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Topological indices of Join graph of zero divisor graphs of direct product of three finite fields

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Abstract

In this paper, we calculate the Weiner index, some degree-based topological indices, and some eccentricity-based topological indices of Join graph of $\Gamma(F_1 \times F_2 \times F_3)$ and $\Gamma(J_1 \times J_2 \times J_3)$, where $F_1, F_2, F_3, J_1, J_2, J_3$ are finite fields of order at least two. The vertex set of $\Gamma(F_1 \times F_2 \times F_3)$ ($\Gamma(J_1 \times J_2 \times J_3)$) is $Z^*(F_1 \times F_2 \times F_3)$ ($Z^*(J_1 \times J_2 \times J_3)$), the set of non-zero zero divisors, and two distinct vertices $(x_1, x_2, x_3), (y_1, y_2, y_3)$ are adjacent if and only if $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (0, 0, 0)$, the additive identity of the ring $F_1 \times F_2 \times F_3$ ($J_1 \times J_2 \times J_3$). The vertex set of Join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ is $V(\Gamma(F_1 \times F_2 \times F_3)) \cup V(\Gamma(J_1 \times J_2 \times J_3))$ and edge set is $E(\Gamma(F_1 \times F_2 \times F_3)) \cup E(\Gamma(J_1 \times J_2 \times J_3)) \cup \{(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) : (x_1, x_2, x_3) \in V(\Gamma(F_1 \times F_2 \times F_3)), (y_1, y_2, y_3) \in V(\Gamma(J_1 \times J_2 \times J_3))\}$.

Key words: Zero divisor graph, Join graph, Weiner index, degree based topological indices, eccentricity based topological indices.

AMS classification: 13A70, 05C07, 05C09, 05C25.

1 Introduction

Consider the rings $F_1 \times F_2 \times F_3$ and $J_1 \times J_2 \times J_3$, where $F_1, F_2, F_3, J_1, J_2, J_3$ are finite fields of order at least two. In this paper we consider the zero divisor graph defined by Anderson and Livingston [2] for the rings $F_1 \times F_2 \times F_3$ and $J_1 \times J_2 \times J_3$, denoted by $\Gamma(F_1 \times F_2 \times F_3)$ and $\Gamma(J_1 \times J_2 \times J_3)$. The vertex set of $\Gamma(F_1 \times F_2 \times F_3)$ ($\Gamma(J_1 \times J_2 \times J_3)$) is $Z^*(F_1 \times F_2 \times F_3)$ ($Z^*(J_1 \times J_2 \times J_3)$) where two distinct non-zero zero divisors $(x_1, x_2, x_3), (y_1, y_2, y_3)$ are adjacent if and only if $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (0, 0, 0)$, the additive identity of the ring $F_1 \times F_2 \times F_3$ ($J_1 \times J_2 \times J_3$). The Join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ is the graph with vertex set $V(\Gamma(F_1 \times F_2 \times F_3)) \cup V(\Gamma(J_1 \times J_2 \times J_3))$ and edge set is $E(\Gamma(F_1 \times F_2 \times F_3)) \cup E(\Gamma(J_1 \times J_2 \times J_3)) \cup \{(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) : (x_1, x_2, x_3) \in V(\Gamma(F_1 \times F_2 \times F_3)), (y_1, y_2, y_3) \in V(\Gamma(J_1 \times J_2 \times J_3))\}$. [12]
For more information on graphs defined on rings which are products of finite fields, we refer to [6, 7, 8, 9]. For basic concepts of graph theory, we refer to [17] and for ring theory we refer to [10].

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2 Weiner index and some degree based topological indices

In this section, we calculate the Weiner index, some degree based topological indices such as [16] Sombor index, [15] the first and second Zagreb indices, forgotten topological index, the Narumi-Katayama index, the first and second multiplicative Zagreb indices, atom-bond-connectivity index of the Join graph of $\Gamma(F_1 \times F_2 \times F_3)$ and $\Gamma(J_1 \times J_2 \times J_3)$.

Example 2.1. *[9] Let* $V(\Gamma(F_1 \times F_2 \times F_3)) = \cup_{i=1}^{6} A_i$*, where*
$A_1 = \{(0, a_2, a_3) : a_2 \in F_2 \setminus \{0\}, a_3 \in F_3 \setminus \{0\}\}$.
$A_2 = \{(a_1, 0, a_3) : a_1 \in F_1 \setminus \{0\}, a_3 \in F_3 \setminus \{0\}\}$.
$A_3 = \{(a_1, a_2, 0) : a_1 \in F_1 \setminus \{0\}, a_2 \in F_2 \setminus \{0\}\}$.
$A_4 = \{(0, 0, a_3) : a_3 \in F_3 \setminus \{0\}\}$.
$A_5 = \{(0, a_2, 0) : a_2 \in F_2 \setminus \{0\}\}$.
$A_6 = \{(a_1, 0, 0) : a_1 \in F_1 \setminus \{0\}\}$.
We have, $|A_1| = (|F_2| - 1)(|F_3| - 1),\ |A_2| = (|F_1| - 1)(|F_3| - 1),$
$|A_3| = (|F_1| - 1)(|F_2| - 1),\ |A_4| = |F_3| - 1,\ |A_5| = |F_2| - 1,\ |A_6| = |F_1| - 1.$
[9] Let $V(\Gamma(J_1 \times J_2 \times J_3)) = \cup_{i=1}^{6} B_i$*, where*
$B_1 = \{(0, b_2, b_3) : b_2 \in J_2 \setminus \{0\}, b3 \in J_3 \setminus \{0\}\}$.
$B_2 = \{(b_1, 0, b_3) : b_1 \in J_1 \setminus \{0\}, b_3 \in J_3 \setminus \{0\}\}$.
$B_3 = \{(b_1, b_2, 0) : b_1 \in J_1 \setminus \{0\}, b_2 \in J_1 \times J_2 \times J_3 \setminus \{0\}\}$.
$B_4 = \{(0, 0, b_3) : b_3 \in J_3 \setminus \{0\}\}$.
$B_5 = \{(0, b_2, 0) : b_2 \in J_2 \setminus \{0\}\}$.
$B_6 = \{(b_1, 0, 0) : b_1 \in J_1 \setminus \{0\}\}$.
We have, $|B_1| = (|J_2 - 1)(|J_3| - 1),\ |B_2| = (|J_1| - 1)(|J_3| - 1),$
$|B_3| = (|J_1| - 1)(|J_2| - 1),\ |B_4| = |J_3| - 1,\ |B_5| = |J_2| - 1,\ |B_6| = |J_1| - 1.$
Now we consider the Join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ *with vertex set* $V(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)) = V(\Gamma(F_1 \times F_2 \times F_3)) \cup V(\Gamma(J_1 \times J_2 \times J_3))$ *and edge set* $E(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)) =$
$E(\Gamma(F_1 \times F_2 \times F_3)) \cup E(\Gamma(J_1 \times J_2 \times J_3)) \cup \{(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) : (x_1, x_2, x_3) \in V(\Gamma(F_1 \times F_2 \times F_3)),\ (y_1, y_2, y_3) \in V(\Gamma(J_1 \times J_2 \times J_3))\}$*. [12]*
The degree of each vertex in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ *can be determined as*

$$d_x = \begin{cases} |F_1| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))| & x \in A_1 \\ |F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))| & x \in A_2 \\ |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))| & x \in A_3 \\ |F_1| \cdot |F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))| & x \in A_4 \\ |F_1| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))| & x \in A_5 \\ |F_2| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))| & x \in A_6. \\ |J_1| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))| & x \in B_1 \\ |J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))| & x \in B_2 \\ |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))| & x \in B_3 \\ |J_1| \cdot |J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))| & x \in B_4 \\ |J_1| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))| & x \in B_5 \\ |J_2| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))| & x \in B_6. \end{cases}$$

