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Stochastic Fractional Differential Equations With Generalized Caputo's Derivative and Impulsive Effects

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Abstract

In this paper, impulsive stochastic fractional differential equations (ISFDEs) in $L^p(p \geq 2)$ space are introduced. We present a general framework for finding solution for ISFDEs. Then, by using the Burkholder - Davis - Gundy inequality and Holder's inequality, we prove the existence and uniqueness of solution to ISFDE by fixed point theorem. We also investigate Lipschitz continuity of solutions with respect to initial values by using Gronwall inequality. Finally, we provide an application to illustrate the results we obtained.

Key words: Stochastic fractional differential equations, Impulsive condition, Generalized Caputo's derivative, Existence and uniqueness of solutions, Continuity of solutions.

AMS classification: [2010] 26A33, 34A12, 60H10, 60H20, 47H10

1 Introduction

Fractional differential equations (FDEs) involve the derivatives of the form $\frac{d^\alpha}{dt^\alpha}$, where $\alpha > 0$ is not necessarily an integer, have received much attention from researchers. This rising interest is motivated both by important applications of the theory, and by the difficulties involved in the mathematical structure. Fractional evolution equations appear in many physical phenomena arising from various scientific fields including analysis of viscoelastic materials, electrical engineering, control theory of dynamical systems, electrodynamics with memory, quantum mechanics, heat conduction in materials with memory, signal processing, economics, and many other

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fields. For comprehensive study of fractional differential equations, we refer the books by Podlubny [20], Hilfer [16] and the papers [10, 11, 8, 9, 2] and references there in.

In order to describe and forecast a real phenomenon, it is necessary to introduce a component that captures the random behaviour caused by a major source of uncertainty, that usually propagates in time. When we add such a component, the model obtain is now governed by stochastic fractional differential equations (SFDEs). SFDEs are emerging in various fields of science and engineering such as economics, control physics, mechanics and many other areas (see [22, 23, 1, 12, 18, 17, 30]).

Recently, impulsive SFDEs (ISFDEs) naturally emerging in the models to describe the case where deterministic changes with impulses are interwoven with noisy fluctuations. In the nature there are lot of systems in which the time evolution of the state variable depend on trajectory subject to abrupt changes are modeled in ISFDEs. For more details on existence and uniqueness results for the ISFDEs can be found in [1, 12] and reference therein. However, the theory of mild solution of IFDEs are studying in two aspacts, one is based on classical Caputo's derivative and other is genralized caputo derivative. Under classical Caputo's derivative, authors (see [21, 26, 27, 5, 19, 13]) described mild solution as integrals over $(t_k, t_{k+1}](k = 1, 2, \ldots, m)$ and $[0, t_1]$. On the other hand, the mild solutions of IFDEs under generalized Caputo's derivative were expressed as integral over $[0, t]$ (see [14, 15, 28, 29]). Moreover generalized Caputo's derivative is more reasonable since under generalized Caputo's derivative, the obtained solution satisfies the given IFDEs. For more details see [14, 15, 7]. In [17], the authors have investigated the well posed-ness for solutions of Caputo's SFDE in $L^p(p \geq 2)$ space:

$$ {}^cD_t^\alpha X(t) = b(t, X(t)) + \sigma(t, X(t))\frac{dW_t}{dt}, \qquad (*) $$

where $\alpha \in (\frac{1}{2}, 1)$, $b, \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $(W_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion.

We have motivated by the work of Huong et al. [17, 14, 15, 28, 29] and [14, 15, 28, 29] respectively, we have introduced the impulsive function and sectorial operator A with generalized Caputo's derivative in $(*)$, which is not study yet. The purpose of this paper is to study the existence, uniqueness and continuous dependence of

2456-8686, vi(ii), 2022:093-115
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solution for ISFDE of the form

$$ {}^{C}D_t^{\alpha} X_t = AX_t + \mu(X_t, t) + \sigma(X_t, t)\frac{dB_t}{dt}, \quad t \in J = (0, T],\ t \neq t_k, \tag{1} $$

$$ \Delta X_{t_k} = I_k(X_{t_k^-}), \quad k = 1, 2, ...m, \tag{2} $$

$$ X(0) = X_0 \tag{3} $$

where X_t is stochastic process and ${}^{C}D_t^{\alpha}$ is generalized Caputo's fractional derivative of order $\alpha \in (1/2, 1)$. Linear operator A, defined from the domain $D(A) \subset \mathbb{R}^n$ into $\mathbb{R}^n$, is such that A generates $\alpha-$ resolvent family $\{S_\alpha(t) : t \geq 0\}$ of bounded linear operators in $\mathbb{R}^n$. The functions $\mu : \mathbb{R}^n \times J \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times J \to \mathbb{R}^n$ are given and satisfy some assumptions and $(B_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion defined on complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$. Here $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, and the functions $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, ...m$, are bounded and the notation $\triangle X_{t_k} = X_{t_k^+} - X_{t_k^-}$ where $X_{t_k^+}$ and $X_{t_k^-}$ represent the right and left- hand limits of X_t at $t = t_k$ respectively, also we take $X_{t_i^-} = X_{t_i}$.

The work of this paper is based on [17, 14, 7]. This paper is concerned with ISFDE with generalized Caputo's derivative in L^p space with $p \geq 2$. The rest of this article is organized as follows: In section 2, we provide some basic definitions and essential preliminary results that will be used in the subsequent sections. Section 3 is devoted to the main results. In section 4, one example is given for validation of results. Finally we give conclusions in section 5.

2 Preliminaries

In the present section, we review some basic definitions, properties and lemmas which are required for establishing our results.

Definition 2.1 The fractional integral of order $\alpha > 0$ for a function $f \in L^1_{loc}(\mathbb{R}^+, X)$ is given as

$$ J_t^0 f(t) = f(t),\ J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,\ t > 0, \tag{4} $$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 The Caputo's fractional derivative of order α for a function $f \in$

2456-8686, vi(ii), 2022:093-115
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$C^n(\mathbb{R}^+, X)$ can be defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = J^{n-\alpha} f^{(n)}(t), \tag{5}$$

for $n-1 < \alpha < n,\ n \in N$. If $0 < \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds. \tag{6}$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.3 A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi\iota} \int_c \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu,\ \alpha,\ \beta > 0,\ z \in \mathbb{C},$$

where c is a contour which starts and ends at ∞ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counter clockwise. The most interesting properties of the Mittag-Lefller functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega},\ Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0.$$

For more details one can see the monographs of I. Podlubny [20].

Definition 2.4 ([4], Definition 2.3) Let A be a closed and linear operator with domain $D(A)$ defined on X and α>0. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an α- order resolvent operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha x dt,\ Re\lambda > \omega,\ x \in X.$$

In this case $\{S_\alpha(t)\}_{t\geq 0}$ is called α-order resolvent operator generated by A.

Lemma 2.5 [7] Suppose A is a sectorial operator and $\alpha \in (0,1)$, then

$$^{c}D_t^{\alpha}[T_\alpha(t)x_0] = A[T_\alpha(t)x_0]$$

and

$$\begin{aligned} &^{c}D_t^{\alpha}\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] \\ &= A\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] + \mu(X_t,t)+\sigma(X_t,t)\frac{dB_t}{dt}, \end{aligned}$$

where $T_\alpha(t) = E_{\alpha,1}(At^\alpha)$, $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$.

Lemma 2.6 If the functions $\mu : \mathbb{R}^d \times J \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times J \to \mathbb{R}^d$ are $\mathcal{F}_t-$ adapted process and A is a sectorial operator, then a piecewise $\mathcal{F}_t-$ adapted stochastic process X_t is called a solution of (1)-(3) if $X(0) = X_0$ and the following equality holds on $\mathbb{X}_t^p$ for $t \in J = [0, T,]$

$$X_t = \begin{cases} T_\alpha(t)X_0 + \sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-}) + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds \\ + \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s \end{cases} \quad (7)$$

Proof: If $t \in [0, t_1]$, then

$$^{c}D_t^{\alpha}X_t = AX_t + \mu(X_t,t) + \sigma(X_t,t)\frac{dB_t}{dt}, \quad (8)$$

$$X(0) = X_0. \quad (9)$$

Now applying the Riemann-Liouville fractional integral operator on both side, we have

$$\begin{aligned} X_t + c_1 &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mu(X_s,s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(X_s,s)dB_s, \end{aligned}$$

using initial condition, we get $c_1 = -X_0$, then

$$\begin{aligned} X_t &= X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s. \end{aligned}$$

Now, if $t \in (t_1, t_2]$, then

$$\begin{aligned} {}^cD_t^\alpha X_t &= AX_t + \mu(X_t, t) + \sigma(X_t, t)\frac{dB_t}{dt}, \\ X_{t_1^+} &= X_{t_1^-} + I_1(X_{t_1^-}). \end{aligned}$$

Again applying the Riemann-Liouville fractional integral operator on both side, we have

$$\begin{aligned} X_t + c_2 &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s, \end{aligned}$$

using initial condition $X_{t_1^+} = X_{t_1^-} + I_1(X_{t_1^-})$, we get $c_2 = -X_0 - I_1(X_{t_1^-})$, then we have

$$\begin{aligned} X_t &= X_0 + I_1(X_{t_1^-}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s. \end{aligned}$$

Similarly, if $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} X_t &= X_0 + \sum_{i=1}^{k} I_i(X_{t_i^-}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s. \end{aligned} \tag{10}$$

Now for $t \in [0, T]$, Eq. (10) can be written as

$$\begin{aligned} X_t &= X_0 + \sum_{i=1}^{m} \chi_i(t) I_i(X_{t_i^-}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A X_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s, \end{aligned} \tag{11}$$

where

$$\chi_i(t) = \begin{cases} 0, \ t \leq t_i \\ 1, \ t > t_i. \end{cases}$$

Now we adopt the idea used in [24] and taking the Laplace transform of the eq. (11) on both sides, we obtain

$$\begin{aligned} L\{X_t\} &= \frac{1}{\lambda}(X_0) + \sum_{i=1}^{m} \frac{e^{-\lambda t_i}}{\lambda} I_i(X_{t_i^-}) + \frac{1}{\lambda^\alpha} A L\{X_t\} + \frac{1}{\lambda^\alpha} L\{\mu(X_t, t)\} \\ &\quad + \frac{1}{\lambda^\alpha} L\{\sigma(X_t, t) \frac{dB_t}{dt}\} \end{aligned} \tag{12}$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is $\lambda^\alpha \in \rho(A)$, from eq. (12), we obtain

$$\begin{aligned} L\{X_t\} &= \frac{\lambda^{\alpha-1}}{\lambda^\alpha I - A}(X_0) + \sum_{i=1}^{m} \frac{\lambda^{\alpha-1}}{\lambda^\alpha I - A} e^{-\lambda t_i} I_i(X_{t_i^-}) + \frac{1}{\lambda^\alpha I - A} L\{\mu(X_t, t)\} \\ &\quad + \frac{1}{\lambda^\alpha I - A} L\{\sigma(X_t, t) \frac{dB_t}{dt}\}. \end{aligned} \tag{13}$$

Taking the inverse Laplace transform on both sides of eq. (13), we get

$$\begin{aligned} X_t = E_{\alpha,1}(At^\alpha) X_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \mu(X_s, s) ds \\ + \sum_{i=1}^{m} E_{\alpha,1}(A(t-t_i)^\alpha) \chi_i(t) I_i(X_{t_i^-}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \sigma(X_s, s) dB_s. \end{aligned} \tag{14}$$

Now we rewrite the eq. (14) as

$$\begin{aligned} X_t &= T_\alpha(t)X_0 + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds + \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s \\ &\quad + \sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-}), \quad t\in[0,T], t\neq t_1,\ldots,t_m. \end{aligned} \tag{15}$$

where $T_\alpha(t) = E_{\alpha,1}(At^\alpha)$, $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$. Now, we show that the solution (15) satisfies the system (1)-(3).
Step 1. We apply the Caputo's derivative on both side of eq. (15)

$$\begin{aligned} &{}^cD_t^\alpha X_t = \\ &{}^cD_t^\alpha\left[T_\alpha(t)X_0\right] + {}^cD_t^\alpha\left[\sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-})\right] \\ &+{}^cD_t^\alpha\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] \\ &= AT_\alpha(t)X_0 + A\sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-}) \\ &+A\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] + \mu(X_t,t)+\sigma(X_t,t)\frac{dB_t}{dt}. \end{aligned}$$

Therefore

$${}^cD_t^\alpha X_t = AX_t + \mu(X_t,t) + \sigma(X_t,t)\frac{dB_t}{dt}.$$

Step 2. From solution (15), we have

$$\begin{aligned} X_{t_k^+} &= T_\alpha(t_k)X_0 + \int_0^{t_k} S_\alpha(t_k-s)\mu(X_s,s)ds + \int_0^{t_k} S_\alpha(t_k-s)\sigma(X_s,s)dB_s \\ &\quad + \sum_{0<t_i\le t_k} T_\alpha(t_k-t_i)I_i(X_{t_i^-}), \end{aligned}$$

and

$$\begin{aligned} X_{t_k^-} &= T_\alpha(t_k)X_0 + \int_0^{t_k} S_\alpha(t_k-s)\mu(X_s,s)ds + \int_0^{t_k} S_\alpha(t_k-s)\sigma(X_s,s)dB_s \\ &\quad + \sum_{0<t_i\le t_{k-1}} T_\alpha(t_k-t_i)I_i(X_{t_i^-}), \end{aligned}$$

So

$$\begin{aligned} X_{t_k^+} - X_{t_k^-} &= T_\alpha(t_k - t_k) I_k(x(t_k^-)) \\ &= T_\alpha(0) I_k(x(t_k^-)) \\ &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots m. \end{aligned}$$

Thus we observe that the solution (15) satisfies the system (1)-(3). □

We impose the following assumptions to developed our results.

