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Stability Of 4D Alternate Additive Functional Equation

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Abstract

In this paper, we confer the generalized Ulam-Hyers stability of a 4D alternate additive functional equation in Banach and Random Banach space via direct and fixed point methods.

Key words: Additive functional equation, Ulam stability, Hyers method.

AMS classification: 39B52, 32B72, 32B82

1 Introduction

The stability of functional equations is a hot topic that was delt in the last eight decades. In 1940, S.M. Ulam, gave a widespread of talk before a Mathematical Colloquium at the University of Wisconsin in which he introduced the number of important unsolved problems. One of them is the first point of a new line of investigation, the Stability Problem.
The Cauchy additive functional equation is the famous functional equation

$$P(u_1 + u_2) = P(u_1) + P(u_2) \tag{1}$$

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by M.Arunkumar etal., by considering the summation of both the sum and the product of two $p-$norms.

In this paper, we confer the generalized Ulam-Hyers stability of a 4D alternate additive functional equation

$$P\left(\sum_{i=1}^{4} i\ u_i\right) + P\left(\sum_{i=1;i\neq 3}^{4} i\ u_i - 3\ u_3\right) + P\left(\sum_{i=1;i\neq 2}^{4} i\ u_i - 2\ u_2\right) + P\left(\sum_{i=2}^{4} iu_i - 1u_1\right)$$

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$$+P\left(\sum_{i=1;i\neq 2,3}^{4} iu_i-\sum_{i=2}^{3} iu_i\right)+P\left(\sum_{i=2;i\neq 3}^{4} iu_i-\sum_{i=2;i\neq 2}^{3} iu_i\right)$$

$$+P\left(\sum_{i=3}^{4} iu_i-\sum_{i=1}^{2} iu_i\right)+P\left(-\sum_{i=1}^{3} iu_i+4u_4\right)=32P(u_4) \tag{2}$$

in Banach and Random Banach space via direct and fixed point methods.

2 General Solution

In this segment, the author discuss about the general solution of functional equation (2). By considering V and W as real vector spaces.

Theorem 2.1 If $P: V \to W$ fulfilling with the functional equation (1) if and only if $P: V \to W$ fulfilling the functional equation (2) for all $u_4, u_3, u_2, u_1 \in V$.

Proof. Suppose $P: V \to W$ fulfilling with the functional equation (1). Let $u_1 = u_2 = 0$ in (1), we get $P(0) = 0$. Let $u_2 = -u_1$ in (1), we obtain $P(-u_1) = -P(u_1)$, for all $u_1 \in V$. Let $u_2 = u_1$ in (1), we have $P(2u_1) = 2P(u_1)$, for all $u_1 \in V$. Let $u_2 = 2u_1$ in (1), we have $P(3u_1) = 3P(u_1)$, for all $u_1 \in V$. In general for a positive integer N, such that $P(Nu_1) = NP(u_1)$, for all $u_1 \in V$.
Replace $u_1 = 4u_4 + 3u_3$ and $u_2 = 2u_2 + u_1$ in (1), we get

$$P(4u_4+3u_3+2u_2+u_1)=4P(u_4)+3P(u_3)+2P(u_2)+P(u_1) \tag{3}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 - 3u_3$ and $u_2 = 2u_2 + u_1$ in (1), we obtain

$$P(4u_4-3u_3+2u_2+u_1)=4P(u_4)-3P(u_3)+2P(u_2)+P(u_1) \tag{4}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 + 3u_3$ and $u_2 = -2u_2 + u_1$ in (1), we arrive

$$P(4u_4+3u_3-2u_2+u_1)=4P(u_4)+3P(u_3)-2P(u_2)+P(u_1) \tag{5}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 + 3u_3$ and $u_2 = 2u_2 - u_1$ in (1), we have

$$P(4u_4+3u_3+2u_2-u_1)=4P(u_4)+3P(u_3)+2P(u_2)-P(u_1) \tag{6}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 - 3u_3$ and $u_2 = -2u_2 + u_1$ in (1), we get

$$P(4u_4 - 3u_3 - 2u_2 + u_1) = 4P(u_4) - 3P(u_3) - 2P(u_2) + P(u_1) \tag{7}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 - 3u_3$ and $u_2 = 2u_2 - u_1$ in (1), we obtain

$$P(4u_4 - 3u_3 + 2u_2 - u_1) = 4P(u_4) - 3P(u_3) + 2P(u_2) - P(u_1) \tag{8}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 + 3u_3$ and $u_2 = -2u_2 - u_1$ in (1), we have

$$P(4u_4 + 3u_3 - 2u_2 - u_1) = 4P(u_4) + 3P(u_3) - 2P(u_2) - P(u_1) \tag{9}$$

for all $u_4, u_3, u_2, u_1 \in V$. Replace $u_1 = 4u_4 - 3u_3$ and $u_2 = -2u_2 - u_1$ in (1), we get

$$P(4u_4 - 3u_3 - 2u_2 - u_1) = 4P(u_4) - 3P(u_3) - 2P(u_2) - P(u_1) \tag{10}$$

for all $u_4, u_3, u_2, u_1 \in V$. Adding all the equations (3), (4), (5), (6), (7), (8), (9), (10), we arrive (2).
Conversely, if $P : V \to W$ fulfilling with the functional equation (2).
Set $u_1 = u_2 = u_3 = u_4 = 0$ in (2), we have

$$P(0) = 0 \tag{11}$$

Again set $u_1 = -u$ and $u_2 = u_3 = u_4 = 0$ in (2), we have

$$P(-u) = -P(u) \tag{12}$$

for all $u \in V$. Therefore P is an odd function. Put $u_2 = u$ and $u_1 = u_3 = u_4 = 0$ in (2), we get

