergence criterion that exist in the Neutrosophic Normed space($\mathscr{NNS}$).

2 Preliminaries

Definition 2.1 [9] The 7-tuple $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ is said to be $\mathscr{NNS}$ if $\mathscr{M}$ is a linear space, $\star$ is a continuous t-norm, $\Diamond$ and $\circledast$ are continuous t-conorm, φ, ν and ω are fuzzy sets on $\mathscr{M} \times (0, \infty)$fulfils the following conditions:
For every $x, \mathscr{Y} \in \mathscr{M}$ and $s, \lambda > 0$;

(a) $0 \leq \varphi(x, \lambda) \leq 1; 0 \leq \nu(x, \lambda) \leq 1; 0 \leq \omega(x, \lambda) \leq 1$,
(b) $\varphi(x, \lambda) + \nu(x, \lambda) + \omega(x, \lambda) \leq 3$,
(c) $\varphi(x, \lambda) > 0$,
(d) $\varphi(x, \lambda) = 1$ if and only if $x = 0$,
(e) $\varphi(\alpha x, \lambda) = \varphi\left(x, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(f) $\varphi(x, \lambda) \star \varphi(\mathscr{Y}, s) \leq \varphi(x + \mathscr{Y}, \lambda + s)$,
(g) $\varphi(x, \lambda) : (0, \infty) \to [0, 1]$ is continuous,
(h) $\lim_{\lambda \to \infty} \varphi(x, \lambda) = 1$ and $\lim_{\lambda \to 0} \varphi(x, \lambda) = 0$,
(i) $\nu(x, \lambda) < 1$,
(j) $\nu(x, \lambda) = 0$ if and only if $x = 0$,
(k) $\nu(\alpha x, \lambda) = \nu\left(x, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(l) $\nu(x, \lambda) \Diamond \nu(\mathscr{Y}, s) \geq \nu(x + \mathscr{Y}, \lambda + s)$,
(m) $\nu(x, \lambda) : (0, \infty) \to [0, 1]$ is continuous,
(n) $\lim_{\lambda \to \infty} \nu(x, \lambda) = 0$ and $\lim_{\lambda \to 0} \nu(x, \lambda) = 1$,

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(o) $\omega(x,\lambda) < 1$,
(p) $\omega(x,\lambda) = 0$ if and only if $x = 0$,
(q) $\omega(\alpha x,\lambda) = \omega\left(x, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(r) $\omega(x,\lambda) \circledast \nu(\mathscr{Y}, s) \geq \omega(x+\mathscr{Y}, \lambda+s)$
(s) $\omega(x,\lambda) : (0,\infty) \to [0,1]$ is continuous,
(t) $\lim_{\lambda\to\infty} \omega(x,\lambda) = 0$ and $\lim_{\lambda\to 0} \omega(x,\lambda) = 1$,

Then (φ,ν,ω) is known as Neutrosophic Normed Space $[\mathcal{NNS}]$.

Example 2.2 [9] Let $(\mathscr{M}, \|\circ\|)$ be any $\mathcal{NS}$. For every $\lambda > 0$ and all $y \in \mathscr{M}$, take $\varphi(y,\lambda) = \frac{\lambda}{\lambda+\|y\|}$, $\nu(y,\lambda) = \frac{\|y\|}{\lambda+\|y\|}$ and $\omega(y,\lambda) = \frac{\|y\|}{\lambda}$.
Also, $a \star b = ab, a\Diamond b = \min\{a+b, 1\}$ and $a \circledast b = \min\{a+b, 1\}$, for all $a, b \in [0,1]$. Then, a 7-tuple $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ is an $\mathcal{NNS}$ which fulfills the above mentioned conditions.

Definition 2.3 Let $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be $\mathcal{NNS}$ with norm (φ,ν,ω). A sequence $x = (x_k)$ in $\mathscr{M}$ is called convergent to some $\xi \in \mathscr{M}$ with respect to the $\mathcal{NN}(\varphi,\nu,\omega)$ if there exists $k_0 \in \mathbb{N}$ for each $\varepsilon > 0$ and $\lambda > 0$ such that
$\varphi(x_k - \xi, \lambda) > 1-\varepsilon, \nu(x_k-\xi,\lambda) < \varepsilon$ and $\omega(x_k-\xi,\lambda) < \varepsilon$, for all $k \geq k_0$. It is denoted by $(\varphi,\nu,\omega) - \lim_{k\to\infty} x_k = \xi$.

Definition 2.4 Let $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ,ν,ω). A sequence $x = (x_k)$ in $\mathscr{M}$ is called Statistical Convergence $(\mathcal{SC})$ to some $\xi \in \mathscr{M}$ with respect to the $\mathcal{NN}(\varphi,\nu,\omega)$ if for each $\varepsilon > 0$ and $\lambda > 0$,

$$\delta\left(\{k \in \mathbb{N} : \varphi(x_k-\xi,\lambda) \leq 1-\varepsilon \,\text{or}\, \nu(x_k-\xi,\lambda) \geq \varepsilon, \omega(x_k-\xi,\lambda) \geq \varepsilon\}\right) = 0.$$

It is denoted by $\mathcal{S}^{(\varphi,\nu,\omega)} - \lim_{k\to\infty} x_k = \xi$. A Double Sequence $(\mathscr{DS})$ $x = (x_{jk})\mathcal{SC}$ to ξ, if double natural density of $E(\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : |x_{jk} - \xi| > \varepsilon\}$ is zero. It is denoted by $\mathcal{S}_2 - \lim_{j,k\to\infty} x_{jk} = \xi$.