(H1) The functions $\mu : \mathbb{R}^n \times J \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times J \to \mathbb{R}^n$ are Lipschitz continuous and for all $X_t, Y_t \in \mathbb{R}^n, t \in [0, T]$, there exists constants $L_\mu > 0, L_\sigma > 0$ such that
$|\mu(X_t, t) - \mu(Y_t, t)|_p \le L_\mu |X_t - Y_t|_p$ and $|\sigma(X_t, t) - \sigma(Y_t, t)|_p \le L_\sigma |X_t - Y_t|_p$.

(H2) The functions μ and σ are essentially bounded, i.e. there exists constant $M > 0$, such that
$esssup_{s\in[0,T]} |\mu(0, s)|_p < M$ and $esssup_{s\in[0,T]} |\sigma(0, s)|_p < M$.

(H3) There exist some positive constants $d_i (i = 1, 2, \ldots, m)$ such that

$$|I_i(X_t) - I_i(Y_t)|_p \le d_i |X_t - Y_t|_p.$$

If $\alpha \in (1/2, 1)$, $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then for $t > 0$, we have $\|E_{\alpha,1}(At^\alpha)\|_{L(X)} \le Me^{\omega t}$ and $\|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)\|_{L(X)} \le ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Let $\widetilde{M}_T = \sup_{t\in[0,T]} \|E_{\alpha,1}(At^\alpha)\|_{L(X)}$, $\widetilde{M}_S = \sup_{t\in[0,T]} ce^{\omega t}(1 + t^{1-\alpha})$, where $L(X)$ is the Banach space of bounded linear operators from X into X equipped with its natural topology. So we have $\|E_{\alpha,1}(At^\alpha)\|_{L(X)} \le \widetilde{M}_T$ and $\|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)\|_{L(X)} \le t^{\alpha-1} \widetilde{M}_S$. For details see [10] and references therein.

3 The main results

For $p \ge 2, t \in [0, \infty)$, we denote $\mathbb{X}_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, the space of all $\mathcal{F}_t-$ measurable and p^{th} integrable functions $X_t = ((X_1)_t, (X_2)_t, \ldots, (X_n)_t) : \Omega \to \mathbb{R}^n$, endowed with the norm

$$\|X_t\|_{L^p} = \left(\sum_{i=1}^{n} E|(X_i)_t|^p \right)^{1/p}$$

and $J = [0, T]$. Thus $(\mathbb{X}_t^p, \|.\|_{L^p})$ is a Banach space. Let $PC(J, \mathbb{X}_t^p)$ be the Banach space of the peicewise continuous mapping from J to $\mathbb{X}_t^p$, satisfying the condition

$$esssup_{t\in[0,T]}\|X_t\|_{L^p} < \infty$$

and $\mathbb{H}_p$ be the closed subspace of the $\mathcal{F}_t-$ measurable peicewise continous processes X in $PC(J, \mathbb{X}_t^p)$ such that X_t is It$\hat{o}$ process and $X(0) = X_0$ is $\mathcal{F}_0-$ measurable, endowed with the norm

$$\|X\|_{\mathbb{H}_p} = (esssup_{t\in[0,T]}\|X_t\|_{L^p})^{1/p} < \infty.$$

Obviously, $(\mathbb{H}_p, \|.\|_{\mathbb{H}_p})$ is also a Banach space. However, in several estimates below, we use $\mathbb{R}^n$ with the $p-$norm, i.e. for any vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the p norm of x is defined by $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ and for It$\hat{o}$ process $X_t \in \mathbb{R}^n$ is defined as $|X_t|_p = (\sum_{i=1}^n |(X_i)_t|^p)^{1/p}$. Now we define relation between L^p-norm and $p-$norm as follows:

$$\|X_t\|_{L^p}^p = \sum_{i=1}^n E(|(X_i)_t|^p) = E\left(\sum_{i=1}^n |(X_i)_t|^p\right),$$

which gives

$$\|X_t\|_{L^p}^p = E(|X_t|_p^p). \tag{16}$$

In the proof of our results, we use the following elementary inequality

$$|x + y|_p^p \le 2^{p-1}(|x|_p^p + |y|_p^p)$$

for all $x, y \in \mathbb{R}^n$.

To prove the existence and uniqueness of solution of the equation (1)-(3), we define an operator $N : \mathbb{H}_p \to \mathbb{H}_p$ by

$$\begin{aligned} N(X_t) &= E_{\alpha,1}(At^\alpha)X_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-}) \\ &\quad + \int_0^t S_\alpha(t-s)\mu(X_s, s)ds + \int_0^t S_\alpha(t-s)\sigma(X_s, s)dB_s. \end{aligned} \tag{17}$$

The following lemma is devoted to show the operator N is well defined.

Lemma 3.1 Suppose that the conditions $(H1), (H2)$ and $(H3)$ are true, then for

$t \in [0,T]$, the operator N is well-defined.

Proof: Let $X \in \mathbb{H}_p([0,T])$ be arbitrary, for all $t \in [0,T]$, we have

$$\|N(X_t)\|^p_{L^p} \leq 2^{2p-2}\widetilde{M}^p_T\|X_0\|^p_{L^p} + 2^{2p-2}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-})\right\|^p_{L^p}$$
$$+2^{2p-2}\left\|\int_0^t S_\alpha(t-s)\mu(X_s,s)ds\right\|^p_{L^p} + 2^{2p-2}\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|^p_{L^p}. \quad (18)$$

Now we find individually the norms of each terms of eq. (18).

Step 1 : From (H3), we have

$$\begin{aligned}\|I_i(X_{t_i^-})\|^p_{L^p} = \sum_{j=1}^n E|I_i(X_j)_{t_i}|^p &\leq d_i^p \sum_{j=1}^n E|(X_j)_{t_i}|^p = d_i^p\|X_{t_i}\|^p_{L^p} \\ . &\leq d_i^p esssup_{t\in[0,T]}\|X_t\|^p_{L^p} \\ &\leq d_i^p\|X\|^p_{\mathbb{H}_p}.\end{aligned}$$

So,

$$\begin{aligned}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-})\right\|^p_{L^p} &\leq (2^{p-1})^{\lceil\frac{m}{2}\rceil}\widetilde{M}^p_T\sum_{0<t_i<t}\|I_i(X_{t_i})\|^p_{L^p} \\ &\leq (2^{p-1})^{\lceil\frac{m}{2}\rceil}\widetilde{M}^p_T D\|X\|^p_{\mathbb{H}_p} \quad (19)\end{aligned}$$

where $D = \sum_{i=1}^m d_i^p$ and $\lceil x\rceil$ is least integer function.

Step 2 : By the Holder's inequality, with constants $P = \frac{p}{p-1}, Q = p$ such that $\frac{1}{P}+\frac{1}{Q} = 1$, we obtain that

$$\begin{aligned}\left\|\int_0^t S_\alpha(t-s)\mu(X_s,s)ds\right\|^p_{L^p} &\leq \widetilde{M}^p_s\sum_{i=1}^n E\left(\int_0^t (t-s)^{\alpha-1}|\mu_i(X_s,s)|ds\right)^p \\ &\leq \widetilde{M}^p_s\sum_{i=1}^n E\left(\left(\int_0^t (t-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{p-1}\left(\int_0^t |\mu_i(X_s,s)|^p ds\right)\right) \\ &\leq \frac{\widetilde{M}^p_s T^{(\alpha p-1)}(p-1)^{p-1}}{(\alpha p-1)^{p-1}}\int_0^t\sum_{i=1}^n E(|\mu_i(X_s,s)|^p)ds \\ &\leq \frac{\widetilde{M}^p_s T^{(\alpha p-1)}(p-1)^{p-1}}{(\alpha p-1)^{p-1}}\int_0^t \|\mu(X_s,s)\|^p_{L^p}ds.\end{aligned}$$

Now, using result (16) and hypothesis (H1), we obtain

$$\begin{aligned}\|\mu(X_s,s)\|_{L^p}^p &= E(|\mu(X_s,s)|_p^p)\\ &\le 2^{p-1}E\left(|\mu(X_s,s)-\mu(0,s)|_p^p+|\mu(0,s)|_p^p\right)\\ &\le 2^{p-1}E\left(L_\mu^p|X_s|_p^p+|\mu(0,s)|_p^p\right)\\ &\le 2^{p-1}L_\mu^p\|X_s\|_{L^p}^p+2^{p-1}|\mu(0,s)|_p^p.\end{aligned}$$

So from (H2), we have

$$\begin{aligned}\|\mu(X_s,s)\|_{L^p}^p &\le 2^{p-1}L_\mu^p\|X\|_{\mathbb{H}_p}^p+2^{p-1}esssup_{s\in[0,T]}|\mu(0,s)|_p^p\\ &\le 2^{p-1}L_\mu^p\|X\|_{\mathbb{H}_p}^p+2^{p-1}M^p.\end{aligned}$$

Therefore

$$\left\|\int_0^t S_\alpha(t-s)\mu(X_s,s)ds\right\|_{L^p}^p \le \frac{\widetilde{M}_s^p T^{\alpha p}(2p-2)^{p-1}}{(\alpha p-1)^{p-1}}\left(L_\mu^p\|X\|_{\mathbb{H}_p}^p+M^p\right), \tag{20}$$

Step 3 : In this step we use the Burkholder-Davis-Gundy and the Holder's inequalities, with holder constants $P=\frac{p}{p-2}, Q=\frac{p}{2}$, s.t. $\frac{1}{P}+\frac{1}{Q}=1$, then we obtain

$$\begin{aligned}\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L^p}^p &= \sum_{i=1}^n E\left(\left|\int_0^t S_\alpha(t-s)\sigma_i(X_s,s)dB_s)\right|^p\right)\\ &\le \sum_{i=1}^n C_pE\left(\left|\int_0^t |S_\alpha(t-s)|^2|\sigma_i(X_s,s)|^2ds)\right|^{\frac{p}{2}}\right)\\ &\le C_p\widetilde{M}_s^p\sum_{i=1}^n E\left(\left|\int_0^t (t-s)^{2\alpha-2}|\sigma_i(X_s,s)|^2ds)\right|^{\frac{p}{2}}\right)\\ \le C_p\widetilde{M}_s^p\sum_{i=1}^n E\left(\left|\int_0^t \left((t-s)^{2\alpha-2}\right)^{\frac{p-2}{p}}\left((t-s)^{2\alpha-2}\right)^{\frac{2}{p}}|\sigma_i(X_s,s)|^2ds)\right|^{\frac{p}{2}}\right),\end{aligned}$$

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where $C_p = \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}$. Now, by Holder's inequality, we have

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L^p}^p \\
&\le C_p\widetilde{M}_S^p\sum_{i=1}^n\left(\left(\int_0^t(t-s)^{2\alpha-2}ds\right)^{\frac{p-2}{2}}\left(\int_0^t(t-s)^{2\alpha-2}|\sigma_i(X_s,s)|^pds\right)\right) \\
&\le C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t(t-s)^{2\alpha-2}\left(\sum_{i=1}^n E|\sigma_i(X_s,s)|^p\right)ds \\
&\le C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t(t-s)^{2\alpha-2}\|\sigma(X_s,s)\|_{L^p}^pds
\end{aligned}$$

Now from result (16) and assumptions (H1) and (H2), we obtain

$$\begin{aligned}
\|\sigma(X_s,s)\|_{L^p}^p &\le 2^{p-1}E\left(|\sigma(X_s,s)-\sigma(0,s)|_p^p+|\sigma(0,s)|_p^p\right) \\
&\le 2^{p-1}(L_\sigma^p\|X_s\|_{L_p}^p+|\sigma(0,s)|_P^p) \\
&\le 2^{p-1}(L_\sigma^p esssup_{s\in[0,T]}\|X_s\|_{L^p}^p+esssup_{s\in[0,T]}|\sigma(0,s)|_p^p \\
&\le \le 2^{p-1}(L_\sigma^p\|X\|_{\mathbb{H}_p}^p+M^p)
\end{aligned}$$

and

$$\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L_p}^p \le 2^{p-1}C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}(L_\sigma^p\|X\|_{\mathbb{H}_p}^p+M^p). \qquad (21)$$

Now, from inequalities (19), (20) and (21), we get

$$\|N(X)\|_{\mathbb{H}_p}<\infty.$$

Which implies that the operator N is well defined.