The edge set $E(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))$ *is given below :*
$xy \in E(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))\ if$

$$\begin{cases} x \in A_1, y \in A_6 & and\ number\ of\ such\ edges\ is\ (|F_2| - 1)(|F_3| - 1)(|F_1| - 1) \\ x \in A_2, y \in A_5 & and\ number\ of\ such\ edges\ is\ (|F_1| - 1)(|F_3| - 1)(|F_2| - 1) \\ x \in A_3, y \in A_4 & and\ number\ of\ such\ edges\ is\ (|F_1| - 1)(|F_2| - 1)(|F_3| - 1) \\ x \in A_4, y \in A_5 & and\ number\ of\ such\ edges\ is\ (|F_1| \cdot |F_2| - 1)(|F_1| \cdot |F_3| - 1) \\ x \in A_4, y \in A_6 & and\ number\ of\ such\ edges\ is\ (|F_1| \cdot |F_2| - 1)(|F_2| \cdot |F_3| - 1) \\ x \in A_5, y \in A_6 & and\ number\ of\ such\ edges\ is\ (|F_1| \cdot |F_3| - 1)(|F_2| \cdot |F_3| - 1). \\ x \in B_1, y \in B_6 & and\ number\ of\ such\ edges\ is\ (|J_2| - 1)(|J_3| - 1)(|J_1| - 1) \\ x \in B_2, y \in B_5 & and\ number\ of\ such\ edges\ is\ (|J_1| - 1)(|J_3| - 1)(|J_2| - 1) \\ x \in B_3, y \in B_4 & and\ number\ of\ such\ edges\ is\ (|J_1| - 1)(|J_2| - 1)(|J_3| - 1) \\ x \in B_4, y \in B_5 & and\ number\ of\ such\ edges\ is\ (|J_1| \cdot |J_2| - 1)(|J_1| \cdot |J_3| - 1) \\ x \in B_4, y \in B_6 & and\ number\ of\ such\ edges\ is\ (|J_1| \cdot |J_2| - 1)(|J_2| \cdot |J_3| - 1) \\ x \in B_5, y \in B_6 & and\ number\ of\ such\ edges\ is\ (|J_1| \cdot |J_3| - 1)(|J_2| \cdot |J_3| - 1). \end{cases}$$

$x \in V(\Gamma(F_1 \times F_2 \times F_3)), y \in V(\Gamma(J_1 \times J_2 \times J_3))\ and\ number\ of\ such\ edges\ is$
$|V(\Gamma(F_1 \times F_2 \times F_3))| \cdot |V(\Gamma(J_1 \times J_2 \times J_3))|$.

1. *The Weiner index of join graph* $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ *is*
   $$\begin{aligned} &W(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)) \\ &= |E(\Gamma(F_1 \times F_2 \times F_3))| \ + \ |E(\Gamma(J_1 \times J_2 \times J_3))| \\ &+ |V(\Gamma(F_1 \times F_2 \times F_3))| \cdot |V(\Gamma(J_1 \times J_2 \times J_3))| \\ &+ \left[\Big(|V(\Gamma(F_1 \times F_2 \times F_3))|,\ 2\Big) - |E(\Gamma(F_1 \times F_2 \times F_3))|\right] \\ &+ \left[C\Big(|V(\Gamma(J_1 \times J_2 \times J_3))|,\ 2\Big) - |E(\Gamma(J_1 \times J_2 \times J_3))|\right]. \end{aligned}$$

Proof. According to [9, Example 3.1],
$diam(\Gamma(F_1 \times F_2 \times F_3)) = 3,\ diam(\Gamma(J_1 \times J_2 \times J_3)) = 3$. Let $x, y \in V(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))$. If both $x, y \in V(\Gamma(F_1 \times F_2 \times F_3))$ and $xy \notin E(\Gamma(F_1 \times F_2 \times F_3)$, then, since both x, y are adjacent to each and every vertex of $V(\Gamma(J_1 \times J_2 \times J_3))$ in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$, it follows that $d(x, y) = 2$. Similarly, if both $x, y \in V(\Gamma(J_1 \times J_2 \times J_3))$ and $xy \notin E(\Gamma(J_1 \times J_2 \times J_3)$, then, $d(x, y) = 2$ in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$.
If $x \in V(\Gamma(F_1 \times F_2 \times F_3)), y \in V(\Gamma(J_1 \times J_2 \times J_3))$, in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$, then, clearly, $d(x, y) = 1$. Therefore, the diameter of the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ is 2. We determine the number of pairs of vertices which are at distance 1 and 2 in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$. The number of pairs of vertices which are at distance 1 in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$ is $E(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))$ which is

$$3\Big[(|F_1|-1)(|F_2|-1)(|F_3|-1)\Big]+\Big[(|F_1|\cdot|F_2|-1) + (|F_1|\cdot|F_3|-1) + (|F_2|\cdot|F_3|-1)\Big] \\ + 3\Big[(|J_1|-1)(|J_2|-1)(|J_3|-1)\Big] + \Big[(|J_1|\cdot|J_2|-1) + (|J_1|\cdot|J_3|-1) + (|J_2|\cdot|J_3|-1)\Big] \\ + |V(\Gamma(F_1 \times F_2 \times F_3))| \cdot |V(\Gamma(J_1 \times J_2 \times J_3))| = |E(\Gamma(F_1 \times F_2 \times F_3))| + \\ |E(\Gamma(J_1 \times J_2 \times J_3))| + |V(\Gamma(F_1 \times F_2 \times F_3))| \cdot |V(\Gamma(J_1 \times J_2 \times J_3))|.$$

Now we compute the number of pairs of vertices which are at distance 2 in the join graph $\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3)$. The number of pairs of vertices which are at distance 2, with $x, y \in V(\Gamma(F_1 \times F_2 \times F_3))$, $xy \notin E(\Gamma(F_1 \times F_2 \times F_3)$ is

$$C\Big(|V(\Gamma(F_1 \times F_2 \times F_3))|,\ 2\Big) - 3\Big[(|F_1|-1)(|F_2|-1)(|F_3|-1)\Big] + \Big[(|F_1|\cdot|F_2|-1) \\ + (|F_1|\cdot|F_3|-1) + (|F_2|\cdot|F_3|-1)\Big] \\ = C\Big(|V(\Gamma(F_1 \times F_2 \times F_3))|,\ 2\Big) - |E(\Gamma(F_1 \times F_2 \times F_3))|.$$

Similarly, number of pairs of vertices which are at distance 2, with $x, y \in V(\Gamma(J_1 \times J_2 \times J_3)),\ xy \notin E(\Gamma(J_1 \times J_2 \times J_3)$ is

$$C\Big(|V(\Gamma(J_1 \times J_2 \times J_3))|,\ 2\Big) - 3\Big[(|J_1|-1)(|J_2|-1)(|J_3|-1)\Big] + \Big[(|J_1|\cdot|J_2|-1) \\ + (|J_1|\cdot|J_3|-1) + (|J_2|\cdot|J_3|-1)\Big] \\ = C\Big(|V(\Gamma(J_1 \times J_2 \times J_3))|,\ 2\Big) - |E(\Gamma(J_1 \times J_2 \times J_3))|.$$