Definition 2.5 Let $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ,ν,ω). A $\mathscr{DS}$ $x = (x_{jk})$ in $\mathscr{M}$ is called $\mathcal{SC}$ to some $\xi \in \mathscr{M}$ with respect to the $\mathcal{NN}(\varphi,\nu,\omega)$ if for each $\varepsilon > 0$ and $\lambda > 0$,

$$\delta_2\left(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \varphi(x_{jk}-\xi,\lambda) \leq 1-\varepsilon \,\text{or}\, \nu(x_{jk}-\xi,\lambda) \geq \varepsilon, \omega(x_{jk}-\xi,\lambda) \geq \varepsilon\}\right) = 0.$$

It is denoted by $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} x_{jk} = \xi$.

3 Main Results

In this article, we explain the Statistical $\Delta^{\mathfrak{M}}$ - Convergence and related ideas as indicated by the arrangement of $\mathcal{NNS}$. First we define the following terms.

Definition 3.1 Let $(\mathcal{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ, ν, ω). A $\mathcal{DS}$ $x = (x_{jk})$ in $\mathcal{M}$ is called $\Delta^{\mathfrak{M}} - \mathcal{SC}$ to some $\xi \in \mathcal{M}$ with respect to the $\mathcal{NNS}(\varphi, \nu, \omega)$ if for each $\varepsilon > 0$ and $\lambda > 0$,

$$\delta_2\left(\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)\leq 1-\varepsilon \text{ or}\\ \nu(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)\geq\varepsilon, \omega(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)\geq\varepsilon\end{array}\right\}\right) = 0.$$

It is denoted by $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$.

Definition 3.2 Let $(\mathcal{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ, ν, ω). A $\mathcal{DS}$ $x = (x_{jk})$ in $\mathcal{M}$ is called $\Delta^{\mathfrak{M}}$- Statistical Cauchy ($\mathcal{SC}a$) with respect to the $\mathcal{NNS}(\varphi, \nu, \omega)$ if there exists $j_0, k_0 \in \mathbb{N}$ for each $\varepsilon > 0$ and $\lambda > 0$ such that for all $j, r \geq j_0$ and $k, s \geq k_0$, we have

$$\delta_2\left(\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}}x_{jk}-\Delta^{\mathfrak{M}}x_{rs},\lambda)\leq 1-\varepsilon \text{ or}\\ \nu(\Delta^{\mathfrak{M}}x_{jk}-\Delta^{\mathfrak{M}}x_{rs},\lambda)\geq\varepsilon, \omega(\Delta^{\mathfrak{M}}x_{jk}-\Delta^{\mathfrak{M}}x_{rs},\lambda)\geq\varepsilon\end{array}\right\}\right) = 0.$$

Lemma 3.3 Let $(\mathcal{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ, ν, ω). Then the following statements are equivalent for $\mathcal{DS}$ $y = (y_{jk})$ in $\mathcal{M}$ whenever $\varepsilon > 0$ and $\lambda > 0$,

1. $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$,
2. $\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda) > 1-\varepsilon\right\}\right)$
   $= \delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \nu(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda) < \varepsilon\right\}\right)$
   $= \delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \omega(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda) < \varepsilon\right\}\right)$
   $= 1$,
3. $\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda) \leq 1-\varepsilon\right\}\right)$

$= \delta_2 \left(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \geq \varepsilon\}\right)$
$= \delta_2 \left(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \geq \varepsilon\}\right)$
$= 0,$

4. $\mathcal{S}_2 - \lim_{j,k\to\infty} \varphi\left(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda\right) = 1, \mathcal{S}_2 - \lim_{j,k\to\infty} \nu\left(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda\right) = 0$ and $\mathcal{S}_2 - \lim_{j,k\to\infty} \omega\left(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda\right) = 0.$

Theorem 3.4 Let $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ, ν, ω). If $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$, then the limit ξ is unique.

Proof: Suppose that $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi_1$ and $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi_2$. For given $\varepsilon \in (0,1)$ and $\lambda > 0$, take $\rho > 0$ such that $(1-\rho) \star (1-\rho) > 1 - \varepsilon, \rho \Diamond \rho < \varepsilon$ and $\rho \circledast \rho < \varepsilon$. Consider

$$\begin{aligned}
\mathscr{K}_{1,\varphi}(\rho,\lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \varphi(\Delta^{\mathfrak{M}} x_{jk} - \xi_1, \lambda/2) \leq 1 - \rho\}, \\
\mathscr{K}_{2,\varphi}(\rho,\lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \varphi(\Delta^{\mathfrak{M}} x_{jk} - \xi_2, \lambda/2) \leq 1 - \rho\}, \\
\mathscr{K}_{3,\nu}(\rho,\lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\Delta^{\mathfrak{M}} x_{jk} - \xi_1, \lambda/2) \geq \rho\}, \\
\mathscr{K}_{4,\nu}(\rho,\lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\Delta^{\mathfrak{M}} x_{jk} - \xi_2, \lambda/2) \geq \rho\}, \\
\mathscr{K}_{5,\omega}(\rho,\lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\Delta^{\mathfrak{M}} x_{jk} - \xi_1, \lambda/2) \geq \rho\} \text{ and} \\
\mathscr{K}_{6,\omega}(\rho,\lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\Delta^{\mathfrak{M}} x_{jk} - \xi_2, \lambda/2) \geq \rho\}.
\end{aligned}$$