Our next result is based on the Banach contraction principle.
Theorem 3.2 Assume that the assumptions (H1), and (H3) are satisfied. If $A \in \mathcal{A}^\alpha(\theta_0,\omega_0)$, then the system (1)-(3) has a unique solution in J if $2^{\frac{2p-2}{p}}\mathcal{V}<1$, where

$$\mathcal{V}=\left[(2^{p-1})^{\lceil\frac{m}{2}\rceil-1}\widetilde{M}_T^pD+\widetilde{M}_S^p\left(L_\mu^pT^{\alpha p}\left(\frac{p-1}{2\alpha-1}\right)^{p-1}+c_pL_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}}.$$

Proof: Consider the operator $N : \mathbb{H}_p \to \mathbb{H}_p$ defined by

$$\begin{aligned} N(X_t) &= E_{\alpha,1}(At^\alpha)X_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-}) \\ &\quad + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds + \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s. \end{aligned} \tag{22}$$

To prove that N has a unique fixed point. Let $X, Y \in \mathbb{H}_p$, then for all $t \in [0,T]$, we have

$$\begin{aligned} \|N(X_t) - N(Y_t)\|_{L_p}^p &\le 2^{p-1} \left\| \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)(I_i(X_{t_i^-}) - I_i(Y_{t_i^-})) \right\|_{L_p}^p \\ &\quad + 2^{2p-2} \left\| \int_0^t S_\alpha(t-s)(\mu(X_s,s) - \mu(Y_s,s))ds \right\|_{L_p}^p \\ &\quad + 2^{2p-2} \left\| \int_0^t S_\alpha(t-s)(\sigma(X_s,s) - \sigma(Y_s,s))dB_s \right\|_{L_p}^p \end{aligned} \tag{23}$$

Further proof is given in the following steps
Step 1. From assumption (H3), we have

$$\begin{aligned} \|I_i(X_{t_i^-}) - I_i(Y_{t_i^-}))\|_{L^p}^p &\le d_i^p \|X_{t_i^-} - Y_{t_i^-}\|_{L^p}^p \\ &\le d_i^p \; esssup_{t_i \in [0,T]} \|X_{t_i^-} - Y_{t_i^-}\|_{L^p}^p \\ &\le d_i^p \|X - Y\|_{\mathbb{H}_p}^p. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\| \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)(I_i(X_{t_i^-}) - I_i(Y_{t_i^-})) \right\|_{L_p}^p \\ &\qquad \le (2^{p-1})^{\lceil \frac{m}{2} \rceil} \widetilde{M}_T^p D \|X - Y\|_{\mathbb{H}_p}^p, \end{aligned} \tag{24}$$

Step 2. By the Holder's inequality, we obtain

$$\begin{aligned}&\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L^p}^p\\&=\sum_{i=1}^n E\left(\left|\int_0^t S_\alpha(t-s)\{\mu_i(X_s,s)-\mu_i(Y_s,s)\}ds\right|^p\right)\\&\le \widetilde{M}_S^p\sum_{i=1}^n E\left(\int_0^t (t-s)^{\alpha-1}|\mu_i(X_s,s)-\mu_i(Y_s,s)|\right)^p.\end{aligned}$$

Using Holder's inequality, we get

$$\begin{aligned}&\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L^p}^p\\&\le \widetilde{M}_S^p\left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{p}{p-1}}ds\right)^{p-1}\times\int_0^t\sum_{i=1}^n E|\mu_i(X_s,s)-\mu_i(Y_s,s)|^p ds\\&\le \widetilde{M}_S^p\left(\frac{p-1}{\alpha p-1}\right)^{p-1}T^{\alpha p-1}\int_0^t\|\mu(X_s,s)-\mu(Y_s,s)\|_{L^p}^p ds\\&\le \widetilde{M}_S^p\left(\frac{p-1}{\alpha p-1}\right)^{p-1}T^{\alpha p-1}L_\mu^p\int_0^t\|X_s-Y_s\|_{L^p}^p ds\end{aligned}$$

So

$$\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L_p}^p\le \widetilde{M}_S^pT^{\alpha p}L_\mu^p\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\|X-Y\|_{\mathbb{H}^p}^p. \quad (25)$$

Step 3.

$$\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L^p}^p=\sum_{i=1}^n E\left(\left|\int_0^t S_\alpha(t-s)\{\sigma_i(X_s,s)-\sigma_i(Y_s,s)\}dB_s\right|^p\right).$$

Using Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L^p}^p \\
&\leq \sum_{i=1}^n C_p E\left(\int_0^t |S_\alpha(t-s)|^2|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^2 ds\right)^{\frac{p}{2}} \\
&\leq C_p\widetilde{M}_S^p \sum_{i=1}^n E\left(\int_0^t (t-s)^{2\alpha-2}|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^2 ds\right)^{\frac{p}{2}} \\
&\leq C_p\widetilde{M}_S^p \sum_{i=1}^n E\left(\int_0^t \{(t-s)^{2\alpha-2}\}^{\frac{p-2}{p}}\{(t-s)^{2\alpha-2}\}^{\frac{2}{p}}|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^2 ds\right)^{\frac{p}{2}}.
\end{aligned}
$$

Now by Holder's inequality, we get

$$
\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L^p}^p \\
&\leq C_p\widetilde{M}_S^p\left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{\frac{p-2}{2}}\left(\int_0^t (t-s)^{2\alpha-2}\sum_{i=1}^n E|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^p ds\right) \\
&\leq C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}} L_\sigma^p \int_0^t (t-s)^{2\alpha-2}\|X_s-Y_s\|_{L^p}^p ds.
\end{aligned}
$$

So

$$
\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L_p}^p \leq C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}} L_\sigma^p\|X-Y\|_{\mathbb{H}_p}^p. \quad (26)
$$

Now from inequalities (24), (25) and (26), we have

$$
\|N(X)-N(Y)\|_{\mathbb{H}_p} \leq 2^{\frac{2p-2}{p}}\mathcal{V}\|X-Y\|_{\mathcal{H}_p}, \quad (27)
$$

where

$$
\mathcal{V} = \left[\left(2^{p-1}\right)^{\lceil\frac{m}{2}\rceil-1}\widetilde{M}_T^p D + \widetilde{M}_S^p\left(L_\mu^p T^{\alpha p}\left(\frac{p-1}{2\alpha-1}\right)^{p-1} + (C_p)L_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}}.
$$

Since $2^{\frac{2p-2}{p}}\mathcal{V} < 1$, implies that the map N is a contraction map.Therefore the map

N has a unique fixed point $X_t \in \mathbb{H}_p$, that is a solution of the system (1)-(3) on $[0, T]$. Hence the proof of Theorem is completed.
In the next result we prove the continuous dependence of solutions on the initial values.

Definition 3.3 If $X_t, \widehat{X}_t$ be different solutions of the problem (1)-(3) with initial values $X_0, \widehat{X}_0$ respectively and for all $\epsilon > 0$, there exist $\delta > 0$ such that $\|X_t - \widehat{X}_t\| \leq \epsilon$ when $\|X_0 - \widehat{X}_0\| < \delta$ for all $t \in [0, T]$, then X_t is said to be continuous with respect to initial values (see definition 4.1 in [3]).

Theorem 3.4 Assume that the assumptions (H1), (H 2) are satisfied and

$$\left[(2^{p-1})^{\lceil \frac{m}{2} \rceil + 2} \widetilde{M}_T^p D\right] < 1.$$

Then the solution of the system (1)-(3) depends continuously on initial values.

Proof: Let for each initial values X_0, Y_0, there exist corresponding solutions X_t and Y_t of the system (1)-(3). Then for $t \in [0, T]$, we have

$$\begin{aligned} X_t &= E_{\alpha,1}(At^\alpha)X_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-}) + \int_0^t S_\alpha(t-s)\mu(X_s, s)ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(X_s, s)dB_s \end{aligned}$$

and

$$\begin{aligned} Y_t &= E_{\alpha,1}(At^\alpha)Y_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(Y_{t_i^-}) + \int_0^t S_\alpha(t-s)\mu(Y_s, s)ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(Y_s, s)dB_s \end{aligned}$$

Now it follows that

$$
\begin{aligned}
&\|X_t - Y_t\|_{L_p}^p \\
&\leq 2^{2p-2}\widetilde{M}_T^p\|X_0 - Y_0\|_{L_p}^p + 2^{2p-2}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)(I_i(X_{t_i^-}) - I_i(Y_{t_i^-}))\right\|_{L_p}^p \\
&+2^{2p-2}\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s) - \mu(Y_s,s))ds\right\|_{L_p}^p \\
&+2^{2p-2}\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s) - \sigma(Y_s,s))dB_s\right\|_{L_p}^p.
\end{aligned}
$$

By using Holder and the Burkholder - Davis - Gundy inequalities and assumption (H1), we get

$$
\begin{aligned}
&\|X_t - Y_t\|_{L_p}^p \\
&\leq 2^{2p-2}\widetilde{M}_T^p\|X_0 - Y_0\|_{L_p}^p + (2^{p-1})^{\lceil \frac{m}{2}\rceil+2}\widetilde{M}_T^p D\|X_t - Y_t\|_{L_p}^p \\
&+\frac{2^{2p-2}\widetilde{M}_S^p T^{(\alpha p-2\alpha+1)}(p-1)^{(p-1)}L_\mu^p}{(\alpha p-2\alpha+1)^{p-1}}\int_0^t (t-s)^{2\alpha-2}\|X_s - Y_s\|_{L_p}^p ds \\
&+2^{2p-2}C_p\widetilde{M}_S^p L_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t (t-s)^{2\alpha-2}\|X_s - Y_s\|_{L_p}^p ds.
\end{aligned}
$$

(for detail see Theorem 3.2)
Thus

$$
\|X_t - Y_t\|_{L_p}^p \leq \nu_1\|X_0 - Y_0\|_{L_p}^p + \nu_2\int_0^t (t-s)^{2\alpha-2}\|X_s - Y_s\|_{L_p}^p ds \tag{28}
$$

where

$$
\begin{aligned}
\nu_1 &= \frac{2^{2p-2}\widetilde{M}_T^p}{\left[1 - (2^{p-1})^{\lceil \frac{m}{2}\rceil+2}\widetilde{M}_T^p D\right]} \\
\nu_2 &= \frac{2^{2p-2}\widetilde{M}_S^p}{\left[1 - (2^{p-1})^{\lceil \frac{m}{2}\rceil+2}\widetilde{M}_T^p D\right]}\left\{\frac{L_\mu^p T^{(\alpha p-2\alpha+1)}(p-1)^{p-1}}{(\alpha p-2\alpha+1)} + C_p L_\sigma^p\left(\frac{T^{\alpha p-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\right\}
\end{aligned}
$$

Now applying the Gronwall inequality on equation (28) (see [25], corollary 2), we obtain

$$\|X_t - Y_t\|_{L_p}^p \leq E_{2\alpha-1}(\nu_2\Gamma(2\alpha-1)t^{2\alpha-1})\nu_1\|X_0 - Y_0\|_{L_p}^p.$$

Hence,

$$lim_{X_0\to Y_0}\|X_t - Y_t\|_{L_p} = 0.$$

The theorem is proved.

4 Application

To illustrate our results, we consider the following ISFDE

$$\frac{\partial^{2/3}X(t,x)}{\partial t^{2/3}} = \frac{\partial^2 X(t,x)}{\partial x^2} + \frac{e^{t/2}X(t,x)}{13+X(t,x)} + e^t sin\left(\frac{x(t,x)}{21+X(t,x)}\right)\frac{dB_t}{dt}, \quad (29)$$

$$t \in [0,1], x \in (0,\pi)$$

$$X(t,0) = X(t,\pi) = 0, t \geq 0 \quad (30)$$

$$\Delta X(t,x)|_{t=\frac{1}{2}^-} = sin\left(\frac{1}{11}X\left(\frac{1}{2}^-, x\right)\right) \quad (31)$$

$$X(0,x) = X_0(x), \quad (32)$$

where $X_0 \in \mathbb{R}^n$ and B_t is a standard scalar Brownian motion. Let $\mathbb{X}_t^p = L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ and define the operator $A : D(A) \subset \mathbb{X}_t^p \to \mathbb{X}_t^p$ by $Au = u''$ with the domain $D(A) = \{u \in \mathbb{X}_t^p : u, u'$ are absolutely continuous,$u'' \in \mathbb{X}_t^p, u(0) = 0 = u(\pi)\}$. Then $Au = \sum_{n=1}^{\infty} n^2(u,u_n)u_n, u \in D(A)$, where $u_n(x) = \sqrt{\frac{2}{\pi}}sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A, it is well known from [31] A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}_t^p$ and given by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u,u_n)u_n$$

for all $u \in \mathbb{X}_t^p$ and every $t > 0$. From these expression it follows that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda^\alpha, A) = (\lambda^\alpha I - A)^{-1}$ is a compact operator for all $\lambda^\alpha \in \rho(A)$ i.e. $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$. Therefore, from the subordination principal [[6] Theorems 3.1 and 3.3], we know that A generates α-order resolvent operator $S_\alpha(t)_{t\geq 0}$.