Therefore, the Weiner index is $W(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))$
$= |E(\Gamma(F_1 \times F_2 \times F_3))| + |E(\Gamma(J_1 \times J_2 \times J_3))|$
$+ |V(\Gamma(F_1 \times F_2 \times F_3))| \cdot |V(\Gamma(J_1 \times J_2 \times J_3))|$

$$+\left[C\Big(|V(\Gamma(F_1\times F_2\times F_3))|,\ 2\Big)-|E(\Gamma(F_1\times F_2\times F_3))|\right]$$
$$+\left[C\Big(|V(\Gamma(J_1\times J_2\times J_3))|,\ 2\Big)-|E(\Gamma(J_1\times J_2\times J_3))|\right].\qquad\square$$

2. *The first Zagreb index [11, 14] is* $M_1(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))$
$= \sum_{x\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))}(deg(x))^2$
$= \sum_{x\in A_1}(deg(x))^2 + \sum_{x\in A_2}(deg(x))^2 + \sum_{x\in A_3}(deg(x))^2 + \sum_{x\in A_4}(deg(x))^2$
$+ \sum_{x\in A_5}(deg(x))^2 + \sum_{x\in A_6}(deg(x))^2 + \sum_{x\in B_1}(deg(x))^2 + \sum_{x\in B_2}(deg(x))^2$
$+ \sum_{x\in B_3}(deg(x))^2 + \sum_{x\in B_4}(deg(x))^2 + \sum_{x\in B_5}(deg(x))^2 + \sum_{x\in B_6}(deg(x))^2$
$= (|F_2|-1)(|F_3|-1)(|F_1|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^2 + (|F_1|-1)$
$(|F_3|-1)(|F_2|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^2$
$+(|F_1|-1)(|F_2|-1)(|F_3|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^2 + (|F_3|-1)$
$(|F_1|\cdot|F_2|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^2$
$+(|F_2|-1)(|F_1|\cdot|F_3|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^2 + (|F_1|-1)$
$(|F_2|\cdot|F_3|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^2$
$+(|J_2|-1)(|J_3|-1)(|J_1|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^2 + (|J_1|-1)$
$(|J_3|-1)(|J_2|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^2$
$+(|J_1|-1)(|J_2|-1)(|J_3|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^2 + (|J_3|-1)$
$(|J_1|\cdot|J_2|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^2$
$+(|J_2|-1)(|J_1|\cdot|J_3|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^2 + (|J_1|-1)$
$(|J_2|\cdot|J_3|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^2$.

3. *The second Zagreb index [11, 14] is* $M_2(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))$
$= \sum_{xy\in E(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))} deg(x)deg(y)$
$= \sum_{x\in A_1,y\in A_6} deg(x)deg(y) + \sum_{x\in A_2,y\in A_5} deg(x)deg(y) +$
$\sum_{x\in A_3,y\in A_4} deg(x)deg(y)$
$+\sum_{x\in A_4,y\in A_5} deg(x)deg(y) + \sum_{x\in A_4,y\in A_6} deg(x)deg(y) +$
$\sum_{x\in A_5,y\in A_6} deg(x)deg(y)$
$+\sum_{x\in B_1,y\in B_6} deg(x)deg(y) + \sum_{x\in B_2,y\in B_5} deg(x)deg(y) +$
$\sum_{x\in B_3,y\in B_4} deg(x)deg(y)$
$+\sum_{x\in B_4,y\in B_5} deg(x)deg(y) + \sum_{x\in B_4,y\in B_6} deg(x)deg(y) +$
$\sum_{x\in B_5,y\in B_6} deg(x)deg(y)$
$+\sum_{x\in A_1,y\in B_1} deg(x)deg(y) + \sum_{x\in A_1,y\in B_2} deg(x)deg(y) +$
$\sum_{x\in A_1,y\in B_3} deg(x)deg(y)$
$+\sum_{x\in A_1,y\in B_4} deg(x)deg(y) + \sum_{x\in A_1,y\in B_5} deg(x)deg(y) +$
$\sum_{x\in A_1,y\in B_6} deg(x)deg(y)$
$+\sum_{x\in A_2,y\in B_1} deg(x)deg(y) + \sum_{x\in A_2,y\in B_2} deg(x)deg(y) +$
$\sum_{x\in A_2,y\in B_3} deg(x)deg(y)$
$+\sum_{x\in A_2,y\in B_4} deg(x)deg(y) + \sum_{x\in A_2,y\in B_5} deg(x)deg(y) +$
$\sum_{x\in A_2,y\in B_6} deg(x)deg(y)$
$+\sum_{x\in A_3,y\in B_1} deg(x)deg(y) + \sum_{x\in A_3,y\in B_2} deg(x)deg(y) +$
$\sum_{x\in A_3,y\in B_3} deg(x)deg(y)$
$+\sum_{x\in A_3,y\in B_4} deg(x)deg(y) + \sum_{x\in A_3,y\in B_5} deg(x)deg(y) +$