Using Lemma (3.3) we have

$$\begin{aligned}
\delta_2(\mathscr{K}_{1,\varphi}(\rho,\lambda)) = \delta_2(\mathscr{K}_{3,\nu}(\rho,\lambda)) = \delta_2(\mathscr{K}_{5,\omega}(\rho,\lambda)) = 0. \\
\delta_2(\mathscr{K}_{2,\varphi}(\rho,\lambda)) = \delta_2(\mathscr{K}_{4,\nu}(\rho,\lambda)) = \delta_2(\mathscr{K}_{6,\omega}(\rho,\lambda)) = 0.
\end{aligned}$$

Let $\mathscr{K}_{\varphi,\nu,\omega}(\rho,\lambda) = \left\{ \begin{array}{c} [\mathscr{K}_{1,\varphi}(\rho,\lambda) \cup \mathscr{K}_{2,\varphi}(\rho,\lambda)] \cap [\mathscr{K}_{3,\nu}(\rho,\lambda) \cup \mathscr{K}_{4,\nu}(\rho,\lambda)] \cap \\ [\mathscr{K}_{5,\omega}(\rho,\lambda) \cup \mathscr{K}_{6,\omega}(\rho,\lambda)] \end{array} \right\}$
Clearly, $\delta_2(\mathscr{K}_{\varphi,\nu,\omega}(\rho,\lambda)) = 0$. Whenever $(j,k) \in \mathbb{N} \times \mathbb{N} - \mathscr{K}_{\varphi,\nu,\omega}(\rho,\lambda)$, we have three possibilities, either $(j,k) \in \mathbb{N} \times \mathbb{N} - [\mathscr{K}_{1,\varphi}(\rho,\lambda) \cup \mathscr{K}_{2,\varphi}(\rho,\lambda)]$ or
$(j,k) \in \mathbb{N} \times \mathbb{N} - [\mathscr{K}_{1,\nu}(\rho,\lambda) \cup \mathscr{K}_{2,\nu}(\rho,\lambda)], (j,k) \in \mathbb{N} \times \mathbb{N} - [\mathscr{K}_{1,\omega}(\rho,\lambda) \cup \mathscr{K}_{2,\omega}(\rho,\lambda)]$.
First we consider $(j,k) \in \mathbb{N} \times \mathbb{N} - [\mathscr{K}_{1,\varphi}(\rho,\lambda) \cup \mathscr{K}_{2,\varphi}(\rho,\lambda)]$.

Then

$$\varphi\left(\xi_1-\xi_2,\lambda\right) \geq \varphi\left(\Delta^{\mathfrak{M}}x_{jk}-\xi_1,\frac{\lambda}{2}\right)\star\varphi\left(\Delta^{\mathfrak{M}}x_{jk}-\xi_2,\frac{\lambda}{2}\right)$$
$$> (1-\rho)\star(1-\rho) > 1-\varepsilon.$$

As given $\varepsilon\in(0,1)$ was arbitrary, then $\varphi(\xi_1-\xi_2,\lambda)=1$, for all $\lambda>0$, then $\xi_1=\xi_2$.
Secondly, if $(j,k)\in\mathbb{N}\times\mathbb{N}-[\mathscr{K}_{3,\nu}(\rho,\lambda)\cup\mathscr{K}_{4,\nu}(\rho,\lambda)]$,

$$\nu(\xi_1-\xi_2,\lambda)\leq\nu\left(\Delta^{\mathfrak{M}}x_{jk}-\xi_1,\frac{\lambda}{2}\right)\Diamond\nu\left(\Delta^{\mathfrak{M}}x_{jk}-\xi_2,\frac{\lambda}{2}\right)<\rho\Diamond\rho<\varepsilon.$$

Since $\varepsilon\in(0,1)$ was arbitrary, then $\nu(\xi_1-\xi_2,\lambda)=0$, for all $\lambda>0$, i.e., $\xi_1=\xi_2$.
Therefore, $\mathcal{S}_2^{(\varphi,\nu,\omega)}-\lim\limits_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}$ exists uniquely.
Thirdly, if $(j,k)\in\mathbb{N}\times\mathbb{N}-[\mathscr{K}_{5,\omega}(\rho,\lambda)\cup\mathscr{K}_{6,\omega}(\rho,\lambda)]$,

$$\omega(\xi_1-\xi_2,\lambda)\leq\omega\left(\Delta^{\mathfrak{M}}x_{jk}-\xi_1,\frac{\lambda}{2}\right)\circledast\omega\left(\Delta^{\mathfrak{M}}x_{jk}-\xi_2,\frac{\lambda}{2}\right)<\rho\circledast\rho<\varepsilon.$$

Since $\varepsilon\in(0,1)$ was arbitrary, then $\omega(\xi_1-\xi_2,\lambda)=0$, for all $\lambda>0$, i.e., $\xi_1=\xi_2$.
Therefore, $\mathcal{S}_2^{(\varphi,\nu,\omega)}-\lim\limits_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}$ exists uniquely.