Let $X(t,x) = X(t)(x) = X_t(x)$, where X_t is it$\widehat{o}$-process, and

$$\begin{aligned} \frac{e^{t/2}X_t(x)}{13+X_t(x)} &= \mu(X_t,t)(x) \\ e^t sin\left(\frac{X_t(x)}{21+X_t(x)}\right) &= \sigma(X_t,t)(x) \\ sin\left(\frac{1}{11}X_{1/2^-}(x)\right) &= I_k(X_{t_k^-})(x), t_k = 1/2, k = 1. \end{aligned}$$

Then with these settings problem (1)-(3) is an abstract version of problem (29)-(32). Now, for $t \in [0,1], X_t, Y_t \in \mathbb{X}_t^p$, we have

$$\begin{aligned} \|\mu(X_t,t) - \mu(Y_t,t)\|_{L^p} &= \left(\sum_{i=1}^{n} E\left|\frac{e^{t/2}(X_i)_t}{13+(X_i)_t} - \frac{e^{t/2}(Y_i)_t}{13+(Y_i)_t}\right|^p\right)^{1/p} \\ &\leq \frac{e^{t/2}}{13}\left(\sum_{i=1}^{n}|(X_i)_t - (Y_i)_t|^p\right)^{1/p} \\ &\leq \frac{e^{t/2}}{13}\|X_t - Y_t\|_{L^p} \end{aligned}$$

Similiarly,

$$\begin{aligned} \|\sigma(X_t,t) - \sigma(Y_t,t)\|_{L_p} &\leq \frac{e}{21}\|X_t - Y_t\|_{L_p} \\ \|I_k(X_{t_k^-}) - I_k(Y_{t_k^-})\|_{L_p} &\leq \frac{1}{11}\|X_{t_k} - Y_{t_k}\|_{L_p}. \end{aligned}$$

Thus the functions μ, σ, and I_k, are satisfied the conditions (H1)-(H3) with $T = 1, L_\mu = \frac{e^{1/2}}{13}, L_\sigma = \frac{e}{21}$ and $D = \frac{1}{11}$. Now we take $\widehat{M_T} = 1, \widehat{M_S} = \frac{1}{\Gamma(\frac{2}{3})}$, and for $p = 2, C_p = 4$. Further

$$\begin{aligned} \mathcal{V} &= \left[\left(2^{p-1}\right)^{\lceil \frac{m}{2}\rceil - 1}\widetilde{M}_T^p D + \widetilde{M}_S^p\left(L_\mu^p T^{\alpha p}\left(\frac{p-1}{2\alpha-1}\right)^{p-1} + (C_p)L_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}}. \\ &= 0.47539. \end{aligned}$$

So, $2^{\frac{2p-2}{p}}\mathcal{V} = 0.9507 < 1$. Thus all conditions of Theorem 3.2 are fulfilled. So we deduce that problem (29) - (32) has a unique solution on $[0,1]$.

5 Conclusion

In this paper, we have firstly defined solution based on Laplace transform method in order to study the existence and uniqueness of ISFDE. Then, as a lemma, we proved that the operator, used in fixed point theorem, is well defined. In main results, by using Burkholder Davis Gundy and Holder's inequalities, we first proved the existence and uniqueness of solutions of ISFDE under Banach contraction theorem and then we showed Lipschitz continuity of solutions with respect to initial values. Finally we have given one example to illustrate our results.

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2456-8686, vi(ii), 2022:116-120
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ORIGINAL RESEARCH Open Access

On Paired Domination of Some Graphs

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Abstract

For a graph a subset of the vertex set is called a dominating set if every vertex in is adjacent to some vertex in D. The domination number is the minimum cardinality of a dominating set of a graph G. The paired dominating set of a graph is a dominating set and the subgraph induced by it contains a perfect matching. The paired domination number is the minimum cardinality of a paired dominating set in G. In this paper, we discuss the paired domination number of the graphs obtained by the k^{th} power of path and cycle and degree splitting graphs of some standard graphs.

Key words: Paired domination, Paired domination number, k^{th} Power of a graph, Degree splitting of a graph.

AMS classification: 05C38, 05C76, 05C90.

1 Introduction

We begin with finite, connected and undirected graph $G = (V, E)$ with vertex set V and edge set E, without loops and multiple edges. The notations and terminology used here in the sense of Clark and Holtan [2]. The subset $D \subseteq V$ is called dominating set if every vertex of V is either an element of D or is adjacent to an element of D. A dominating set D of a graph G is minimal if no proper subset of D is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a minimal dominating set of graph G.

In graph theory the study of dominating sets was introduced by Ore [6] in 1962 and Berge [1] in 1958. They introduced the term dominating set and domination number of a graph. The concept of domination in graphs and its many variations are studied by many researchers and the literature on this subject has been surveyed in the books by T. Haynes, S. Hedetniemi and P. Slater [3] [4]. In this paper we focus on

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a variation named paired domination of graphs, obtained by some graph operations.

The degree of vertex v, denoted by $d(v)$, is the number of edges incident to v with counting loop twice. The set of edges with no common end vertex is called a matching. A matching of a graph G is perfect if every vertex of the graph is incident to an edge of the matching. The paired dominating set D of graph G is the subset of vertex set V of graph G such that the subset D is dominating set and the subgraph induced by D contains a perfect matching. A paired dominating set with minimum cardinality is called minimal paired dominating set. The paired domination number $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set D of graph G. Paired domination was introduced by Haynes and P. Slater [3]. Paired domination numbers of complete graph, star graph, wheel graph and complete bipartite graph are studied by Sangeetha and Swarnamalya [8]. They also studied paired domination of the corona graphs obtained by the standard graphs. Locating and paired dominating sets in graphs are studied by J. McCoy and Henning in [5] while the upper paired domination numbers of graphs are done by Ulatowski [9].

Definition 1.1 The distance between two vertices is the number of edges in a shortest path connecting them.

Definition 1.2 The k^{th} power of a graph G, denoted by G^k, is a graph having same vertex set as of G and two vertices are adjacent if and only if distance between them is at most k.

Definition 1.3 Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of all vertices of the same degree, with at least two elements and $T = V - \cup_{i=1}^{t} S_i$. The degree splitting graph of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i; $1 \leqslant i \leqslant t$.

Proposition 1.4 [3] If P_n, C_n, K_n, $K_{m,n}$ and $W_n : C_n + K_1$ are respectively path, cycle, complete graph, complete bipartite graph and wheel graph then,

i. $\gamma(P_n) = \left\lfloor \dfrac{n+2}{3} \right\rfloor$.

ii. $\gamma(C_n) = \left\lfloor \dfrac{n+2}{3} \right\rfloor$.

iii. $\gamma(K_n) = 1$.

2456-8686, vi(ii), 2022:116-120
https://doi.org/10.26524/cm153

iv. $\gamma(K_{m,n}) = 2$.
v. $\gamma(W_n) = 1$.

Proposition 1.5 [8]
i. $\gamma_{pr}(P_n) = 2 \cdot \left\lceil \frac{n}{4} \right\rceil$.
ii. $\gamma_{pr}(C_n) = 2 \cdot \left\lceil \frac{n}{4} \right\rceil$.
iii. $\gamma_{pr}(K_n) = 2$.
iv. $\gamma_{pr}(K_{m,n}) = 2$.
v. $\gamma_{pr}(W_n) = 2$.

Proposition 1.6 [7] If G is $k-$regular graph then $DS(G) \cong G + K_1$.

2 Main results

Theorem 2.1 The paired domination number $\gamma_{pr}(P_n^k) = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$.

Proof: Let $V = \{v_1, v_2, v_3, \cdots, v_n\}$ be the set of vertices in P_n^k. Consider any two vertices v_i and v_j such that $v_iv_j \in E(P_n^k)$. Clearly, distance between v_i & v_j is atmost k in P_n. It is also clear that the vertex v_i can dominate atmost $2k$ vertices and similarly the vertex v_j can dominate $2k$ vertices among which at most $k-1$ vertices are same. Thus any two vertices of the set V can dominate at most $3k+1$ vertices.
Case 1: $n < 3k+1$
Consider the vertices v_i and v_j such that
(i) $d(v_i)$ is maximum,
(ii) $d(v_j)$ is at most one less than $d(v_i)$ and
(iii) $v_1v_i,\ v_jv_n$ and $v_iv_j \in E(P_n^k)$.
Clearly, the set $\{v_i, v_j\}$ can dominate V and thus, it forms a minimal paired dominating set.
Thus, $\gamma_{pr}(P_n^k) = 2 = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$ for $n < 3k+1$.
Case 2: $n \geqslant 3k+1$.
Consider two vertices v_{k+1} and v_{2k+1}. Clearly $v_{k+1}v_{2k+1} \in E(P_n^k)$ and the vertex v_{k+1} dominates set of $2k$ vertices $\{v_1, v_2, v_3, \cdots, v_k\} \cup \{v_{k+2}, v_{k+3}, \cdots, v_{2k+1}\}$. Similarly the vertex v_{2k+1} dominates $2k$ vertices among which $k-1$ vertices are dominated by v_{k+1}. Thus the set $D = \{v_{k+1}, v_{2k+1}\}$ dominates $2k + 2k - (k-1) = 3k+1$ vertices.
Subcase 1: $n \equiv 0(\text{mod } (3k+1))$

2456-8686, vi(ii), 2022:116-120
https://doi.org/10.26524/cm153

Take $n = t \cdot (3k+1)$.
As the vertices v_{k+1} and v_{2k+1} can dominate $3k+1$ vertices, the set V can be dominated by $2 \cdot t$ vertices which forms a minimal paired dominating set.

$$\begin{aligned} \therefore \gamma_{pr}(P_n^k) &= 2 \cdot t \\ &= 2 \cdot \frac{n}{3k+1} \quad \text{if } n = t \cdot (3k+1) \end{aligned}$$

Subcase 2: $n \equiv a (\text{mod } (3k+1)), a < 3k+1 \ \& \ a \neq 0$.
Take $n = a + t \cdot (3k+1)$.
We require $2 \cdot t$ vertices in order to dominate $t \cdot (3k+1)$ vertices as in subcase 1.
For the remaining vertices $a < 3k+1$, we require two vertices by case 1.

$$\begin{aligned} \therefore \gamma_{pr}(P_n^k) &= 2 + 2 \cdot t \\ &= 2 \cdot (t+1) \\ &= 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil \quad \text{for } n = a + t \cdot (3k+1) \text{ where } a < 3k+1 \end{aligned}$$

Thus, $\gamma_{pr}(P_n^k) = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$.

Theorem 2.2 The paired domination number $\gamma_{pr}(C_n^k) = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$.

Proof: The proof is similar to theorem 2.1.

Theorem 2.3 $\gamma_{pr}(DS(P_n)) = 4; \ n \geqslant 4$.

Proof: Consider path P_n with $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_1, e_2, e_3, \cdots, e_{n-1}\}$. There are two types of vertices in P_n: vertices v_1 & v_n with degree 1 and vertices $v_2, v_3, \cdots, v_{n-1}$ with degree 2. In order to get $DS(P_n)$, consider $S_1 = \{v_2, v_3, \cdots, v_{n-1}\}$, $S_2 = \{v_1, v_n\}$ and $T = \phi$. Take w_i corresponds to S_i and join w_i to S_i; $i = 1, 2$. If we consider $D = \{w_1, v_k, w_2, v_i; \ v_k \in S_1, v_i \in S_2\}$, we get D as a dominating set that induces a perfect matching. It is very clear to observe that by removing any pair of vertices from it, it will not remain as a dominating set. Hence D forms a minimal paired dominating set. Thus $\gamma_{pr}(DS(P_n)) = 4$.

Theorem 2.4 $\gamma_{pr}(DS(C_n)) = 2; \ n \geqslant 3$.

Proof: By proposition 1.6, $DS(C_n) \cong C_n + K_1 = W_n$.
$\therefore \gamma_{pr}(DS(C_n)) = 2$ by proposition 1.5.

2456-8686, vi(ii), 2022:116-120
https://doi.org/10.26524/cm153

3 Conclusion

The paired domination is mainly applied in networking and military surveillance. In this paper we have found the paired domination number of k^{th} power of path and cycle as well as that of degree splitting graph of path and cycle.

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2456-8686, vi(ii), 2022:121-127
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ORIGINAL RESEARCH Open Access

A Study on Fuzzy Simply Lindelof Spaces

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Abstract

In this paper, the concept of fuzzy simply Lindelof spaces is introduced. Several characterizations of fuzzy simply Lindelof spaces are given.

Key words: Fuzzy dense set, Fuzzy nowhere dense set, Fuzzy simply open set, fuzzy second category space, Fuzzy submaximal space, Strongly irresolvable space.

AMS classification: 03E72, 16Y80, 03B52, 20N25.

1 Introduction

Many generalizations of Lindelof spaces have been introduced and studied by several authors. Among the various covering properties of topological spaces, a lot of attention has been made to those covers which involve open and regular open sets in classical topology. In 1982 G.Balasubramanian introduced and studied the notion of nearly Lindeof spaces. In 1984 S.Willard and U.N.B. Dissanayake gave the notion of almost Lindelof spaces and in 1996 F. Cammaroto and G. Santoro introduced the notion of weakly regular Lindelof spaces on using regular covers. In 1965, L.A.Zadeh introduced the concept of fuzzy sets as a new approach for modelling uncertainties. In 1968, C.L.Chang introduced the concept of fuzzy topological spaces. The paper of Chang proved the way for the subsequent tremendous growth of the numerous fuzzy topological spaces. In this paper by means of fuzzy simply open sets, the notion of fuzzy simply Lindelof spaces, is introduced and studied. Several examples are given to illustrate the concepts introduced in this paper.