$$\sum_{x\in A_3,y\in B_6} deg(x)deg(y)$$
$$+\sum_{x\in A_4,y\in B_1} deg(x)deg(y) + \sum_{x\in A_4,y\in B_2} deg(x)deg(y) + \sum_{x\in A_4,y\in B_3} deg(x)deg(y)$$
$$+\sum_{x\in A_4,y\in B_4} deg(x)deg(y) + \sum_{x\in A_4,y\in B_5} deg(x)deg(y) + \sum_{x\in A_4,y\in B_6} deg(x)deg(y)$$
$$+\sum_{x\in A_5,y\in B_1} deg(x)deg(y) + \sum_{x\in A_5,y\in B_2} deg(x)deg(y) + \sum_{x\in A_5,y\in B_3} deg(x)deg(y)$$
$$+\sum_{x\in A_5,y\in B_4} deg(x)deg(y) + \sum_{x\in A_5,y\in B_5} deg(x)deg(y) + \sum_{x\in A_5,y\in B_6} deg(x)deg(y)$$
$$+\sum_{x\in A_6,y\in B_1} deg(x)deg(y) + \sum_{x\in A_6,y\in B_2} deg(x)deg(y) + \sum_{x\in A_6,y\in B_3} deg(x)deg(y)$$
$$+\sum_{x\in A_6,y\in B_4} deg(x)deg(y) + \sum_{x\in A_6,y\in B_5} deg(x)deg(y) + \sum_{x\in A_6,y\in B_6} deg(x)deg(y)$$
$$= [(|F_1|-1)(|F_2|\cdot|F_3|-1)][(|F_2|-1)(|F_3|-1)(|F_1|-1)]$$
$$+[(|F_2|-1)(|F_1|\cdot|F_3|-1)][(|F_1|-1)(|F_3|-1)(|F_2|-1)]$$
$$+[(|F_3|-1)(|F_1|\cdot|F_2|-1)][(|F_1|-1)(|F_2|-1)(|F_3|-1)]$$
$$+[(|F_1|\cdot|F_2|-1)(|F_1|\cdot|F_3|-1)][(|F_3|-1)(|F_2|-1)]$$
$$+[(|F_1|\cdot|F_2|-1)(|F_2|\cdot|F_3|-1)][(|F_3|-1)(|F_1|-1)]$$
$$+[(|F_1|\cdot|F_3|-1)(|F_2|\cdot|F_3|-1)][(|F_2|-1)(|F_1|-1)]$$
$$+[(|J_1|-1)(|J_2|\cdot|J_3|-1)][(|J_2|-1)(|J_3|-1)(|J_1|-1)]$$
$$+[(|J_2|-1)(|J_1|\cdot|J_3|-1)][(|J_1|-1)(|J_3|-1)(|J_2|-1)]$$
$$+[(|J_3|-1)(|J_1|\cdot|J_2|-1)][(|J_1|-1)(|J_2|-1)(|J_3|-1)]$$
$$+[(|J_1|\cdot|J_2|-1)(|J_1|\cdot|J_3|-1)][(|J_3|-1)(|J_2|-1)]$$
$$+[(|J_1|\cdot|J_2|-1)(|J_2|\cdot|J_3|-1)][(|J_3|-1)(|J_1|-1)]$$
$$+[(|J_1|\cdot|J_3|-1)(|J_2|\cdot|J_3|-1)][(|J_2|-1)(|J_1|-1)]$$
$$+[(|F_1|-1)(|J_1|-1)][(|F_2|-1)(|F_3|-1)(|J_2|-1)(|J_3|-1)]$$
$$+[(|F_1|-1)(|J_2|-1)][(|F_2|-1)(|F_3|-1)(|J_1|-1)(|J_3|-1)]$$
$$+[(|F_1|-1)(|J_3|-1)][(|F_2|-1)(|F_3|-1)(|J_1|-1)(|J_2|-1)]$$
$$+[(|F_1|-1)(|J_1|\cdot|J_2|-1)][(|F_2|-1)(|F_3|-1)(|J_3|-1)]$$
$$+[(|F_1|-1)(|J_1|\cdot|J_3|-1)][(|F_2|-1)(|F_3|-1)(|J_2|-1)]$$
$$+[(|F_1|-1)(|J_2|\cdot|J_3|-1)][(|F_2|-1)(|F_3|-1)(|J_1|-1)]$$
$$+[(|F_2|-1)(|J_1|-1)][(|F_1|-1)(|F_3|-1)(|J_2|-1)(|J_3|-1)]$$
$$+[(|F_2|-1)(|J_2|-1)][(|F_1|-1)(|F_3|-1)(|J_1|-1)(|J_3|-1)]$$
$$+[(|F_2|-1)(|J_3|-1)][(|F_1|-1)(|F_3|-1)(|J_1|-1)(|J_2|-1)]$$
$$+[(|F_2|-1)(|J_1|\cdot|J_2|-1)][(|F_1|-1)(|F_3|-1)(|J_3|-1)]$$
$$+[(|F_2|-1)(|J_1|\cdot|J_3|-1)][(|F_1|-1)(|F_3|-1)(|J_2|-1)]$$
$$+[(|F_2|-1)(|J_2|\cdot|J_3|-1)][(|F_1|-1)(|F_3|-1)(|J_1|-1)]$$
$$+[(|F_3|-1)(|J_1|-1)][(|F_1|-1)(|F_2|-1)(|J_2|-1)(|J_3|-1)]$$
$$+[(|F_3|-1)(|J_2|-1)][(|F_1|-1)(|F_2|-1)(|J_1|-1)(|J_3|-1)]$$
$$+[(|F_3|-1)(|J_3|-1)][(|F_1|-1)(|F_2|-1)(|J_1|-1)(|J_2|-1)]$$
$$+[(|F_3|-1)(|J_1|\cdot|J_2|-1)][(|F_1|-1)(|F_2|-1)(|J_3|-1)]$$

$+[(|F_3|-1)(|J_1|\cdot|J_3|-1)][(|F_1|-1)(|F_2|-1)(|J_2|-1)]$
$+[(|F_3|-1)(|J_2|\cdot|J_3|-1)][(|F_1|-1)(|F_2|-1)(|J_1|-1)]$
$+[(|F_1|\cdot|F_2|-1)(|J_1|-1)][(|F_3|-1)(|J_2|-1)(|J_3|-1)]$
$+[(|F_1|\cdot|F_2|-1)(|J_2|-1)][(|F_3|-1)(|J_1|-1)(|J_3|-1)]$
$+[(|F_1|\cdot|F_2|-1)(|J_3|-1)][(|F_3|-1)(|J_1|-1)(|J_2|-1)]$
$+[(|F_1|\cdot|F_2|-1)(|J_1|\cdot|J_2|-1)][(|F_3|-1)(|J_3|-1)]$
$+[(|F_1|\cdot|F_2|-1)(|J_1|\cdot|J_3|-1)][(|F_3|-1)(|J_2|-1)]$
$+[(|F_1|\cdot|F_2|-1)(|J_2|\cdot|J_3|-1)][(|F_3|-1)(|J_1|-1)]$
$+[(|F_1|\cdot|F_3|-1)(|J_1|-1)][(|F_2|-1)(|J_2|-1)(|J_3|-1)]$
$+[(|F_1|\cdot|F_3|-1)(|J_2|-1)][(|F_2|-1)(|J_1|-1)(|J_3|-1)]$
$+[(|F_1|\cdot|F_3|-1)(|J_3|-1)][(|F_2|-1)(|J_1|-1)(|J_2|-1)]$
$+[(|F_1|\cdot|F_3|-1)(|J_1|\cdot|J_2|-1)][(|F_2|-1)(|J_3|-1)]$
$+[(|F_1|\cdot|F_3|-1)(|J_1|\cdot|J_3|-1)][(|F_2|-1)(|J_2|-1)]$
$+[(|F_1|\cdot|F_3|-1)(|J_2|\cdot|J_3|-1)][(|F_2|-1)(|J_1|-1)]$
$+[(|F_2|\cdot|F_3|-1)(|J_1|-1)][(|F_1|-1)(|J_2|-1)(|J_3|-1)]$
$+[(|F_2|\cdot|F_3|-1)(|J_2|-1)][(|F_1|-1)(|J_1|-1)(|J_3|-1)]$
$+[(|F_2|\cdot|F_3|-1)(|J_3|-1)][(|F_1|-1)(|J_1|-1)(|J_2|-1)]$
$+[(|F_2|\cdot|F_3|-1)(|J_1|\cdot|J_2|-1)][(|F_1|-1)(|J_3|-1)]$
$+[(|F_2|\cdot|F_3|-1)(|J_1|\cdot|J_3|-1)][(|F_1|-1)(|J_2|-1)]$
$+[(|F_2|\cdot|F_3|-1)(|J_2|\cdot|J_3|-1)][(|F_1|-1)(|J_1|-1)]$.