Theorem 3.5 Let $(\mathscr{M},\varphi,\nu,\omega,\star,\Diamond,\circledast)$ be an $\mathcal{NNS}$ with norm (φ,ν,ω). If $(\varphi,\nu,\omega)-\lim\limits_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}=\xi$, then $\mathcal{S}_2^{(\varphi,\nu,\omega)}-\lim\limits_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}=\xi$. But converse not true in general.

Proof: Let $(\varphi,\nu,\omega)-\lim\limits_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}=\xi$. Then, there exists $j_0,k_0\in\mathbb{N}$ for given $\varepsilon>0$ and any $\lambda>0$, such that for all $j\geq j_0$ and $k\geq k_0$ we have
$\varphi\left(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda\right)>1-\varepsilon,\nu\left(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda\right)<\varepsilon$ and $\omega\left(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda\right)<\varepsilon$.
Additionally, the set

$$\mathscr{A}(\varepsilon,\lambda)=\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda\right)\leq1-\varepsilon\text{ or}\\ \nu\left(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda\right)\geq\varepsilon,\omega\left(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda\right)\geq\varepsilon\end{array}\right\},$$

has finite number of elements. We know that the natural density of any finite set is always zero.
Therefore, $\delta_2(\mathscr{A}(\varepsilon,\lambda))=0$ i.e., $\mathcal{S}_2^{(\varphi,\nu,\omega)}-\lim\limits_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}=\xi$. The next example might be used to demonstrate that the converse of the above result is not true.

Example 3.6 Consider the space of all real numbers with the usual norm i.e. $(\mathscr{R}, |\cdot|)$. Define $a \star b = ab, a \Diamond b = \min\{a+b, 1\}$ and $a \circledast b = \min\{a+b, 1\}$, for all $a, b \in [0,1]$. For all $\lambda > 0$ and every $y \in \mathscr{R}$, consider $\varphi(y,\lambda) = \frac{\lambda}{\lambda+|y|}, \nu(y,\lambda) = \frac{|y|}{\lambda+|y|}$ and $\omega(y,\lambda) = \frac{|y|}{\lambda}$. Then, clearly $(\mathscr{R}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ is $\mathscr{NNS}$. Define the double sequence $\Delta^{\mathfrak{M}} x_{jk} = \begin{cases} jk & j \text{ and } k \text{ are squares} \\ 0 & \text{otherwise} \end{cases}$. Then for given $\varepsilon > 0$ and any $\lambda > 0$, we get the below set for $\xi = 0$.

$$\begin{aligned}\mathscr{K}(\varepsilon,\lambda) &= \{(j,k) \in \mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}} x_{jk}, \lambda) \le 1-\varepsilon \text{ or } \nu(\Delta^{\mathfrak{M}} x_{jk}, \lambda) \ge \varepsilon, \omega(\Delta^{\mathfrak{M}} x_{jk}, \lambda) \ge \varepsilon\} \\ &= \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : |\Delta^{\mathfrak{M}} x_{jk}| \ge \varepsilon\frac{\lambda}{1-\varepsilon} > 0\right\} \\ &= \{(j,k) \in \mathbb{N}\times\mathbb{N} : |\Delta^{\mathfrak{M}} x_{jk}| = jk\} \\ &= \{(j,k) \in \mathbb{N}\times\mathbb{N} : j \text{ and } k \text{ are squares}\}\end{aligned}$$

Thus, $\frac{1}{\mathfrak{MN}}|\mathscr{K}(\varepsilon,\lambda)| \le \frac{\sqrt{\mathfrak{MN}}}{\mathfrak{MN}} \Rightarrow \lim_{\mathfrak{M},\mathfrak{N}\to\infty} \frac{1}{\mathfrak{MN}}|\mathscr{K}(\varepsilon,\lambda)| = 0$.
Hence, $\mathscr{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = 0$.
By the above defined double sequence $(\Delta^{\mathfrak{M}} x_{jk})$, we get

$$\varphi\left(\Delta^{\mathfrak{M}} x_{jk}, \lambda\right) = \begin{cases} \frac{\lambda}{\lambda+|jk|} & j \text{ and } k \text{ are squares} \\ 0 & \text{otherwise} \end{cases} \quad \text{i.e., } \varphi(\Delta^{\mathfrak{M}} x_{jk}, \lambda) \le 1, \text{ for all } j,k,$$

$$\nu(\Delta^{\mathfrak{M}} x_{jk}, \lambda) = \begin{cases} \frac{|jk|}{\lambda+|jk|} & j \text{ and } k \text{ are squares} \\ 0 & \text{otherwise} \end{cases} \quad \text{i.e., } \nu(\Delta^{\mathfrak{M}} x_{jk}, \lambda) \ge 0, \text{ for all } j,k \text{ and}$$

$$\omega(\Delta^{\mathfrak{M}} x_{jk}, \lambda) = \begin{cases} \frac{|jk|}{\lambda} & j \text{ and } k \text{ are squares} \\ 0 & \text{otherwise} \end{cases} \quad \text{i.e., } \omega(\Delta^{\mathfrak{M}} x_{jk}, \lambda) \ge 0, \text{ for all j,k.}$$

This implies that $(\varphi,\nu,\omega) - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} \ne 0$.