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2 Preliminaries

Definition 2.1 Let (X,T) be a fuzzy topological space and λ be any fuzzy in (X,T). The interior and the closure of λ are defined as follows

(i) $\text{Int}(\lambda) = \vee\{\ \mu/\mu \leq \lambda, \mu \in T\}$
(ii) $Cl(\lambda) = \wedge\{\ \mu/\lambda \leq \mu, 1-\mu \in T\}$

Definition 2.2 A fuzzy set λ in a fuzzy topological space (X,T), is called fuzzy dense if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$. That is, $\text{cl}(\lambda) = 1$, in (X,T).

Definition 2.3 A fuzzy set λ in a fuzzy topological space (X,T), is called fuzzy nowhere dense if there exists no non zero fuzzy open set μ in (X,T) such that $\mu <$ $\text{cl}(\lambda)$. That is, $\text{intcl}(\lambda) = 0$, in (X,T).

Definition 2.4 A fuzzy set λ in a fuzzy topological space (X,T), is called a fuzzy simply open set if $\text{Bd}(\lambda)$ is a fuzzy nowhere dense set in (X,T). That is, λ is a fuzzy simply open set in (X,T) if $\text{cl}\ (\lambda) \wedge \text{cl}(1-\lambda)$, is a fuzzy nowhere dense set in (X,T).

Definition 2.5 A fuzzy topological space (X,T) is said to be fuzzy Lindelof if every fuzzy open cover of X has a countable subcover. That is, for every fuzzy open cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of X, there exists $\{\lambda_{\alpha_n}\}_{n\in N}$ of fuzzy open sets in (X,T) such that $\bigvee_{n\in N}$ $\{\lambda_{\alpha_n}\} = 1$.

Theorem 2.6 If λ is a fuzzy open and fuzzy dense set in a fuzzy topological space (X,T), then λ is a fuzzy simply open set in (X,T).

Definition 2.7 Let (X,T) be a fuzzy topological space. A fuzzy set λ in a (X,T) is called a fuzzy first category set if $\lambda = \bigvee_{\alpha=1}^{\infty}(\lambda_\alpha)$, where (λ_α) 's are fuzzy nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of fuzzy second category.

Definition 2.8 A fuzzy topological space (X,T) is called fuzzy first category space if $1_x = V_{\alpha=1}^{\infty}(\lambda_\alpha)$, where (λ_α)'s are fuzzy nowhere dense sets in (X,T). A fuzzy topological space which is not of fuzzy first category is said to be of fuzzy second category.

Definition 2.9 A fuzzy topological space (X, T) is called a fuzzy submaximal space if for each fuzzy set λ in (X, T) such that cl $(\lambda) = 1$, then $\lambda \in T$

Theorem 2.10 If λ is a fuzzy closed set with $\text{int}(\lambda) = 0$, in a fuzzy topological space (X, T), then λ is a fuzzy simply open set in (X, T).

Theorem 2.11 If λ is a fuzzy simply open set in a fuzzy topological (X, T), then $\lambda \wedge (1 - \lambda)$ is a fuzzy nowhere dense set in (X, T).

Theorem 2.12 If λ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is a fuzzy simply open set in (X, T).

Theorem 2.13 If $\lambda = \mu \vee \delta$ where μ is a fuzzy open and fuzzy dense set and δ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is a fuzzy simply open set in (X, T).

Theorem 2.14 If int $(\lambda) = 0$ for a fuzzy set λ in a fuzzy strongly irresolvable space (X, T), then λ is a fuzzy simply open set in (X, T).

3 Fuzzy Simply Lindelof Spaces

Definition 3.1 A fuzzy topological space (X,T) is said to be fuzzy simply Lindelof if each cover of X by fuzzy simply open sets has a countable subcover. That is, (X, T) is a fuzzy simply Lindelof space if $V_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$ where intcl $[\text{bd}(\lambda_\alpha)] = 0$ in (X, T), then $V_{n\in\mathbb{N}} \{\lambda_{\alpha_n}\} = 1$ in (X, T).

Proposition 3.2 If (X,T) is a fuzzy simply Lindelof space and if $\bigvee_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$ where $\{\lambda_\alpha\}$'s are fuzzy closed sets with int $(\lambda_\alpha) = 0$ in (X, T), then $\bigvee_{n\in N} \{\lambda_{\alpha_n}\} = 1$, in (X, T).

proof:

Let (X, T) be a fuzzy simply Lindelof space. By hypothesis, the fuzzy sets $\{\lambda_\alpha\}$'s are fuzzy closed sets with int $(\lambda_\alpha) = 0$ in (X, T). Then by Theorem 2.10, $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T). Now $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T), implies that $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a fuzzy simply open cover of X. Since (X,T) is a fuzzy simply Lindelof space, there exist a countable subcover $\{\lambda_{\alpha_n}\}_{n\in\mathbb{N}}$ of fuzzy simply open sets, for X. Hence $\bigvee_{n\in\mathbb{N}}\{\lambda_{\alpha_n}\}=1$, where $(1-\lambda_{\alpha_n})\in T$ and int $\{\lambda_{\alpha_n}\}=0$, in (X,T).

Proposition 3.3 If (X,T) is a fuzzy simply Lindelof space and if $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\lambda_\alpha\in T$ and $\mathrm{cl}\,(\lambda_\alpha)=1$ in (X,T), then $\bigwedge_{n\in N}\{\mu_{\alpha_n}\}=0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X,T).

proof:

Let (X,T) be a fuzzy simply Lindelof space. By hypothesis, the fuzzy sets $\{\lambda_\alpha\}$'s are fuzzy open and fuzzy dense sets in (X,T).Then by Theorem 2.10, $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T). Now $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T), implies that $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a fuzzy simply open cover of X. Since (X,T) is a fuzzy simply Lindelof space, there exist a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ of fuzzy simply open sets for X and then $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$ in (X,T). Then $1-\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=0$ and hence $\bigwedge_{n\in N}(1-\lambda_{\alpha_n})=0$. Let $\mu_{\alpha_n}=1-\lambda_{\alpha_n}$. Now intcl $(1-\lambda_{\alpha_n})=1-cl$ int $(\lambda_{\alpha_n})=1-\mathrm{cl}\,(\lambda_{\alpha_n})=1-1=0$, and thus $(1-\lambda_{\alpha_n})$'s are fuzzy nowhere dense set in (X,T).Hence $\bigwedge_{n\in N}\{\mu_{\alpha_n}\}=0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X,T).

Proposition 3.4 If (X,T) is a fuzzy simply Lindelof space, then (X,T) is a fuzzy second category space.

proof:

Let (X,T) be a fuzzy simply Lindelof space and $\{\lambda_\alpha\}_{\alpha\in\Delta}$ be a cover of X by fuzzy simply open sets in (X,T). By Theorem 2.11, $\{\lambda_{\alpha_n}\wedge((1-\lambda_{\alpha_n})\}$'s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X. Then $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$ in (X,T). Now $[\lambda_{\alpha_n}\wedge(1-\ \lambda_{\alpha_n})]\leq\lambda_{\alpha_n}$ in (X,T) implies that $\bigvee_{n\in N}[\lambda_{\alpha_n}\wedge(1-\lambda_{\alpha_n})]\leq\bigvee_{n\in N}\{\lambda_{\alpha_n}\}$ and then $\bigvee_{n\in N}\ [\lambda_{\alpha_n}\wedge(1-\lambda_{\alpha_n})]\ \leq\ 1$. Then $\bigvee_{n\in N}[\lambda_{\alpha_n}\wedge((1-\lambda_{\alpha_n})]\neq 1$, where $\{\lambda_{\alpha_n}\wedge(1-\lambda_{\alpha_n})\}$'s are fuzzy nowhere dense sets, implies that (X,T) is not a fuzzy first category space and hence (X,T) is a fuzzy second category space.

Proposition 3.5 If $\{\lambda_a\}_{\alpha\in\Delta}$ is a cover of X by fuzzy nowhere dense sets in a fuzzy simply Lindelof space (X,T), then there is a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X.

proof:

If $\{\lambda_a\}_{\alpha\in\Delta}$ is a cover of X by fuzzy nowhere dense sets in (X,T), then $\bigvee_{\alpha\epsilon\Delta}$ $\{\lambda_\alpha\}=1$, where $\operatorname{intcl}(\lambda_\alpha)=0$ in (X,T). By theorem 2.12, the fuzzy nowhere dense sets $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T) and thus $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a cove of X by fuzzy simply open sets. Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ of X. That is, $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$ where intcl $(\lambda_{\alpha_n})=0$ in (X,T).

Proposition 3.6 If $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\lambda_\alpha=\mu_\alpha\vee\delta_\alpha$ and $\{\mu_\alpha\}$'s are fuzzy open and fuzzy dense sets and $\{\delta_\alpha\}$'s are fuzzy nowhere dense sets in a fuzzy simply Lindelof space (X,T), then $\eta\vee\delta=1$, where $\eta\in T$ and $cl(\eta)=1$ and δ is a fuzzy first category set in (X,T).

Proof:

Let (X,T) be a fuzzy simply Lindelof space such that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ and $\lambda_\alpha=\mu_\alpha\vee\delta_\alpha$, where $\mu_\alpha\in T$ and $cl(\mu_\alpha)=1$ and intcl $(\delta_\alpha)=0$ in (X,T) By theorem 2.13, $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ of X. That is, $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$. Then, $\bigvee_{n\in N}[\mu_{\alpha_n}\vee\delta_{\alpha_n}]=\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]\bigvee\left[\bigvee_{n\in N}(\delta_{\alpha_n})\right]$ implies that $\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]\bigvee\left[\bigvee_{n\in N}(\delta_{\alpha_n})\right]=1$. Since $\{\delta_\alpha\}$'s are fuzzy nowhere dense sets in $(X,T),\bigvee_{n\in\mathbb{N}}\{\delta_{\alpha_n}\}=\delta$, implies that δ is a fuzzy first category set,in (X,T). Since $(\mu_{\alpha_n})\in T,\bigvee_{n\in N}(\mu_{\alpha_n})\in T$. Also $cl\left[\bigvee_n\in\mathbb{N}\ \ (\mu_{\alpha_n})\right]\geq\bigvee_{n\in N}cl\left[(\mu_{\alpha_n})\right]$ implies that $cl\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]\geq\bigvee 1$ and hence $cl\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]=1$. Let $\eta=\bigvee_{n\in N}(\mu_{\alpha_n})$. Then η is a fuzzy open and fuzzy dense set in (X,T). Thus $\delta\vee\eta=1$ in (X,T).

Proposition 3.7 If $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a cover of X by fuzzy sets with $\operatorname{int}(\lambda_\alpha)=0$, in a fuzzy strongly irresolvable and fuzzy simply Lindelof space (X,T), then $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$.

Proof:

Suppose that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$, where $\operatorname{int}(\lambda_\alpha)=0$ in (X,T).Since (X,T) is a fuzzy strongly irresolvable space, by Theorem 2.14, $\{\lambda_{\alpha_n}\}$'s are fuzzy simply open sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, $V_{\alpha\in\Delta}\{\lambda_\alpha\}=1$, where $\{\lambda_\alpha\}$'s are fuzzy simply open sets implies that there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X by fuzzy simply open sets. That is, $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$, where $\operatorname{int}(\lambda_{\alpha_n})=0$ in (X,T).

Proposition 3.8 If (X,T) is a fuzzy simply Lindelof space and fuzzy submaximal space and if $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\{\lambda_\alpha\}$'s are fuzzy dense sets in (X,T), then $\bigwedge_{n\in N}$

$\{\mu_{\alpha_n}\} = 0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X,T).

proof:

Let (X,T) be a fuzzy simply Lindelof and fuzzy submaximal space. By hypothesis $\bigvee_{\alpha\in\Delta}\{\lambda_a\} = 1$, where $\text{cl}(\lambda_\alpha) = 1$ in (X,T). Since (X,T) is a fuzzy submaximal space, the fuzzy dense sets $\{\lambda_a\}$'s are fuzzy open sets in (X,T). Hence $\{\lambda_\alpha\}$'s are fuzzy open sets and fuzzy dense sets in (X,T). Then, by proposition 3.3, $\bigwedge_{n\in N}\{\mu_{\alpha_n}\} = 0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X,T).

Proposition 3.9 If $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$, where intcl $(\lambda_\alpha) = 0$ in a fuzzy simply Lindelof space (X,T), then there exist a fuzzy first category set λ in (X,T) such that cl $(\lambda) = 1$ in (X,T).

proof:

Let (X,T) be a fuzzy simply Lindelof space such that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$ and $\text{intcl}(\lambda_\alpha) = 0$. Now $\text{int cl}(\lambda_\alpha) = 0$ in (X,T) implies that (λ_a)'s are fuzzy nowhere dense sets in (X,T). Then, by theorem 2.7, $\{\text{cl}(\lambda_\alpha)\}$'s are fuzzy simply open sets in (X,T). Now $\lambda_\alpha \leq \text{cl}(\lambda_\alpha)$ implies that $\bigvee_\alpha \in \Delta\{\lambda_\alpha\} \leq \bigvee_\alpha \in \Delta\,\text{cl}(\lambda_\alpha)$ and then $1 \leq \bigvee_{\alpha\in\Delta}\text{cl}(\lambda_\alpha)$. That is $\bigvee_{\alpha\in\Delta}\text{cl}(\lambda_\alpha) = 1$ where $\{\text{cl}(\lambda_\alpha)\}$'s are fuzzy simply open sets in the fuzzy Lindelof space (X,T). Then there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X by fuzzy simply open sets in (X,T). That is, $\bigvee_{n\in N}\text{cl}(\lambda_{\alpha_n}) = 1$. Now, by lemma, $\bigvee_{n\in N} cl(\lambda_{\alpha_n}) \leq cl\left(\bigvee_{n\in N}\{\lambda_{\alpha_n}\}\right)$ implies that $1 \leq cl\left(\bigvee_{n\in N}\{\lambda_{\alpha_n}\}\right)$. That is, $cl\left(\bigvee_{n\in N}\{\lambda_{\alpha_n}\}\right) = 1$. Let $\bigvee_{n\in N}\{\lambda_{\alpha_n}\} = \lambda$. Then λ is a fuzzy first category set in (X,T) such that $\text{cl}(\lambda) = 1$, in (X,T).