4. *The Forgotten topological index [5] is*
$F(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))=\sum_{x\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))}(deg(x))^3$
$=\sum_{x\in A_1}(deg(x))^3+\sum_{x\in A_2}(deg(x))^3+\sum_{x\in A_3}(deg(x))^3+\sum_{x\in A_4}(deg(x))^3$
$+\sum_{x\in A_5}(deg(x))^3+\sum_{x\in A_6}(deg(x))^3+\sum_{x\in B_1}(deg(x))^3+\sum_{x\in B_2}(deg(x))^3$
$+\sum_{x\in B_3}(deg(x))^3+\sum_{x\in B_4}(deg(x))^3+\sum_{x\in B_5}(deg(x))^3+\sum_{x\in B_6}(deg(x))^3$
$=(|F_1|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^3(|F_2|-1)(|F_3|-1)+(|F_2|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^3(|F_1|-1)(|F_3|-1)$
$+(|F_3|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^3(|F_1|-1)(|F_2|-1)+(|F_1|\cdot|F_2|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^3(|F_3|-1)$
$+(|F_1|\cdot|F_3|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^3(|F_2|-1)+(|F_2|\cdot|F_3|-1+|V(\Gamma(J_1\times J_2\times J_3))|)^3(|F_1|-1)$
$+(|J_1|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^3(|J_2|-1)(|J_3|-1)+(|J_2|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^3(|J_1|-1)(|J_3|-1)$
$+(|J_3|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^3(|J_1|-1)(|J_2|-1)+(|J_1|\cdot|J_2|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^3(|J_3|-1)$
$+(|J_1|\cdot|J_3|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^3(|J_2|-1)+(|J_2|\cdot|J_3|-1+|V(\Gamma(F_1\times F_2\times F_3))|)^3(|J_1|-1)$.

5. *The Narumi-Katayama index [13] is*
$NK(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))=\prod_{x\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))}deg(x)$
$=\prod_{x\in A_1}deg(x)\cdot\prod_{x\in A_2}deg(x)\cdot\prod_{x\in A_3}deg(x)\cdot\prod_{x\in A_4}deg(x)\cdot\prod_{x\in A_5}deg(x)\cdot\prod_{x\in A_6}deg(x)$

$\prod_{x\in B_1} deg(x) \cdot \prod_{x\in B_2} deg(x) \cdot \prod_{x\in B_3} deg(x) \cdot \prod_{x\in B_4} deg(x) \cdot \prod_{x\in B_5} deg(x) \cdot \prod_{x\in B_6} deg(x)$
$= [(|F_1| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)(|F_2| - 1)(|F_3| - 1)] \cdot [(|F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)(|F_1| - 1)(|F_3| - 1)] \cdot [(|F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)(|F_1| - 1)(|F_2| - 1)] \cdot [(|F_1| \cdot |F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)(|F_3| - 1)] \cdot [(|F_1| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)(|F_2| - 1)] \cdot [(|F_2| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)(|F_1| - 1)]$
$\cdot [(|J_1| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)(|J_2| - 1)(|J_3| - 1)] \cdot [(|J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)(|J_1| - 1)(|J_3| - 1)] \cdot [(|J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)(|J_1| - 1)(|J_2| - 1)] \cdot [(|J_1| \cdot |J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)(|J_3| - 1)] \cdot [(|J_1| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)(|J_2| - 1)] \cdot [(|J_2| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)(|J_1| - 1)]$
$= (|F_1| - 1)^3 \cdot (|F_2| - 1)^3 \cdot (|F_3| - 1)^3 \cdot (|F_1| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|) \cdot (|F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|) \cdot [(|F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|) \cdot (|F_1| \cdot |F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|) \cdot (|F_1| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|) \cdot (|F_2| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|) \cdot (|J_1| - 1)^3 \cdot (|J_2| - 1)^3 \cdot (|J_3| - 1)^3 \cdot (|J_1| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|) \cdot (|J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|) \cdot [(|J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|) \cdot (|J_1| \cdot |J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|) \cdot (|J_1| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|) \cdot (|J_2| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)$.

6. *The first multiplicative Zagreb index [1, 18] is*
$\Pi_1(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))$
$= \prod_{x\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))} (deg(x))^2$
$= NK(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))^2$
$= (|F_1| - 1)^6 \cdot (|F_2| - 1)^6 \cdot (|F_3| - 1)^6 \cdot (|F_1| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)^2 \cdot (|F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)^2$
$\cdot [(|F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)^2 \cdot (|F_1| \cdot |F_2| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)^2 \cdot (|F_1| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)^2$
$\cdot (|F_2| \cdot |F_3| - 1 + |V(\Gamma(J_1 \times J_2 \times J_3))|)^2 \cdot (|J_1| - 1)^6 \cdot (|J_2| - 1)^6 \cdot (|J_3| - 1)^6 \cdot (|J_1| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)^2$
$\cdot (|J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)^2 \cdot [(|J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)^2 \cdot (|J_1| \cdot |J_2| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)^2$
$\cdot (|J_1| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)^2 \cdot (|J_2| \cdot |J_3| - 1 + |V(\Gamma(F_1 \times F_2 \times F_3))|)^2$.

7. *The second multiplicative Zagreb index [1, 18] is*
$\Pi_2(\Gamma(F_1 \times F_2 \times F_3) + \Gamma(J_1 \times J_2 \times J_3))$
$= \prod_{xy\in E(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))} deg(x)deg(y)$
$= \prod_{x\in A_1,y\in A_6} deg(x)deg(y) \cdot \prod_{x\in A_2,y\in A_5} deg(x)deg(y) \cdot \prod_{x\in A_3,y\in A_4} deg(x)deg(y)$
$\cdot \prod_{x\in A_4,y\in A_5} deg(x)deg(y) \cdot \prod_{x\in A_4,y\in A_6} deg(x)deg(y) \cdot \prod_{x\in A_5,y\in A_6} deg(x)deg(y)$
$\cdot \prod_{x\in B_1,y\in B_6} deg(x)deg(y) \cdot \prod_{x\in B_2,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in B_3,y\in B_4} deg(x)deg(y)$
$\cdot \prod_{x\in B_4,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in B_4,y\in B_6} deg(x)deg(y) \cdot \prod_{x\in B_5,y\in B_6} deg(x)deg(y)$
$\cdot \prod_{x\in A_1,y\in B_1} deg(x)deg(y) \cdot \prod_{x\in A_1,y\in B_2} deg(x)deg(y) \cdot \prod_{x\in A_1,y\in B_3} deg(x)deg(y)$
$\cdot \prod_{x\in A_1,y\in B_4} deg(x)deg(y) \cdot \prod_{x\in A_1,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in A_1,y\in B_6} deg(x)deg(y)$
$\cdot \prod_{x\in A_2,y\in B_1} deg(x)deg(y) \cdot \prod_{x\in A_2,y\in B_2} deg(x)deg(y) \cdot \prod_{x\in A_2,y\in B_3} deg(x)deg(y)$
$\cdot \prod_{x\in A_2,y\in B_4} deg(x)deg(y) \cdot \prod_{x\in A_2,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in A_2,y\in B_6} deg(x)deg(y)$
$\cdot \prod_{x\in A_3,y\in B_1} deg(x)deg(y) \cdot \prod_{x\in A_3,y\in B_2} deg(x)deg(y) \cdot \prod_{x\in A_3,y\in B_3} deg(x)deg(y)$