Theorem 3.7 Let $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathscr{NNS}$ with norm (φ,ν,ω). Let $x = (x_{jk})$ and $z = (z_{jk})$ be any two $\mathscr{DS}s \in \mathscr{M}$. Then

(i) If $\mathscr{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$ then $\mathscr{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} \alpha x_{jk} = \alpha\xi; \alpha \in \mathscr{R}$,

(ii) If $\mathscr{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi_1$ and $\mathscr{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} z_{jk} = \xi_2$ then $\mathscr{S}_2^{(\varphi,\nu,\omega)} -$

$$\lim_{j,k\to\infty} \Delta^{\mathfrak{M}}(x_{jk}+z_{jk}) = \xi_1 + \xi_2.$$

Proof: Proof of the above is obvious so we leave it.

Theorem 3.8 Let $(\mathscr{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathscr{NNS}$ with norm (φ, ν, ω). Then $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$ iff there exists a set $\mathscr{J} = \{(j_p, k_p) : p, q = 1, 2, 3, \ldots\} \subseteq \mathbb{N}\times\mathbb{N}$ such that $\delta(\mathscr{J}) = 1$ and $(\varphi, \nu, \omega) - \lim_{j_p,k_q\to\infty} \Delta^{\mathfrak{M}} x_{j_p k_q} = \xi$.

Proof:

Necessary part:

Assume $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$. For $\lambda > 0$ and $\rho \in \mathbb{N}$, we take

$$\mathscr{L}(\rho,\lambda) = \left\{ \begin{array}{c} (j,k) \in \mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) > 1 - \frac{1}{\rho}, \\ \nu(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) < \frac{1}{\rho} \,\text{and}\, \omega(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) < \frac{1}{\rho} \end{array} \right\}$$

and

$$\mathscr{K}(\rho,\lambda) = \left\{ \begin{array}{c} (j,k) \in \mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \le 1 - \frac{1}{\rho} \,\text{or} \\ \nu(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \ge \frac{1}{\rho},\, \omega(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \ge \frac{1}{\rho} \end{array} \right\}$$

As $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$, then $\delta_2(\mathscr{K}(\rho,\lambda)) = 0$.

Also for any $\lambda > 0$ and $\rho \in \mathbb{N}$, evidently we get $\mathscr{L}(\rho,\lambda) \supset \mathscr{L}(\rho+1,\lambda)$ and

$$\delta_2(\mathscr{L}(\rho,\lambda)) = 1. \tag{1}$$

For $(j,k) \in \mathscr{L}(\rho,\lambda)$, we prove that $(\varphi, \nu, \omega) - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$.

We show this contrary. Suppose that $\mathscr{DS}\, y = (x_{jk})$ is not $\Delta^{\mathfrak{M}}$-convergent to ξ, for all $(j,k) \in \mathscr{L}(\rho,\lambda)$.

So, the existence of $\alpha > 0$ and $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\varphi(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \le 1 - \alpha \,\text{or} \\ &\nu(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \ge \alpha, \\ &\omega(\Delta^{\mathfrak{M}} x_{jk} - \xi, \lambda) \ge \alpha, \,\text{for all}\, j,k \ge k_0. \end{aligned}$$

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$$\Rightarrow \varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)>1-\alpha,$$
$$\nu(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)<\alpha \text{ and}$$
$$\omega(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)<\alpha, \text{ for all } j,k<k_0.$$

Therefore,

$$\delta_2\left(\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N}:\varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)>1-\alpha,\\ \nu(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)<\alpha \text{ and } \omega(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)<\alpha\end{array}\right\}\right)=0.$$

i.e. $\delta_2(\mathscr{L}(\alpha,\lambda))=0$.
Since $\alpha>\frac{1}{\rho}$, then $\delta_2(\mathscr{L}(\rho,\lambda))=0$ as $\mathscr{L}(\rho,\lambda)\subset\mathscr{L}(\alpha,\lambda)$, which is a contradiction to (1).
This implies that the existence a set $\mathscr{L}(\rho,\lambda)$ for which $\delta_2(\mathscr{L}(\rho,\lambda))=1$ and the $\mathscr{DS}$ $y=(x_{jk})$ is statistically $\Delta^{\mathfrak{M}}$-convergent to ξ.
Sufficient Part:
Now, there is a subset $\mathscr{J}=\{(j_p,k_q):p,q=1,2,3,\dots\}\subseteq\mathbb{N}\times\mathbb{N}$ with $\delta_2(\mathscr{J})=1$ and $(\varphi,\nu,\omega)-\lim_{j_p,k_q\to\infty}\Delta^{\mathfrak{M}}x_{j_pk_q}=\xi$, i.e. for given $\alpha>0$ and any $\lambda>0$, we have $\mathbb{N}_0\in\mathbb{N}$ which gives $\varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)>1-\alpha,\nu(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)<\alpha$ and $\omega(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)<\alpha$, for all $j,k\geq\mathbb{N}_0$.
Now, let

$$\mathscr{K}(\alpha,\lambda)=\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N}:\varphi(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)\leq 1-\alpha \text{ or}\\ \nu(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)\geq\alpha,\omega(\Delta^{\mathfrak{M}}x_{jk}-\xi,\lambda)\geq\alpha\end{array}\right\}.$$

Then, $\mathscr{K}(\alpha,\lambda)\subseteq\mathbb{N}-\{(j_{\mathbb{N}_{0+1}},k_{\mathbb{N}_{0+1}}),(j_{\mathbb{N}_{0+2}},k_{\mathbb{N}_{0+2}}),\dots\}$.
As $\delta_2(\mathscr{J})=1\Rightarrow\delta_2(\mathscr{K}(\alpha,\lambda))\leq 0$. Hence, $\mathcal{S}_2^{(\varphi,\nu,\omega)}-\lim_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}=\xi$.