Proposition 3.10 If $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$ where (λ_α)'s are fuzzy nowhere dense sets in a fuzzy simply Lindelof and fuzzy Baire space (X,T) then (X,T) is a fuzzy resolvable space.

Proof:

Let (X,T) be a fuzzy simply Lindelof space such that $\bigvee_\alpha \epsilon\Delta\{\lambda_\alpha\} = 1$ where $\text{intcl}(\lambda_\alpha) = 0$ in (X,T). Then, by proposition 3.9, there exist a fuzzy first category set λ in (X,T) such that $\text{cl}(\lambda) = 1$. Since (X,T) is a fuzzy Baire space, by theorem int $(\lambda) = 0$ in (X,T). Now $\text{cl}(1-\lambda) = 1 - \text{int}(\lambda) = 1 - 0 = 1$.Thus $\text{cl}(\lambda) = 1$ and $\text{cl}(1-\lambda) = 1$ in (X,T). Hence (X,T) is a fuzzy resolvable space.

Proposition 3.11 If $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$, where (λ_α)'s are fuzzy open sets in a fuzzy simply Lindelof and fuzzy hyper connected space (X,T) then there exists a countable subcover $\{\lambda_{\alpha_n}\}_{\mathrm{n}\in\mathrm{N}}$ for X.

Proof:

Let (X,T) be a fuzzy simply Lindelof and fuzzy hyper connected space such that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$, where $\lambda_\alpha \in T$. Since (X,T) is a fuzzy hyper connected space, the fuzzy open sets (λ_α)'s are fuzzy dense sets in (X,T). Then (λ_α)'s are fuzzy open and fuzzy dense sets in (X,T). Then, by theorem 2.6, (λ_a)'s are fuzzy simply open sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X.

4 Conclusion

In this paper we have presented a Fuzzy Simply Lindelof Spaces through definitions and also stated important theorems. Using that we have proved some propositions.

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ORIGINAL RESEARCH Open Access

A Study on Intuitionistic Fuzzy Baire Spaces

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Abstract

In this paper, the concepts of intuitionistic fuzzy Baire spaces are introduced and characterizations of intuitionistic fuzzy Baire spaces are studied.

Key words: Intuitionistic fuzzy first category, Intuitionistic fuzzy second category, Intuitionistic fuzzy residual set, Intuitionistic fuzzy Baire spaces.

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1 Introduction

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy sets by L.A.Zadeh. The theory of fuzzy topological space was introduced and developed by C.L.Chang and since then various notions in classical topology have been extended to fuzzy topological space. The idea of "intuitionistic fuzzy set" was first published by Atanassov and many works by the same author and his colleagues appeared in the literature. Later, this concept was generalized to "intuitionistic L - fuzzy sets" by Atanassov and Stoeva. The concept of somewhat fuzzy continuous functions and somewhat fuzzy open hereditarily irresolvable by G.Thangaraj and G.Balasubramanian.

In this paper the concepts of intuitionistic fuzzy Baire spaces are introduced and characterizations of intuitionistic fuzzy Baire spaces are studied.

Definition 1.1 Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS for short) A is an object having the form $A = \{\langle x, \mu_A(x), \delta_A(x)\rangle : x \in X\}$ where the function $\mu_A : X \to I$ and $\delta_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership ($\delta_A(x)$) of each element $x \in X$ to the set

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A, respectively, and $0 \leq \mu_A(x) + \delta_A(x) \leq 1$ for each $x \in X$.

Definition 1.2 Let X be a nonempty set and the intuitionistic fuzzy sets A and B in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;
(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$;
(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;
(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;
(f) $[\ ]\ A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$
(g) $\langle\ \rangle\ A = \{\langle x, 1 - \gamma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.3 $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$.

Definition 1.4 An intuitionistic fuzzy topology (IFT) on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$,
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \tau$.

In this case the ordered pair (X, τ) or simply X is called an intuitionistic fuzzy topological space (IFTS) and each IFS in τ is called an intuitionistic fuzzy open set (IFOS). The complement $\bar{A}$ of an IFOS A in X is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 1.5 Let A be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space X. Then $\text{int}(A) = \bigcup\{G \mid G$ is an IFOS in X and $G \subseteq A\}$ is called the intuitionistic fuzzy interior of A; $cl(A) = \bigcap\{G \mid G$ is an IFCS in X and $G \supseteq A\}$ is called the intuitionistic fuzzy closure of A.

Definition 1.6 An intuitionistic fuzzy set A in intuitionistic fuzzy topological space (X, T) is called intuitionistic fuzzy dense if there exists no intuitionistic fuzzy closed set B in (X, T) such that $A \subset B \subset 1_\sim$

2456-8686, vi(ii), 2022:128-134
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Proposition 1.7 If A is an intuitionistic fuzzy nowhere dense set in (X, T), then A is an intuitionistic fuzzy dense set in (X, T).

Proposition 1.8 Let A be an intuitionistic fuzzy set. If A is an intuitionistic fuzzy closed set in (X, T) with IFint $A) = 0_{\sim}$, then A is an intuitionistic fuzzy nowhere dense set in (X, T).

2 Intuitionistic Fuzzy Baire Spaces

Definition 2.1 Let (X, T) be an intuitionistic fuzzy topological space. An intuitionistic fuzzy set A in (X, T) is called intuitionistic fuzzy first category if $A = \bigcup_{i=1}^{\infty} B_i$, where B_i's are intuitionistic fuzzy nowhere dense sets in (X, T). Any other intuitionistic fuzzy set in (X, T) is said to be of intuitionistic fuzzy second category.

Definition 2.2 An intuitionistic fuzzy topological space (X, T) is called intuitionistic fuzzy first category space if the intuitionistic fuzzy set $1_{\sim}$ is an intuitionistic fuzzy first category set in (X, T). That is, $1_{\sim} = \bigcup_{i=1}^{\infty} A_i$ where A_i's are intuitionistic fuzzy nowhere dense sets in (X, T). Otherwise (X, T) will be called an intuitionistic fuzzy second category space.

Proposition 2.3 If A be an intuitionistic fuzzy first category set in (X, T), then $\bar{A} = \bigcap_{i=1}^{\infty} B_i$ where $\mathrm{IFcl}\,(B_i) = 1_{\sim}$.

Proof:

Let A be an intuitionsitic fuzzy first category set in (X, T). Then $A = \bigcup_{i=1}^{\infty} A_i$, where A_i's are intuitionistic fuzzy nowhere dense sets in (X, T). Now $\bar{A} = \bigcup_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \left(\overline{A_i}\right)$. Now A_i is an intuitionistic fuzzy nowhere dense set in (X, T). Then, by Proposition 1.7, we have $\overline{A_i}$ is an intuitionistic fuzzy dense set in (X, T). Let us put $B_i = \overline{A_i}$. Then $\bar{A} = \bigcap_{i=1}^{\infty} B_i$ where $\mathrm{IFcl}\,(B_i) = 1_{\sim}$.

Definition 2.4 Let A be an intuitionistic fuzzy first category set in (X, T). Then $\bar{A}$ is called an intuitionistic fuzzy residual set in (X, T).

Definition 2.5 Let (X, T) be an intuitionistic fuzzy topological space. Then (X, T) is said to intuitionistic fuzzy Baire space if $\mathrm{IFint}\,(\bigcup_{i=1}^{\infty} A_i) = 0_{\sim}$, where A_i's are intuitionistic fuzzy nowhere dense sets in (X, T).

2456-8686, vi(ii), 2022:128-134
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Proposition 2.6 If IFint $\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_\sim$ where IFint $(A_i) = 0_\sim$ and $A_i \in T$, then (X,T) is an intuitionistic fuzzy Baire space.

proof:

Now $A_i \in T$ implies that A_i is an intuitionistic fuzzy closed set in (X,T). Since IFin $t\,(A_i) = 0_\sim$. By Proposition 1.8, A_i is an intuitionistic fuzzy nowhere dense set in (X,T). Therefore IFint $\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_\sim$. where A_i's are intuitionistic fuzzy nowhere dense set in (X,T). Hence (X,T) is an intuitionistic fuzzy Baire space.

Proposition 2.7 IF IFcl $\left(\bigcap_{i=1}^{\infty} A_i\right) = 1_\sim$ where A_i 's are intuitionistic fuzzy dense and intuitionistic fuzzy open sets in (X,T), then (X,T) is an intuitionistic fuzzy Baire Space.

Now IFcl $\left(\bigcap_{i=1}^{\infty} A_i\right) = 1_\sim$ implies that $\overline{\text{IFcl}\left(\bigcap_{i=1}^{\infty} A_i\right)} = 0_\sim$. Then we have IFint $\left(\overline{\bigcap_{i=1}^{\infty} A_i}\right) = 0_\sim$. Which implies that IFint $\left(\bigcup_{i=1}^{\infty} \overline{A_i}\right) = 0_\sim$. Let $B_i = \overline{A_i}$. Then IFin $t\left(\bigcup_{i=1}^{\infty} B_i\right) = 0_\sim$. Now $A_i \in T$ implies that $\overline{A_i}$ is an intuitionistic fuzzy closed set in (X,T) and hence B_i is an intuitionistic fuzzy closed and IFint $(B_i) =$ IFint $\left(\overline{A_i}\right) = \overline{\text{IFcl}\,(A_i)} = 0_\sim$. Hence By Proposition 1.8, B_i is an intuitionistic fuzzy nowhere dense set in (X,T). Hence IFint $\left(\bigcup_{i=1}^{\infty} B_i\right) = 0_\sim$ where B_i's are intuitionistic fuzzy nowhere dense sets, implies that (X,T) is an intuitionistic fuzzy Baire space.

Proposition 2.8 Let (X,T) be an intuitionistic fuzzy topological space. Then the following are equivalent

(i) (X,T) is an intuitionistic fuzzy Baire space.

(ii) IFint $(A) = 0_\sim$, for every intuitionistic fuzzy first category set A in (X,T).

(iii) IFcl$(B) = 1_\sim$, for every intuitionistic fuzzy residual set B in (X,T).

Proof:

(i) $\Rightarrow$ (ii) Let A be an intuitionistic fuzzy first category set in (X,T).
Then $A = \left(\bigcup_{i=1}^{\infty} A_i\right)$ where A_i's are intuitionistic fuzzy nowhere dense sets in (X,T). Now IFint$(A) =$ IFint $\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_\sim$. Since (X,T) is an intuitionistic fuzzy Baire space. Therefore IFint$(A) = 0_\sim$.

(ii) $\Rightarrow$ (iii) Let B be an intuitionistic fuzzy residual set in (X,T). Then $\bar{B}$ is an intuitionistic fuzzy first category set in (X,T). By hypothesis IFint$(\bar{B}) = 0_\sim$ which implies that $\overline{\text{IFcl}(A)} = 0_\sim$. Hence IFcl$(A) = 1_\sim$.

(iii) $\Rightarrow$ (i) Let A be an intuitionistic fuzzy first category set in (X,T). Then $A = \left(\bigcup_{i=1}^{\infty} A_i\right)$ where A_i's are intuitionistic fuzzy nowhere dense sets in (X,T). Now A

2456-8686, vi(ii), 2022:128-134
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is an intuitionistic fuzzy first category set implies that $\bar{A}$ is an intuitionistic fuzzy residual set in (X, T). By hypothesis, we have $IFcl(\bar{A}) = 1_{\sim}$, which implies that $\text{IFint}(A) = 1_{\sim}$. Hence $\text{IFint}(A) = 0_{\sim}$. That is, $\text{IFint}\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_{\sim}$, where A_i 's are intuitionistic fuzzy nowhere dense sets in (X, T). Hence (X, T) is an intuitionsitic fuzzy Baire space.

Proposition 2.9 An intuitionistic fuzzy topological space (X, T) is an intuitionistic fuzzy Baire space if and only if $\left(\bigcup_{i=1}^{\infty} A_i\right) = 1_{\sim}$, where A_i's is an intuitionistic fuzzy closed set in (X, T) with $\text{IFint}\ (A_i) = 0_{\sim}$, implies that $\text{IFint}\ \left(\bigcup_{i=1}^{\infty} A_i\right) = 0_{\sim}$.
proof:

Let (X, T) be an intuitionistic fuzzy Baire space. Now A_i is an intuitionistic fuzzy closed in (X, T) and $\text{IFint}\,(A_i) = 0_{\sim}$, implies that A_i is an intuitionistic fuzzy nowhere dense set in (X, T). Now $\bigcup_{i=1}^{\infty} A_i = 1_{\sim}$ implies that $1_{\sim}$ is an intuitionistic fuzzy first category set in (X, T). Since (X, T) is an intuitionistic fuzzy Baire space space, by Proposition 2.8, $\text{IFint}\,(1_{\sim}) = 0_{\sim}$. That is, $\text{IFint}\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_{\sim}$.
Conversely suppose that $\text{IFint}\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_{\sim}$ where A_i. By Proposition 1.8, A_i is an intuitionistic fuzzy nowhere dense set in (X, T). Hence $\text{IFint}\left(\bigcup_{i=1}^{\infty} A_i\right) = 0_{\sim}$ implies that (X, T) is an intuitionistic fuzzy Baire space.