$$
\begin{aligned}
&\cdot \prod_{x\in A_3,y\in B_4} deg(x)deg(y) \cdot \prod_{x\in A_3,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in A_3,y\in B_6} deg(x)deg(y)\\
&\cdot \prod_{x\in A_4,y\in B_1} deg(x)deg(y) \cdot \prod_{x\in A_4,y\in B_2} deg(x)deg(y) \cdot \prod_{x\in A_4,y\in B_3} deg(x)deg(y)\\
&\cdot \prod_{x\in A_4,y\in B_4} deg(x)deg(y) \cdot \prod_{x\in A_4,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in A_4,y\in B_6} deg(x)deg(y)\\
&\cdot \prod_{x\in A_5,y\in B_1} deg(x)deg(y) \cdot \prod_{x\in A_5,y\in B_2} deg(x)deg(y) \cdot \prod_{x\in A_5,y\in B_3} deg(x)deg(y)\\
&\cdot \prod_{x\in A_5,y\in B_4} deg(x)deg(y) \cdot \prod_{x\in A_5,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in A_5,y\in B_6} deg(x)deg(y)\\
&\cdot \prod_{x\in A_6,y\in B_1} deg(x)deg(y) \cdot \prod_{x\in A_6,y\in B_2} deg(x)deg(y) \cdot \prod_{x\in A_6,y\in B_3} deg(x)deg(y)\\
&\cdot \prod_{x\in A_6,y\in B_4} deg(x)deg(y) \cdot \prod_{x\in A_6,y\in B_5} deg(x)deg(y) \cdot \prod_{x\in A_6,y\in B_6} deg(x)deg(y)\\
&= [(|F_1|-1)(|F_2|\cdot|F_3|-1)][(|F_2|-1)(|F_3|-1)(|F_1|-1)]\\
&\cdot[(|F_2|-1)(|F_1|\cdot|F_3|-1)][(|F_1|-1)(|F_3|-1)(|F_2|-1)]\\
&\cdot[(|F_3|-1)(|F_1|\cdot|F_2|-1)][(|F_1|-1)(|F_2|-1)(|F_3|-1)]\\
&\cdot[(|F_1|\cdot|F_2|-1)(|F_1|\cdot|F_3|-1)][(|F_3|-1)(|F_2|-1)]\\
&\cdot[(|F_1|\cdot|F_2|-1)(|F_2|\cdot|F_3|-1)][(|F_3|-1)(|F_1|-1)]\\
&\cdot[(|F_1|\cdot|F_3|-1)(|F_2|\cdot|F_3|-1)][(|F_2|-1)(|F_1|-1)]\\
&\cdot[(|J_1|-1)(|J_2|\cdot|J_3|-1)][(|J_2|-1)(|J_3|-1)(|J_1|-1)]\\
&\cdot[(|J_2|-1)(|J_1|\cdot|J_3|-1)][(|J_1|-1)(|J_3|-1)(|J_2|-1)]\\
&\cdot[(|J_3|-1)(|J_1|\cdot|J_2|-1)][(|J_1|-1)(|J_2|-1)(|J_3|-1)]\\
&\cdot[(|J_1|\cdot|J_2|-1)(|J_1|\cdot|J_3|-1)][(|J_3|-1)(|J_2|-1)]\\
&\cdot[(|J_1|\cdot|J_2|-1)(|J_2|\cdot|J_3|-1)][(|J_3|-1)(|J_1|-1)]\\
&\cdot[(|J_1|\cdot|J_3|-1)(|J_2|\cdot|J_3|-1)][(|J_2|-1)(|J_1|-1)]\\
&\cdot[(|F_1|-1)(|J_1|-1)][(|F_2|-1)(|F_3|-1)(|J_2|-1)(|J_3|-1)]\\
&\cdot[(|F_1|-1)(|J_2|-1)][(|F_2|-1)(|F_3|-1)(|J_1|-1)(|J_3|-1)]\\
&\cdot[(|F_1|-1)(|J_3|-1)][(|F_2|-1)(|F_3|-1)(|J_1|-1)(|J_2|-1)]\\
&\cdot[(|F_1|-1)(|J_1|\cdot|J_2|-1)][(|F_2|-1)(|F_3|-1)(|J_3|-1)]\\
&\cdot[(|F_1|-1)(|J_1|\cdot|J_3|-1)][(|F_2|-1)(|F_3|-1)(|J_2|-1)]\\
&\cdot[(|F_1|-1)(|J_2|\cdot|J_3|-1)][(|F_2|-1)(|F_3|-1)(|J_1|-1)]\\
&\cdot[(|F_2|-1)(|J_1|-1)][(|F_1|-1)(|F_3|-1)(|J_2|-1)(|J_3|-1)]\\
&\cdot[(|F_2|-1)(|J_2|-1)][(|F_1|-1)(|F_3|-1)(|J_1|-1)(|J_3|-1)]\\
&\cdot[(|F_2|-1)(|J_3|-1)][(|F_1|-1)(|F_3|-1)(|J_1|-1)(|J_2|-1)]\\
&\cdot[(|F_2|-1)(|J_1|\cdot|J_2|-1)][(|F_1|-1)(|F_3|-1)(|J_3|-1)]\\
&\cdot[(|F_2|-1)(|J_1|\cdot|J_3|-1)][(|F_1|-1)(|F_3|-1)(|J_2|-1)]\\
&\cdot[(|F_2|-1)(|J_2|\cdot|J_3|-1)][(|F_1|-1)(|F_3|-1)(|J_1|-1)]\\
&\cdot[(|F_3|-1)(|J_1|-1)][(|F_1|-1)(|F_2|-1)(|J_2|-1)(|J_3|-1)]\\
&\cdot[(|F_3|-1)(|J_2|-1)][(|F_1|-1)(|F_2|-1)(|J_1|-1)(|J_3|-1)]\\
&\cdot[(|F_3|-1)(|J_3|-1)][(|F_1|-1)(|F_2|-1)(|J_1|-1)(|J_2|-1)]\\
&\cdot[(|F_3|-1)(|J_1|\cdot|J_2|-1)][(|F_1|-1)(|F_2|-1)(|J_3|-1)]\\
&\cdot[(|F_3|-1)(|J_1|\cdot|J_3|-1)][(|F_1|-1)(|F_2|-1)(|J_2|-1)]\\
&\cdot[(|F_3|-1)(|J_2|\cdot|J_3|-1)][(|F_1|-1)(|F_2|-1)(|J_1|-1)]\\
&\cdot[(|F_1|\cdot|F_2|-1)(|J_1|-1)][(|F_3|-1)(|J_2|-1)(|J_3|-1)]\\
&\cdot[(|F_1|\cdot|F_2|-1)(|J_2|-1)][(|F_3|-1)(|J_1|-1)(|J_3|-1)]\\
&\cdot[(|F_1|\cdot|F_2|-1)(|J_3|-1)][(|F_3|-1)(|J_1|-1)(|J_2|-1)]\\
&\cdot[(|F_1|\cdot|F_2|-1)(|J_1|\cdot|J_2|-1)][(|F_3|-1)(|J_3|-1)]
\end{aligned}
$$