Theorem 3.9 Let $x=(x_{jk})$ be any $\mathscr{DS}$ in $\mathscr{NNS}(\mathscr{M},\varphi,\nu,\omega,\star,\Diamond,\circledast)$ with norm (φ,ν,ω). Then $\mathcal{S}_2^{(\varphi,\nu,\omega)}-\lim_{j,k\to\infty}\Delta^{\mathfrak{M}}x_{jk}=\xi$ iff there is a $\mathscr{DS}$ $y=(y_{jk})$ such that $(\varphi,\nu,\omega)-\lim_{j,k\to\infty}\Delta^{\mathfrak{M}}y_{jk}=\xi$ and $\delta_2\left(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\Delta^{\mathfrak{M}}x_{jk}=\Delta^{\mathfrak{M}}y_{jk}\}\right)=1$.

Proof:
Necessary part:

Let $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$.
By Theorem (3.8) we get a set $\mathcal{J} = \{(j_p,k_q) : p,q = 1,2,3,\ldots\} \subseteq \mathbb{N}\times\mathbb{N}$ with $\delta_2(\mathcal{J}) = 1$ and $(\varphi,\nu,\omega) - \lim_{j_p,k_q\to\infty} \Delta^{\mathfrak{M}} x_{j_p k_q} = \xi$.
Consider the sequence $\Delta^{\mathfrak{M}} y_{jk} = \begin{cases} \Delta^{\mathfrak{M}} x_{jk} & (j,k)\in\mathcal{J} \\ \xi & \text{otherwise} \end{cases}$ which gives the result.

Sufficient Part:

Let $y = (y_{jk})$ and $z = (z_{jk})$ in $\mathcal{M}$ with $(\varphi,\nu,\omega) - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} y_{jk} = \xi$ and $\delta_2\left(\{(j,k)\in\mathbb{N}\times\mathbb{N} : \Delta^{\mathfrak{M}} y_{jk} = \Delta^{\mathfrak{M}} y_{jk}\}\right) = 1$.
Then for every $\varepsilon > 0$ and $\lambda > 0$,

$$\left\{\begin{array}{c} (j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}} x_{jk}-\xi,\lambda)\le 1-\varepsilon \text{ or} \\ \nu(\Delta^{\mathfrak{M}} x_{jk}-\xi,\lambda)\ge\varepsilon, \omega(\Delta^{\mathfrak{M}} x_{jk}-\xi,\lambda)\ge\varepsilon \end{array}\right\} \subseteq \mathcal{A}\cup\mathcal{B},$$

where

$$\mathcal{A} = \left\{\begin{array}{c} (j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}} y_{jk}-\xi,\lambda)\le 1-\varepsilon \text{ or} \\ \nu(\Delta^{\mathfrak{M}} y_{jk}-\xi,\lambda)\ge\varepsilon, \omega(\Delta^{\mathfrak{M}} y_{jk}-\xi,\lambda)\ge\varepsilon \end{array}\right\}$$

and $\mathcal{B} = \{(j,k)\in\mathbb{N}\times\mathbb{N} : (\Delta^{\mathfrak{M}} x_{jk} \neq \Delta^{\mathfrak{M}} y_{jk})\}$.
Since $(\varphi,\nu,\omega) - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$ then the set $\mathcal{A}$ has at most finitely many terms.
Also, $\delta_2(\mathcal{B}) = 0$ as $\delta_2([\mathcal{B}]^c) = 1$ where $\mathcal{B}^c = \{(j,k)\in\mathbb{N}\times\mathbb{N} : \Delta^{\mathfrak{M}} y_{jk} = \Delta^{\mathfrak{M}} x_{jk}\}$.
Therefore

$$\delta_2\left(\left\{\begin{array}{c} (j,k)\in\mathbb{N}\times\mathbb{N} : \varphi(\Delta^{\mathfrak{M}} y_{jk}-\xi,\lambda)\le 1-\varepsilon \text{ or} \\ \nu(\Delta^{\mathfrak{M}} y_{jk}-\xi,\lambda)\ge\varepsilon, \omega(\Delta^{\mathfrak{M}} y_{jk}-\xi,\lambda)\ge\varepsilon \end{array}\right\}\right) = 0$$

We get $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} y_{jk} = \xi$.

Theorem 3.10 Let $y = (y_{jk})$ be a $\mathcal{DS}$ in an $\mathcal{NNS}(\mathcal{M},\varphi,\nu,\omega,\star,\diamondsuit,\circledast)$. Then $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} y_{jk} = \xi$ if and only if there exists two $\mathcal{DS}s$ $z = (z_{jk})$ and $x = (x_{jk})$ in $\mathcal{M}$ such that $\Delta^{\mathfrak{M}} y_{jk} = \Delta^{\mathfrak{M}} z_{jk} + \Delta^{\mathfrak{M}} x_{jk}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$ where $(\varphi,\nu,\omega) - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} z_{jk} = \xi$ and $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$.