Definition 2.10 Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. A map $f : (X, T) \rightarrow (Y, S)$ is said to be an intuitionistic fuzzy open if the image of every intuitionistic fuzzy open set A in (X, T) is intuitionistic fuzzy open $f(A)$ in (Y, S).

Proposition 2.11 Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is an onto intuitionsitic fuzzy contra continuous and intuitionistic fuzzy open then (Y, S) is an intuitionistic fuzzy Baire space.
Proof:

Let A be an intuitionistic fuzzy first category set in (Y, S). Then $A = \left(\bigcup_{i=1}^{\infty} A_i\right)$ where A_i are intuitionistic fuzzy nowhere dense sets in (Y, S). Suppose $\text{IFint}(A) \neq 0_{\sim}$. Then there exists an intuitionistic fuzzy open set $B \neq 0 \sim$ in (Y, S), such that $B \subseteq A$. Then $f^{-1}(B) \subseteq f^{-1}(A) = f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}\left(A_i\right)$. Hence

$$f^{-1}(B) \subseteq \bigcup_{i=1}^{\infty} f^{-1}\left(\text{IFcl}\left(A_i\right)\right) \tag{1}$$

Since f is intuitionistic fuzzy contra continuous and $\text{IFcl}(A_i)$ is an intuitionistic fuzzy closed set in (Y, S), $f^{-1}(\text{IFcl}(A_i))$ is an intuitionistic fuzzy open in (X, T). From Eq (1) we have

$$f^{-1}(B) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(\text{IFcl}(A_i)) = \bigcup_{i=1}^{\infty} \text{IFint}\left(f^{-1}(\text{IFcl}(A_i))\right) \tag{2}$$

Since f is intuitionsitic fuzzy open and onto, $\text{IFint}(f^{-1}(A_i)) \subseteq f^{-1}(\text{IFint}(A_i))$. From Eq (2), we have $f^{-1}(B) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(\text{ IFint } Fcl(A_i)) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(0\sim) = 0_\sim$. Since A_i is an intuitionistic fuzzy nowhere dense. That is, $f^{-1}(B) \subseteq 0_i$ and hence $f^{-1}(B) = 0_\sim$ which implies that $B = 0_\sim$, which is a contradiction to $B \neq 0_\sim$. Hence $\text{Ifint}(A) = 0_\sim$ where A is an intuitionsitic fuzzy first category set in (Y, S). Hence by Proposition 2.8, (Y, S) is an intuitionistic fuzzy Baire space.

3 Conclusion

In this paper we have presented a Intuitionistic Baire spaces with important definitions and also proved required prepositions. Also we discussed the characterizations of Intuitionistic fuzzy Baire spaces.

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On Connectivity of Zero-Divisor Graphs and Complement Graphs of some Semi-Local Rings

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Abstract

Zero-divisor graphs have been a key area of focus for many researchers. For the semi local ring R of finite cartesian product of finite fields, we consider the zero divisor graph of R denoted by $\Gamma(R)$ with vertex set as the non-zero zero-divisors of R where two vertices u and v are adjacent if and only if the product of u and v is the additive identity of the Ring R. The objective of this paper is to determine the number of cut vertices and cut edges, vertex connectivity and edge connectivity of the zero divisor graph $\Gamma(R)$ and complement graph $\overline{\Gamma(R)}$.

Key words: Zero-divisor graphs, Cut vertices, Cut edges, Vertex connectivity, Edge connectivity.

AMS classification: 13A70, 05C25, 05C40.

1 Introduction

Zero divisors graphs have been studied on general rings. The idea of Zero Divisor graphs of a commutative ring R has been introduced by I. Beck [3]. Originally all the elements of the ring R has been considered as vertices of this graph and two vertices u and v are adjacent if and only if $u \cdot v = 0$. This definition is modified by Anderson and Livingston [2] wherein the vertex set is reduced to only the set of non-zero zero divisors of R. Let $n \geq 2$ and $F_1, F_2. \cdots, F_n$ be finite fields then $R = F_1 \times F_2 \times \cdots F_n$ is a semi local ring. Let $Z^*(R)$ denote the set of non-zero, zero-divisors of R and let $\Gamma(R)$ denote the graph with vertex set as $Z^*(R)$ and edge set as $\{xy : x \cdot y = 0, x, y \in Z^*(R)\}$ [4]. Since $Z(R)$ is closed under multiplication, the complement graph $\overline{\Gamma(R)}$ of the zero-divisor graph $\Gamma(R)$ satisfies the property: $rs \in E(\overline{\Gamma(R)})$ if and only if $r \cdot s \in Z^*(R)$.[1]

For basic graph theoretical terminologies we adopt the definitions of [6]. A vertex v is called as cut vertex of a graph G if number of components of $G - v$ is greater than

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the number of components of G and an edge e of a graph G is said to be a cut edge if number of components of $G-e$ is greater than the number of components of G[6]. A subset $S \subset V(G)$ is said to be a vertex cut if $G-S$ is disconnected or has only one vertex and the cardinality of a minimum vertex cut is called as vertex connectivity of a graph G, denoted by $\kappa(G)$[6]. Simlarly $F \subset E(G)$ is said to be an edge cut if $G-F$ is disconnected and minimum size of an edge cut is called the edge connectivity of a graph G, denoted by $\lambda(G)$. [7] In this paper, we determine the number of cut vertices, cut edges, vertex connectivity, edge connectivity. of zero divisor graph over the ring $R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 2)$, where F_1, F_2, $\cdots$, F_n are finite fields.

2 Cut vertices and Cut edges, Vertex connectivity and Edge connectivity

Remark 2.1 If $R = F_1 \times F_2$, and $|F_1| = |F_2| = 2$, then $\Gamma(R)$ is K_2 and does not contain cut vertex. Further, if $R = F_1 \times F_2$ with $|F_1| \geqslant 2$, $|F_2| \geqslant 2$, then it is easy to check $\Gamma(R)$ is a complete bipartite graph $K_{1,|F_2|-1}$ *or* $K_{|F_1|-1,1}$ which is also a tree. Hence, the number of cut vertices in this case is 1. [6]

Theorem 2.2 (i) Let $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_i| = 2$, $(1 \leqslant i \leqslant n)$. If $n \geqslant 3$ then the Zero-divisor graph $\Gamma(R)$ has exactly n cut vertices.
(ii) Let $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$. If $n \geq 2$ and $|F_1| \geqslant 3$ or if $n \geqslant 3$ and $|F_2| \geq 3$ then the Zero-divisor graph $\Gamma(R)$ has no cut vertex.
Proof: (i) Let $n \geqslant 3$ and $|F_i| = 2$, $(1 \leqslant i \leqslant n)$. There are $C(n,r)$ vertices of degree $2^r - 1, (1 \leqslant r \leqslant n-1)$ in $\Gamma(R)$. Let $V_r = \{v \in V(\Gamma(R)) : v$ has exactly r number of zero entries in its coordinates $\}, (1 \leqslant r \leqslant n-1)$. A vertex in the set V_{n-1} has unique $'1'$ say at i^{th} position and all the $n-1$ positions have $'0'$ entries. A vertex in the set V_1 has $'0'$ in i^{th} position and $'1'$ in all other positions. Thus elements of V_1 are of degree $'1'$. A vertex in V_{n-1} is adjacent to an unique vertex in V_1 and thus every vertex of V_{n-1} is a cut vertex. Further V_1 has no cut vertex and $< V_1 \cup V_{n-1} >$ is connected. We will prove that for every $z \in V(\Gamma(R)) \setminus \{V_1 \cup V_{n-1}\}$ is not a cut vertex. Suppose, $z \in V_r, (2 \leqslant r \leqslant n-2)$. We will prove that $\Gamma(R) - z$ is connected. Now $z \in V_r \implies z$ contain non-zero entries, that is $'1'$ in $n-r$ positions, t_1, $\cdots$, t_{n-r}, $(2 \leqslant r \leqslant n-2)$ with $(1 \leqslant t_1 < \cdots < t_{n-r} \leqslant n)$. Let y_1, $y_2 \in \{V_2 \cup \cdots \cup V_{n-2}\} - z$. Since, y_1 and y_2 contains at least two $0's$, both are adjacent to at least two vertices in V_{n-1}. If y_1 and y_2 are adjacent to same vertex, say, x in V_{n-1}, then y_1, x, y_2 is a path connecting y_1 and y_2. If y_1 and y_2 are adjacent to different vertices, say, x_1 and x_2 in V_{n-1}, then

since, x_1 and x_2 are adjacent in V_{n-1}, we have a path y_1,x_1,x_2,y_2 connecting y_1 and y_2 and this path does not contain z. Thus every pair of vertices
(a) $y_1,y_2 \in \{V_1 \cup V_{n-1}\}$
(b) $y_1,y_2 \in \{V_2 \cup \cdots \cup V_{n-2}\} - z$
(c) one of $y_1,y_2 \in \{V_1 \cup V_{n-1}\}$ and the other $\in \{V_2 \cup \cdots \cup V_{n-2}\} - z$
are connected in a path not containing z. Therefore, $\Gamma(R) - z$ is connected. Thus, z is not a cut vertex. Hence, the zero-divisor graph $\Gamma(R)$ of semi-local ring $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| = |F_2| = \cdots = |F_n| = 2, (n \geqslant 3)$, has exactly $|V_{n-1}| = n$ cut vertices.

(ii) Case(1): Let $n \geq 2$, $|F_1| \geqslant 3$. We prove that $v \in V(\Gamma(R))$ is not a cut vertex. That is, we prove that $\Gamma(R) - v$ is connected. Fort $x,y \in \Gamma(R) - v$ if $xy \in E(\Gamma(R) - v)$, then nothing to prove. Suppose $xy \notin E(\Gamma(R) - v)$. Clearly x,y contains at least one entry as $'0'$ in its coordinates. Let x contains $'0'$ in i^{th} position and y contains $'0'$ in j^{th} position. It is possible that $i = j$.
Let $V_{n-1} = \{u \in V(\Gamma(R)) : u$ has exactly $n-1$ number of $'0'$ entries in its coordinates$\}$. Note that there exists $(|F_i| - 1)$ vertices in $V_{n-1} \subset V(\Gamma(R))$ containing non zero entry in i^{th} position. $\Rightarrow$ There exists at least $(|F_i| - 2)$ vertices in $V(\Gamma(R) - v)$ containing non zero entry in i^{th} position.
Since $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$ and $|F_1| \geqslant 3 \Rightarrow |F_i| - 2 \geq 1$, $(1 \leqslant i \leqslant n)$ and hence there is at least one vertex say $z \in V(\Gamma(R) - v)$ with one non-zero entry in i^{th} position and $'0'$ every where else. Clearly $xz \in E(\Gamma(R) - v)$. Similarly there is at least one vertex say $w \in V(\Gamma(R) - v)$ with one non-zero entry in j^{th} position and $'0'$ every where else with $yw \in E(\Gamma(R) - v)$. Further any two vertices in V_{n-1} with non-zero entries in different positions are adjacent. Thus, if $i = j$ then x and y are joined by the path $x - z - y$ and if $i \neq j$ then x and y are joined by the path $x - z - w - y$ in $\Gamma(R) - v$.
$\Rightarrow \Gamma(R) - v$ is connected $\Rightarrow v$ is not a cut vertex in $\Gamma(R)$.
Case(2): Let $n \geqslant 3$, $|F_2| \geq 3$. If $|F_1| \geq 3$, then by case (i) $\Gamma(R)$ has no cut vertex. Suppose $|F_1| = 2$, $|F_2| \geq 3$. For $v \in V(\Gamma(R))$ we prove that $\Gamma(R) - v$ is connected. For $x,y \in \Gamma(R) - v$ if $xy \in E(\Gamma(R) - v)$, we have nothing to prove. Let $xy \notin E(\Gamma(R) - v)$ and V_{n-1} be as defined in case (i). Note that x,y have $'0'$ entry in at least one position. Let x has $'0'$ entry at position i and y has $'0'$ entry in position j where $1 \leq i,j \leq n$. Further there exists a position say $k, 1 \leq k \leq n$ such that x,y both have non zero entry in position k.
If $k = 1$, then there exists at least two vertices in V_{n-1} with $'0'$ entry in position 1 and non zero entry in i and j position. This is possible as $n \geq 3$ and $|F_2| \geq 3$. Thus

there is a vertex $z \in \Gamma(R)-v$, such that xz, $yz \in E(\Gamma(R)-v)$ to get a path $x-z-y$ in $\Gamma(R)-v$.
If $k \neq 1$, then clearly there exists at least two vertices in V_{n-1} with $'0'$ entry in position 1 and non-zero entry in position i as $n \geq 3$ and hence there is a vertex $z_1 \in \Gamma(R)-v$ such that $xz_1 \in E(\Gamma(R)-v)$. Similarly there is a vertex $z_2 \in \Gamma(R)-v$ such that $yz_2 \in E(\Gamma(R)-v)$. If $i = j \implies z_1 = z_2$, then $x-z_1-y$ is a path in $\Gamma(R)-v$. If $z_1 \neq z_2$, then since $z_1z_2 \in E(\Gamma(R)-v)$ we get, a $x-z_1-z_2-y$ path in $\Gamma(R)-v$ and hence v is not a cut vertex. Hence there is no cut vertex in $\Gamma(R)$ in this case also.