$\cdot[(|F_1|\cdot|F_2|-1)(|J_1|\cdot|J_3|-1)][(|F_3|-1)(|J_2|-1)]$
$\cdot[(|F_1|\cdot|F_2|-1)(|J_2|\cdot|J_3|-1)][(|F_3|-1)(|J_1|-1)]$
$\cdot[(|F_1|\cdot|F_3|-1)(|J_1|-1)][(|F_2|-1)(|J_2|-1)(|J_3|-1)]$
$\cdot[(|F_1|\cdot|F_3|-1)(|J_2|-1)][(|F_2|-1)(|J_1|-1)(|J_3|-1)]$
$\cdot[(|F_1|\cdot|F_3|-1)(|J_3|-1)][(|F_2|-1)(|J_1|-1)(|J_2|-1)]$
$\cdot[(|F_1|\cdot|F_3|-1)(|J_1|\cdot|J_2|-1)][(|F_2|-1)(|J_3|-1)]$
$\cdot[(|F_1|\cdot|F_3|-1)(|J_1|\cdot|J_3|-1)][(|F_2|-1)(|J_2|-1)]$
$\cdot[(|F_1|\cdot|F_3|-1)(|J_2|\cdot|J_3|-1)][(|F_2|-1)(|J_1|-1)]$
$\cdot[(|F_2|\cdot|F_3|-1)(|J_1|-1)][(|F_1|-1)(|J_2|-1)(|J_3|-1)]$
$\cdot[(|F_2|\cdot|F_3|-1)(|J_2|-1)][(|F_1|-1)(|J_1|-1)(|J_3|-1)]$
$\cdot[(|F_2|\cdot|F_3|-1)(|J_3|-1)][(|F_1|-1)(|J_1|-1)(|J_2|-1)]$
$\cdot[(|F_2|\cdot|F_3|-1)(|J_1|\cdot|J_2|-1)][(|F_1|-1)(|J_3|-1)]$
$\cdot[(|F_2|\cdot|F_3|-1)(|J_1|\cdot|J_3|-1)][(|F_1|-1)(|J_2|-1)]$
$\cdot[(|F_2|\cdot|F_3|-1)(|J_2|\cdot|J_3|-1)][(|F_1|-1)(|J_1|-1)]$
$=(|F_1|-1)^{30}\cdot(|F_2|-1)^{30}\cdot(|F_3|-1)^{30}\cdot(|F_1||F_2|-1)^9\cdot(|F_1||F_3|-1)^9\cdot(|F_2||F_3|-1)^9$
$\cdot(|J_1|-1)^{30}\cdot(|J_2|-1)^{30}\cdot(|J_3|-1)^{30}\cdot(|J_1||J_2|-1)^9\cdot(|J_1||J_3|-1)^9\cdot(|J_2||J_3|-1)^9.$

3 Eccentricity based topological indices

Example 3.1. *By Example 2.1, (1), the diameter of the join graph $\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3)$ is 2 and for $x,y\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))$ if $xy\notin E(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))$, then $d(x,y)=2$. Therefore, the eccentricity $e(x)$ of each vertex x in the join graph $\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3)$ is 2.*

1. *The total eccentricity [3], [4] of $\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3)$ is*
   $\xi(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))=\sum_{x\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))}e(x)$
   $=2[|V(\Gamma(F_1\times F_2\times F_3))|+|V(\Gamma(J_1\times J_2\times J_3))|]$.
2. *The first Zagreb eccentricity index [1] is*
   $E_1(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))=\sum_{x\in V(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))}(e(x))^2$
   $=4[|V(\Gamma(F_1\times F_2\times F_3))|+|V(\Gamma(J_1\times J_2\times J_3))|]$.
3. *The second Zagreb eccentricity index [1] is*
   $E_2(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))=\sum_{xy\in E(\Gamma(F_1\times F_2\times F_3)+\Gamma(J_1\times J_2\times J_3))}e(x)e(y)$
   $=\sum_{x\in A_1,y\in A_6}e(x)e(y)\ +\ \sum_{x\in A_2,y\in A_5}e(x)e(y)\ +\ \sum_{x\in A_3,y\in A_4}e(x)e(y)$
   $+\sum_{x\in A_4,y\in A_5}e(x)e(y)\ +\ \sum_{x\in A_4,y\in A_6}e(x)e(y)\ +\ \sum_{x\in A_5,y\in A_6}e(x)e(y)$
   $+\sum_{x\in B_1,y\in B_6}e(x)e(y)\ +\ \sum_{x\in B_2,y\in B_5}e(x)e(y)\ +\ \sum_{x\in B_3,y\in B_4}e(x)e(y)$
   $+\sum_{x\in B_4,y\in B_5}e(x)e(y)\ +\ \sum_{x\in B_4,y\in B_6}e(x)e(y)\ +\ \sum_{x\in B_5,y\in B_6}e(x)e(y)$
   $+\sum_{x\in A_1,y\in B_1}e(x)e(y)\ +\ \sum_{x\in A_1,y\in B_2}e(x)e(y)\ +\ \sum_{x\in A_1,y\in B_3}e(x)e(y)$
   $+\sum_{x\in A_1,y\in B_4}e(x)e(y)\ +\ \sum_{x\in A_1,y\in B_5}e(x)e(y)\ +\ \sum_{x\in A_1,y\in B_6}e(x)e(y)$
   $+\sum_{x\in A_2,y\in B_1}e(x)e(y)\ +\ \sum_{x\in A_2,y\in B_2}e(x)e(y)\ +\ \sum_{x\in A_2,y\in B_3}e(x)e(y)$
   $+\sum_{x\in A_2,y\in B_4}e(x)e(y)\ +\ \sum_{x\in A_2,y\in B_5}e(x)e(y)\ +\ \sum_{x\in A_2,y\in B_6}e(x)e(y)$