Proof:
Necessary part:
Let $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim\limits_{j,k\to\infty} \Delta^{\mathfrak{M}} y_{jk} = \xi$.
By Theorem (3.8) we get a set $\mathcal{J} = \{(j_p, k_q) : p, q = 1, 2, 3, \ldots\} \subseteq \mathbb{N} \times \mathbb{N}$ with $\delta_2(\mathcal{J}) = 1$ and $(\varphi, \nu, \omega) - \lim\limits_{j_p,k_q\to\infty} \Delta^{\mathfrak{M}} y_{j_p k_q} = \xi$.
Consider the $\mathcal{DS}s$ $z = (z_{jk})$ and $x = (x_{jk})$

$$\Delta^{\mathfrak{M}} z_{jk} = \begin{cases} \Delta^{\mathfrak{M}} y_{jk} & (j,k) \in \mathcal{J} \\ \xi & \text{otherwise} \end{cases} \quad \text{and} \quad \Delta^{\mathfrak{M}} x_{jk} = \begin{cases} 0 & (j,k) \in \mathcal{J} \\ \Delta^{\mathfrak{M}} y_{jk} - \xi & \text{otherwise} \end{cases}.$$

Sufficient Part:
Consider $x = (x_{jk})$ and $z = (z_{jk})$ in $\mathcal{M}$ with $\Delta^{\mathfrak{M}} y_{jk} = \Delta^{\mathfrak{M}} z_{jk} + \Delta^{\mathfrak{M}} x_{jk}$, for all $(j,k) \in \mathbb{N} \times \mathbb{N}$ where $(\varphi, \nu, \omega) - \lim\limits_{j,k\to\infty} \Delta^{\mathfrak{M}} z_{jk} = \xi$ and $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim\limits_{j,k\to\infty} \Delta^{\mathfrak{M}} x_{jk} = \xi$.
Then, we get result using Theorem (3.5) and Theorem (3.7).

Theorem 3.11 Let $(\mathcal{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ be an $\mathcal{NNS}$ with norm (φ, ν, ω). Then subsequence of a $\mathcal{DS}$ which is $\Delta^{\mathfrak{M}} - \mathcal{SC}$, is also $\Delta^{\mathfrak{M}} - \mathcal{SC}$ with respect to (φ, ν, ω).

Proof: The proof is obvious.
Next we can found the Cauchy criterion for $\mathcal{SC}$ sequences in $\mathcal{NNS}$.

Theorem 3.12 A $\mathcal{DS}$ $y = (y_{jk})$ in $\mathcal{NNS}(\mathcal{M}, \varphi, \nu, \omega, \star, \Diamond, \circledast)$ is $\Delta^{\mathfrak{M}} - \mathcal{SC}$ with respect to (φ, ν, ω) if and only if it is $\Delta^{\mathfrak{M}} - \mathcal{SC}a$ with respect to (φ, ν, ω).

Proof: Let $\mathcal{S}_2^{(\varphi,\nu,\omega)} - \lim\limits_{j,k\to\infty} \Delta^{\mathfrak{M}} y_{jk} = \xi$. Then, for $\varepsilon > 0$ and $\lambda > 0$, take $\rho > 0$ such that $(1-\rho) \star (1-\rho) > 1 - \varepsilon, \rho \Diamond \rho < \varepsilon$ and $\rho \circledast \rho < \varepsilon$.
Let

$$\mathcal{K}(\rho, \lambda) = \left\{ \begin{array}{c} (j,k) \in \mathbb{N} \times \mathbb{N} : \varphi\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) \leq 1 - \rho \,\text{or} \\ \nu\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) \geq \rho, \\ \omega\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) \geq \rho \end{array} \right\}$$

Therefore $\delta_2(\mathscr{K}(\rho,\lambda))=0$ and $\delta_2([\mathscr{K}(\rho,\lambda)]^c)=1$.
Let

$$\mathscr{L}(\varepsilon,\lambda)=\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N}:\varphi(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{rs},\lambda)\leq 1-\varepsilon\text{ or}\\ \nu(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{rs},\lambda)\geq\varepsilon,\\ \omega(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{rs},\lambda)\geq\varepsilon\end{array}\right\}.$$

Now, we prove $\mathscr{L}(\varepsilon,\lambda)\subset\mathscr{K}(\rho,\lambda)$, for this if $(j,k)\in\mathscr{L}(\varepsilon,\lambda)-\mathscr{K}(\rho,\lambda)$.
Then, we get
$\varphi\left(\Delta^{\mathfrak{M}}y_{jk}-\xi,\frac{\lambda}{2}\right)\leq 1-\rho$ or $\nu\left(\Delta^{\mathfrak{M}}y_{jk}-\xi,\frac{\lambda}{2}\right)\geq\rho$, $\omega\left(\Delta^{\mathfrak{M}}y_{jk}-\xi,\frac{\lambda}{2}\right)\geq\rho$.