Theorem 2.3 (i) If $R = F_1 \times F_2$ with $|F_1| = 2$ then there are $|F_2|-1$ cut edges in $\Gamma(R)$ and if $|F_2| = 2$ then there are $|F_1|-1$ cut edges in $\Gamma(R)$.
(ii) If $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_i| = 2$, $(1 \leqslant i \leqslant n)$ and $n \geqslant 3$ then the $\Gamma(R)$ has exactly n cut edges.
(iii) If $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$ and if $n \geq 2$ with $|F_1| \geqslant 3$ or if $n \geqslant 3$ with $|F_2| \geq 3$ then $\Gamma(R)$ has no cut edges.
Proof: (i) If $|F_1| = 2$, then $\Gamma(R) = K_{1,|F_2|-1}$ which contains exactly $|F_2|-1$ edges and each edge is a cut edge. If $|F_2| = 2$, then $\Gamma(R) = K_{|F_1|-1,1}$ which contains exactly $|F_1|-1$ edges and each edge is a cut edge. (ii) Let $n \geqslant 3$ *and* $|F_i| = 2$, $(1 \leqslant i \leqslant n)$. A vertex with $'0'$ in one position and $'1'$ in remaining positions is a degree 1 vertex and hence the unique edge adjacent to this vertex is a cut edge. There are n vertices of degree 1 in $\Gamma(R)$ and hence $\Gamma(R)$ contains at least n cut edges. Let $E_C(\Gamma(R))$ be the set of all these n cut edges.

Now let $e = uv \in E(\Gamma(R)) \setminus E_C(\Gamma(R))$. Then both the endpoints u and v of edge e contain $'0'$ in at least two positions. Suppose, u contains $'0'$ in i^{th} and j^{th} position where $1 \leqslant i < j \leqslant n$ and v contains $'0'$ in l^{th} and m^{th} position where $1 \leqslant l < m \leqslant n$. Note that i, j may be equal to l, m. Further u is adjacent to vertex v_i containing $'1'$ in i^{th} position and $'0'$ in all other positions and v is adjacent to vertex v_l containing $'1'$ in l^{th} position and $'0'$ in all other positions.
If $i = l$ then uv_ivu is a cycle and hence uv is not a cut edge.
If $i \neq l$ then v_iv_l is an edge and uv_iv_lvu is a cycle and hence uv is not a cut edge.
Therefore, the zero divisor graph of Semi-local ring $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| = |F_2| = \cdots = |F_n| = 2, (n \geqslant 3)$ has exactly $|E_C(\Gamma(R))| = n$ cut edges.

(iii) By property of cut edge, $e = uv$ is a cut edge $\Rightarrow$ either u or v is a cut vertex. By theorem, 2.2, for $n \geq 2$ and $|F_1| \geqslant 3$ or $n \geqslant 3$, $|F_1| = 2$, $|F_2| \geqslant 3$, $\Gamma(R)$ has no cut vertex $\Rightarrow \Gamma(R)$ has no cut edge.

Theorem 2.4 $\overline{\Gamma(R)}$ has no cut vertex.

Proof: For $n = 2$, $\overline{\Gamma(R)}$ is disconnected and hence has no cut vertex. If $n \geqslant 3$, we show that $\overline{\Gamma(R)}$ is connected and diameter of $\overline{\Gamma(R)}$ is 2. [5] Let $u,\ v \in V(\overline{\Gamma(R)})$. If $(u,v) \in E(\overline{\Gamma(R)})$, then $d(u,v) = 1$. Suppose, $(u,v) \notin E(\overline{\Gamma(R)})$. Then u and v do not contain non-zero entries in the same position. Let u contains $t_i \in \{1,2,\cdots,m_i-1\}$ in the i^{th} position and v contains $t_j \in \{1,2,\cdots,m_j-1\}$ in the j^{th} position. Clearly, $i \neq j$. Let $z \in V(\overline{\Gamma(R)}) \setminus \{u,v\}$ contain non-zero entry in the i^{th} and j^{th} position and $'0'$ elsewhere. Then, clearly, $(u,z),(v,z) \in E(\overline{\Gamma(R)})$. Therefore, $u-z-v$ is a path connecting u and v. Thus, $d(u,v) = 2$. Hence, $\overline{\Gamma(R)}$ is connected and diameter of $\overline{\Gamma(R)}$ is 2.

Let $v \in \overline{\Gamma(R)}$. We prove that v is not a cut vertex. Let $x,y \in \overline{\Gamma(R)} - v$. We claim that there exists an $x-y$ path in $\overline{\Gamma(R)} - v$.

If $xy \in E(\overline{\Gamma(R)} - v)$ nothing to prove. If $xy \notin E(\overline{\Gamma(R)} - v)$ then x and y contain non-zero entries in different positions. That is, there exists $i \neq j$ such that x contains non-zero entry in i^{th} position and y contains non-zero entry in j^{th} position. Clearly, the vertex w with $'1'$ in i^{th} and j^{th} position and $'0'$ in other positions is such that $xw,\ wy \in E(\overline{\Gamma(R)})$. If $w \neq v$, then $x-w-y$ is a path connecting x and y in $\overline{\Gamma(R)} - v$. If $w = v$, then since $n \geqslant 3$ and $xy \notin E(\overline{\Gamma(R)} - v) \implies$ There exists a position k, $(1 \leqslant k \leqslant n)$ such that x,y have $'0'$ entry in position k. Let w_1 be the vertex with entry $'1'$ in i^{th} and k^{th} position and $'0'$ everywhere else and w_2 be the vertex with entry $'1'$ in j^{th} and k^{th} position and $'0'$ everywhere else. Clearly, $x-w_1-w_2-y$ is a path connecting x and y in $\overline{\Gamma(R)} - v \implies \overline{\Gamma(R)} - v$ is connected and v is not a cut vertex. Hence, $\overline{\Gamma(R)}$ has no cut vertex.

Corollary 2.5 $\overline{\Gamma(R)}$ has no cut edge.

Theorem 2.6 If $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$. then the vertex connectivity $\kappa(\Gamma(R))$ and edge connectivity $\lambda(\Gamma(R))$ of $\Gamma(R)$ is $\delta(\Gamma(R))$. In other words, $\kappa(\Gamma(R)) = \lambda(\Gamma(R)) = \delta(\Gamma(R)) = min\{|F_i| - 1 : 1 \leq i \leq n\}$.

Proof: If $|F_1| = |F_2| = \cdots = |F_n| = 2$, then by theorems 2.2 and 2.3, $\Gamma(R)$ contains cut vertex and cut edge and hence, $\kappa(\Gamma(R)) = \lambda(\Gamma(R)) = \delta(\Gamma(R)) = 1$. Let $|F_i| \geqslant 2$. Consider a vertex with a zero in the i^{th} position and non-zero entries in the remaining positions. The degree of such a vertex is $|F_i| - 1$, $(1 \leqslant i \leqslant n)$. Therefore,

2456-8686, vi(ii), 2022: 135-141
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$\delta(\Gamma(R)) = min\{|F_i| - 1 : 1 \leq i \leq n\}$. Let $S = \{v_1, \cdots, v_{\delta(\Gamma(R))-2}\} \subset V(\Gamma(R))$ with $|S| = \delta(\Gamma(R)) - 2$. We prove that $\Gamma(R) - S$ is connected.
For $x, y \in \Gamma(R) - S$ if $xy \in E(\Gamma(R) - S)$ we have nothing to prove.
Suppose $xy \notin E(\Gamma(R) - S)$. Note that x and y has at least one $'0'$ as its co-ordinates.
Let $T = \{v \in V(G) : v$ contains non-zero entry in exactly one position and '0' in remaining positions $\}$. There are $|F_i| - 1$ vertices having one non-zero entry in i^{th} position, $(1 \leqslant i \leqslant n)$ and $'0'$ elsewhere and $|T| = \sum_{i=1}^{n}(|F_i| - 1)$. Therefore (even if $S \subset T$), there exists a vertex in $T \setminus S$ with a non zero entry in i^{th} position and $'0'$ everywhere else for $i = 1, 2 \cdots, n$.
There exists two vertices z and w in T, such that,
(i) $z, w \notin S$
(ii) $xz, yw \in E(\Gamma(R) - S)$.
It is possible that $z = w$. Further, any two vertices in T with non-zero entries not in same position are adjacent. Thus, if $z = w$, then $x - z - y$ is a path in $\Gamma(R) - S$. If $z \neq w$, then $x - z - w - y$ is a path in $\Gamma(R) - S$ connecting x and y. $\implies$ $\Gamma(R) - S$ is connected. $\implies$ The minimum cardinality of a vertex cut in $\Gamma(R)$ is $min\{|F_i| - 1 : 1 \leq i \leq n\}$.
$\implies \kappa(\Gamma(R)) \geqslant min\{|F_i| - 1 | 1 \leq i \leq n\}$.
But, $\delta(\Gamma(R)) = min\{|F_i| - 1 | 1 \leq i \leq n\}$ and
$\kappa(\Gamma(R)) \leqslant \delta(\Gamma(R)) \implies \kappa(\Gamma(R)) = \delta(\Gamma(R)) = min\{|F_i| - 1 | 1 \leq i \leq n\}$.
Further, $\lambda(\Gamma(R))$- the edge connectivity of $\Gamma(R)$ satisfies,
$\kappa(\Gamma(R)) \leqslant \lambda(\Gamma(R)) \leqslant \delta(\Gamma(R))$[6].
$\implies \kappa(\Gamma(R)) = \lambda(\Gamma(R)) = \delta(\Gamma(R)) = min\{|F_i| - 1 : 1 \leq i \leq n\}$.

Corollary 2.7 (i) If $n = 2$, then $\overline{\Gamma(R)}$ is disconnected, therefore, $\kappa(\overline{\Gamma(R)}) = \lambda(\overline{\Gamma(R)}) = 0$.
(ii) If $R = F_1 \times F_2 \times \cdots \times F_n$, $n \geqslant 3$ then $\kappa(\overline{\Gamma(R)}) = \lambda(\overline{\Gamma(R)}) = \delta(\overline{\Gamma(R)})$.
Proof: (i) Trivial.
(ii) $\delta(\overline{\Gamma(R)}) = (|V(\Gamma(R))| - 1) - \Delta(\Gamma(R)) =$
$((\prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1) - 1) - max\{(\frac{\prod_{i=1}^{n}|F_i|}{|F_j|}) - 1|\ (1 \leq j \leq n)\}$.
The maximum degree vertices in $(\overline{\Gamma(R)})$ contains $'0'$ in exactly one co-ordinate position and non-zero entries in remaining co-ordinate positions. And the number of such maximum degree vertices is
$((\prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i| - 1) - 1) - 1) - min\{|F_i| - 1 | 1 \leq i \leq n\}$.
Therefore, if $S \subset V(\overline{\Gamma(R)})$ with $|S| = \delta(\overline{\Gamma(R)}) - 1$ and $T \subset V(\overline{\Gamma(R)})$ is set of

maximum degree vertices then, clearly, for $n \geq 3$, $|T| - |S| \geqslant 2$. Consider the set $S \subset V(\overline{\Gamma(R)})$ with $|S| = \delta(\overline{\Gamma(R)}) - 1$. Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in S$. Suppose $xy \notin E(\overline{\Gamma(R)})$. Then the non-zero entries of x and y are not in the common co-ordinate positions. That is, if x has non-zero entries in the co-ordinate positions $x_1, \cdots, x_k$ $(1 \leq k \leq n)$, then y contains $'0'$ entries in the co-ordinate positions $y_1, y_2, \cdots, y_k$. Since, $|T| - |S| \geqslant 2$, let $z \in T$, such that $z \notin S$. If $xz \in E(\overline{\Gamma(R)})$ then $yz \notin E(\overline{\Gamma(R)})$, since x and y have non-zero entries that are not in the common co-ordinate positions, and since $|T| - |S| \geqslant 2$, there exists $w \in T$, such that $yw \in E(\overline{\Gamma(R)})$. Since, T is a clique, $zw \in E(\overline{\Gamma(R)})$. Thus, $x - z - w - y$ is a path connecting x and y. Hence, $\overline{\Gamma(R)} \setminus \{S\}$ is connected if $S \subset V(\overline{\Gamma(R)})$ with $|S| = \delta(\overline{\Gamma(R)}) - 1$. Therefore, $\kappa(\overline{\Gamma(R)}) = \delta(\overline{\Gamma(R)})$. Hence, $\kappa(\overline{\Gamma(R)}) = \lambda(\overline{\Gamma(R)}) = \delta(\overline{\Gamma(R)})$.

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