$+\sum_{x\in A_3,y\in B_1} e(x)e(y) + \sum_{x\in A_3,y\in B_2} e(x)e(y) + \sum_{x\in A_3,y\in B_3} e(x)e(y)$
$+\sum_{x\in A_3,y\in B_4} e(x)e(y) + \sum_{x\in A_3,y\in B_5} e(x)e(y) + \sum_{x\in A_3,y\in B_6} e(x)e(y)$
$+\sum_{x\in A_4,y\in B_1} e(x)e(y) + \sum_{x\in A_4,y\in B_2} e(x)e(y) + \sum_{x\in A_4,y\in B_3} e(x)e(y)$
$+\sum_{x\in A_4,y\in B_4} e(x)e(y) + \sum_{x\in A_4,y\in B_5} e(x)e(y) + \sum_{x\in A_4,y\in B_6} e(x)e(y)$
$+\sum_{x\in A_5,y\in B_1} e(x)e(y) + \sum_{x\in A_5,y\in B_2} e(x)e(y) + \sum_{x\in A_5,y\in B_3} e(x)e(y)$
$+\sum_{x\in A_5,y\in B_4} e(x)e(y) + \sum_{x\in A_5,y\in B_5} e(x)e(y) + \sum_{x\in A_5,y\in B_6} e(x)e(y)$
$+\sum_{x\in A_6,y\in B_1} e(x)e(y) + \sum_{x\in A_6,y\in B_2} e(x)e(y) + \sum_{x\in A_6,y\in B_3} e(x)e(y)$
$+\sum_{x\in A_6,y\in B_4} e(x)e(y) + \sum_{x\in A_6,y\in B_5} e(x)e(y) + \sum_{x\in A_6,y\in B_6} e(x)e(y)$
$= [2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|F_1|-1)] + [2(|F_1-1)\cdot(|F_3|-1)]\cdot[2(|F_2|-1)] + [2(|F_1-1)\cdot(|F_2|-1)]\cdot[2(|F_3|-1)] + [2(|F_3|-1)]\cdot[2(|F_2|-1)] + [2(|F_3|-1)]\cdot[2(|F_1|-1)] + [2(|F_2|-1)]\cdot[2(|F_1|-1)]$
$+ [2(|J_2|-1)\cdot(|J_3|-1)]\cdot[2(|J_1|-1)] + [2(|J_1-1)\cdot(|J_3|-1)]\cdot[2(|J_2|-1)] + [2(|J_1-1)\cdot(|J_2|-1)]\cdot[2(|J_3|-1)] + [2(|J_3|-1)]\cdot[2(|J_2|-1)] + [2(|J_3|-1)]\cdot[2(|J_1|-1)] + [2(|J_2|-1)]\cdot[2(|J_1|-1)]$
$+$
$[2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|J_2|-1)(|J_3|-1)] + [2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|J_1|-1)(|J_3|-1)] + [2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|J_1|-1)(|J_2|-1)] + [2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|J_3|-1)] + [2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|J_2|-1)] + [2(|F_2|-1)\cdot(|F_3|-1)]\cdot[2(|J_1|-1)]$
$+$
$[2(|F_1|-1)\cdot(|F_3|-1)]\cdot[2(|J_2|-1)(|J_3|-1)] + [2(|F_1|-1)\cdot(|F_3|-1)]\cdot[2(|J_1|-1)(|J_3|-1)] + [2(|F_1|-1)\cdot(|F_3|-1)]\cdot[2(|J_1|-1)(|J_2|-1)] + [2(|F_1|-1)\cdot(|F_3|-1)]\cdot[2(|J_3|-1)] + [2(|F_1|-1)\cdot(|F_3|-1)]\cdot[2(|J_2|-1)] + [2(|F_1|-1)\cdot(|F_3|-1)]\cdot[2(|J_1|-1)]$
$+$
$[2(|F_1|-1)\cdot(|F_2|-1)]\cdot[2(|J_2|-1)(|J_3|-1)] + [2(|F_1|-1)\cdot(|F_2|-1)]\cdot[2(|J_1|-1)(|J_3|-1)] + [2(|F_1|-1)\cdot(|F_2|-1)]\cdot[2(|J_1|-1)(|J_2|-1)] + [2(|F_1|-1)\cdot(|F_2|-1)]\cdot[2(|J_3|-1)] + [2(|F_1|-1)\cdot(|F_2|-1)]\cdot[2(|J_2|-1)] + [2(|F_1|-1)\cdot(|F_2|-1)]\cdot[2(|J_1|-1)]$
$+ [2(|F_3|-1)]\cdot[2(|J_2|-1)(|J_3|-1)] + [2(|F_3|-1)]\cdot[2(|J_1|-1)(|J_3|-1)] + [2(|F_3|-1)]\cdot[2(|J_1|-1)(|J_2|-1)] + [2(|F_3|-1)]\cdot[2(|J_3|-1)] + [2(|F_3|-1)]\cdot[2(|J_2|-1)] + [2(|F_3|-1)]\cdot[2(|J_1|-1)]$
$+ [2(|F_2|-1)]\cdot[2(|J_2|-1)(|J_3|-1)] + [2(|F_2|-1)]\cdot[2(|J_1|-1)(|J_3|-1)] + [2(|F_2|-1)]\cdot[2(|J_1|-1)(|J_2|-1)] + [2(|F_2|-1)]\cdot[2(|J_3|-1)] + [2(|F_2|-1)]\cdot[2(|J_2|-1)] + [2(|F_2|-1)]\cdot[2(|J_1|-1)]$
$+ [2(|F_1|-1)]\cdot[2(|J_2|-1)(|J_3|-1)] + [2(|F_1|-1)]\cdot[2(|J_1|-1)(|J_3|-1)] + [2(|F_1|-1)]\cdot[2(|J_1|-1)(|J_2|-1)] + [2(|F_1|-1)]\cdot[2(|J_3|-1)] + [2(|F_1|-1)]\cdot[2(|J_2|-1)] + [2(|F_1|-1)]\cdot[2(|J_1|-1)]$.

4. *The eccentricity connectivity [1] is* $\xi^c(\Gamma(R)) = \sum_{x\in V(\Gamma(R))} e(x)deg(x)$
$= 6(|F_1|-1)\cdot(|F_2|-1)\cdot(|F_3|-1) + 6(|J_1|-1)\cdot(|J_2|-1)\cdot(|J_3|-1) + 2(|F_1|\cdot|F_2|-1)[(|F_3|-1)] + 2(|F_1|\cdot|F_3|-1)[(|F_2|-1)] + 2(|F_2|\cdot|F_3|-1)[(|F_1|-1)] + 2(|J_1|\cdot|J_2|-1)[(|J_3|-1)] + 2(|J_1|\cdot|J_3|-1)[(|J_2|-1)] + 2(|J_2|\cdot|J_3|-1)[(|J_1|-1)]$.

4 Conclusion

We have calculated Weiner index, degree based topological indices and eccentricity based topological indices of the join graph of zero divisor graphs of direct product of three finite fields in terms of the order of fields.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of Generative AI and AI-assisted technologies in the writing process

The authors declare that they have not used generative AI and AI-assisted technologies in the writing process.

References

[1] Abdussakir, A. Some topological indices of subgroup graph of symmetric group. Mathematics and Statistics 7, 4 (2019), 98–105.

[2] Anderson, D. F., and Livingston, P. S. The zero-divisor graph of a commutative ring. Journal of Algebra 217, 2 (1999), 434–447.

[3] De, N., Abu Nayeem, S. M., and Pal, A. Total eccentricity index of the generalized hierarchical product of graphs. International Journal of Applied and Computational Mathematics 1 (2015), 503–511.

[4] De, N., Pal, A., and Nayeem, S. M. A. Total eccentricity index of some composite graphs. Malaya Journal of Matematik 3, 04 (2015), 523–529.

[5] Furtula, B., and Gutman, I. A forgotten topological index. Journal of mathematical chemistry 53, 4 (2015), 1184–1190.

[6] Gaded, S. M., and Narayana, N. S. On connectivity of zero-divisor graphs and complement graphs of some semi-local rings. Journal of Computational Mathematica 6, 2 (2022), 135–141.

[7] Gaded, S. M., and Narayana, N. S. On corona product of zero-divisor graphs of direct product of finite fields. Advances and Applications in Discrete Mathematics 40, 1 (2023), 113–123.

[8] Gaded, S. M., and Narayana, N. S. On zero divisors graphs of direct product of finite fields. Journal of Computational Mathematica 7, 1 (2023), 100–109.

[9] Gaded, S. M., and Narayana, N. S. On some topological indices of zero divisor graphs of direct product of three finite fields. Examples and Counterexamples 5 (2024).

[10] Gallian, J. Contemporary abstract algebra. Chapman and Hall/CRC, 2021.

[11] Ghorbani, M., and Hosseinzadeh, M. A. A new version of zagreb indices. Filomat 26, 1 (2012), 93–100.

[12] Go, C. E., and Canoy Jr, S. R. Domination in the corona and join of graphs. In International Mathematical Forum (2011), vol. 6, Citeseer, pp. 763–771.

[13] Hosseinzadeh, M. A., Iranmanesh, A., and Došli'c, T. On the narumi-katayama index of composite graphs. Croatica Chemica Acta 86, 4 (2013), 503–508.

[14] Idrees, N., Saif, M. J., Rauf, A., and Mustafa, S. First and second zagreb eccentricity indices of thorny graphs. Symmetry 9, 1 (2017), 7.

[15] Jauhari, M., Ali, F., et al. Survey on topological indices and graphs associated with a commutative ring. In Journal of Physics: Conference Series (2020), vol. 1562, IOP Publishing, p. 012008.

[16] Ramane, H. S., and Kitturmath, D. V. On the conjecture of sombor energy of a graph. Examples and Counterexamples 3 (2023), 100115.

[17] West, D. B., et al. Introduction to graph theory, vol. 2. Prentice hall Upper Saddle River, 2001.

[18] Zhang, X., Awais, H. M., Javaid, M., and Siddiqui, M. K. Multiplicative zagreb indices of molecular graphs. Journal of Chemistry 2019 (2019), 1–19.