$$\begin{aligned}\text{Also}, 1-\varepsilon\geq\varphi(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{rs},\lambda)&\geq\varphi\left(\Delta^{\mathfrak{M}}y_{jk}-\xi,\frac{\lambda}{2}\right)\star\varphi\left(\Delta^{\mathfrak{M}}y_{rs}-\xi,\frac{\lambda}{2}\right)\\&>(1-\rho)\star(1-\rho)>1-\varepsilon\\ \text{and}\,\varepsilon\leq\nu(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{rs},\lambda)&\leq\nu\left(\Delta^{\mathfrak{M}}y_{jk}-\xi,\frac{\lambda}{2}\right)\lozenge\nu\left(\Delta^{\mathfrak{M}}y_{rs}-\xi,\frac{\lambda}{2}\right)\\&<\rho\lozenge\rho<\varepsilon.\\ \text{Also}, \varepsilon\leq\omega(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{rs},\lambda)&\leq\omega\left(\Delta^{\mathfrak{M}}y_{jk}-\xi,\frac{\lambda}{2}\right)\circledast\omega\left(\Delta^{\mathfrak{M}}y_{rs}-\xi,\frac{\lambda}{2}\right)\\&<\rho\circledast\rho<\varepsilon,\end{aligned}$$

which is impossible.
Therefore $\mathscr{L}(\varepsilon,\lambda)\subset\mathscr{K}(\rho,\lambda)$ and $\delta_2(\mathscr{L}(\varepsilon,\lambda))=0$.
i.e. $y=(y_{jk})$ is $\Delta^{\mathfrak{M}}-\mathcal{S}\mathcal{C}a$ with respect to (φ,ν,ω).
Conversely, assume that $y=(y_{jk})$ is $\Delta^{\mathfrak{M}}-\mathcal{S}\mathcal{C}a$ with respect to (φ,ν,ω) but not $\Delta^{\mathfrak{M}}-\mathcal{S}\mathcal{C}$ with respect to (φ,ν,ω).
Thus for $\varepsilon>0$ and $\lambda>0,\delta_2(\mathscr{L}(\varepsilon,\lambda))=0$, where

$$\mathscr{L}(\varepsilon,\lambda)=\left\{\begin{array}{c}(j,k)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{j_0k_0},\lambda\right)\leq 1-\varepsilon\text{ or}\\ \nu\left(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{j_0k_0},\lambda\right)\geq\varepsilon,\\ \omega\left(\Delta^{\mathfrak{M}}y_{jk}-\Delta^{\mathfrak{M}}y_{j_0k_0},\lambda\right)\geq\varepsilon\end{array}\right\}.$$

Let $\rho > 0$ such that $(1-\rho) \star (1-\rho) > 1-\varepsilon, \rho \Diamond \rho < \varepsilon$ and $\rho \circledast \rho < \varepsilon$.
Also, $\delta_2(\mathscr{K}(\rho, \lambda)) = 0$, where

$$\mathscr{K}(\rho, \lambda) = \left\{ \begin{array}{c} (j,k) \in \mathbb{N} \times \mathbb{N} : \varphi\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) > 1-\rho, \\ \nu\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) < \rho \, \text{and} \\ \omega\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) < \rho \end{array} \right\}$$

$$\begin{aligned} \text{Now}, \varphi(\Delta^{\mathfrak{M}} y_{jk} - \Delta^{\mathfrak{M}} y_{j_0k_0}, \lambda) &\geq \varphi\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) \star \varphi\left(\Delta^{\mathfrak{M}} y_{j_0k_0} - \xi, \frac{\lambda}{2}\right) \\ &> (1-\rho) \star (1-\rho) > 1-\varepsilon \\ \text{and}\, \nu(\Delta^{\mathfrak{M}} y_{jk} - \Delta^{\mathfrak{M}} y_{j_0k_0}, \lambda) &\leq \nu\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) \Diamond \nu\left(\Delta^{\mathfrak{M}} y_{j_0k_0} - \xi, \frac{\lambda}{2}\right) \\ &< \rho \Diamond \rho < \varepsilon. \\ \text{Also}, \omega(\Delta^{\mathfrak{M}} y_{jk} - \Delta^{\mathfrak{M}} y_{j_0k_0}, \lambda) &\leq \omega\left(\Delta^{\mathfrak{M}} y_{jk} - \xi, \frac{\lambda}{2}\right) \circledast \omega\left(\Delta^{\mathfrak{M}} y_{j_0k_0} - \xi, \frac{\lambda}{2}\right) \\ &< \rho \circledast \rho < \varepsilon. \end{aligned}$$

Therefore, $\delta_2\left([\mathscr{L}(\varepsilon, \lambda)]^c\right) = 0$ i.e. $\delta_2(\mathscr{L}(\varepsilon, \lambda)) = 1$, which is a contradiction as $y = (y_{jk})$ is $\Delta^{\mathfrak{M}} - \mathscr{S}\mathscr{C}a$. Hence, $y = (y_{jk})$ is $\Delta^{\mathfrak{M}} - \mathscr{S}\mathscr{C}$ with respect to (φ, ν, ω).

4 Conclusion

We have provided a few instances that demonstrate how generalized this approach of convergence is. Additionally, we described the Statistical $\Delta^{\mathfrak{M}}$- Cauchy sequence and Cauchy convergence criterion that exist in the Neutrosophic Normed space.

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