$$P(2u) = 2P(u) \tag{13}$$

for all $u \in V$. Put $u_3 = u$ and $u_1 = u_2 = u_4 = 0$ in (2), we get

$$P(3u) = 3P(u) \tag{14}$$

for all $u \in V$. Switching $u_4 = u$ and $u_1 = u_2 = u_3 = 0$ in (2), we obtain

$$P(4u) = 4P(u) \tag{15}$$

for all $u \in V$. In general for any integer N such that

$$P(Nu) = NP(u) \tag{16}$$

for all $u \in V$. Setting $u_1 = u_2 = 0$ in (2), we have

$$P(4u_4 + 3u_3) + P(4u_4 - 3u_3) = 8P(u_4) \tag{17}$$

for all $u_4, u_3 \in V$. Put $u_4 = \frac{u_1}{4}$ and $u_3 = \frac{u_2}{3}$ in (15), we arrive

$$P(u_1 + u_2) + P(u_1 - u_2) = 2P(u_1) \tag{18}$$

for all $u_1, u_2 \in V$. Interchanging u_1 and u_2 in (16) , we get

$$P(u_1 + u_2) - P(u_1 - u_2) = 2P(u_2) \tag{19}$$

for all $u_1, u_2 \in V$. Adding (18) and (19), we arrive our result. This completes the proof of the theorem.

3 Stability Results in Banach Space

In this segment, the author discuss about the generalized Ulam - Hyers stability of functional equation (2). Assume that V be a normed space, W be a Banach space and we take $DP : V \to W$ by

$$\begin{aligned} DP(u_4, u_3, u_2, u_1) &= P\left(\sum_{i=1}^{4} i\ u_i\right) + P\left(\sum_{i=1;i\neq 3}^{4} i\ u_i - 3\ u_3\right) \\ &+ P\left(\sum_{i=1;i\neq 2}^{4} i\ u_i - 2\ u_2\right) + P\left(\sum_{i=2}^{4} iu_i - 1u_1\right) + P\left(\sum_{i=1;i\neq 2,3}^{4} iu_i - \sum_{i=2}^{3} iu_i\right) \\ &+ P\left(\sum_{i=2;i\neq 3}^{4} iu_i - \sum_{i=2;i\neq 2}^{3} iu_i\right) + P\left(\sum_{i=3}^{4} iu_i - \sum_{i=1}^{2} iu_i\right) + P\left(-\sum_{i=1}^{3} iu_i + 4u_4\right) - 32P(u_4) \end{aligned}$$

for all $u_4, u_3, u_2, u_1 \in V$, to derive the stability results.

3.1 Direct Method

Theorem 3.1 Let $r = \pm 1$ and $\psi : V^4 \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\psi\left(4^{nr}u_4, 4^{nr}u_3, 4^{nr}u_2, 4^{nr}u_1\right)}{4^{nr}} = 0 \tag{20}$$

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and $P : V \to W$ be a function satisfies the inequality

$$\left\| DP(u_4, u_3, u_2, u_1) \right\| \leq \psi(u_4, u_3, u_2, u_1) \tag{21}$$

for all $u_4, u_3, u_2, u_1 \in V$. Then there exists a unique additive mapping $A : V \to W$ such that

$$A(u) = \lim_{n\to\infty} \frac{P(4^{nr}u)}{4^{nr}} \tag{22}$$

which satisfies the functional equation (2) and

$$\|A(u) - P(u)\| \leq \frac{1}{32} \sum_{i=\frac{1-r}{2}}^{\infty} \frac{\psi(4^{ir}u, 0, 0, 0)}{4^{ir}} \tag{23}$$

for all $u \in V$.

Proof. Assume $r = 1$.

Interchange $u_4 = u$ and $u_3 = u_2 = u_1 = 0$ in (21), we get

$$\|8P(4u) - 32P(u)\| \leq \psi(u, 0, 0, 0) \Longrightarrow \left\| \frac{p(4u)}{4} - P(u) \right\| \leq \frac{1}{32}\psi(u, 0, 0, 0) \tag{24}$$

for all $u \in V$. Replace u by $4u$ in (24) and divided by 4, we obtain

$$\left\| \frac{p(4^2u)}{4^2} - \frac{P(4u)}{4} \right\| \leq \frac{1}{32} \cdot \frac{1}{4}\psi(4u, 0, 0, 0) \tag{25}$$

for all $u \in V$. Adding (24) and (25), we have

$$\left\| \tfrac{p(4^2u)}{4^2} - P(u) \right\| \leq \left\| \frac{p(4^2u)}{4^2} - \frac{P(4u)}{4} \right\| + \left\| \frac{p(4u)}{4} - P(u) \right\|$$

$$\leq \frac{1}{32} \left[\frac{1}{4}\psi(4u, 0, 0, 0) + \psi(u, 0, 0, 0) \right] \tag{26}$$

for all $u \in V$. In general for any positive integer n, we arrive

$$\left\| \frac{P(4^n u)}{4^n} - P(u) \right\| \leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{1}{4^i}\psi(4^i u, 0, 0, 0) \tag{27}$$

for all $u \in V$. Now, we replace u by $4^m u$ and multiply by 4^m in above inequality (27), we get

$$\left\| \frac{P(4^{n+m}u)}{4^{n+m}} - \frac{P(4^m u)}{4^m} \right\| = \frac{1}{4^m} \left\| \frac{P(4^{n+m}u)}{4^n} - P(4^m u) \right\| \leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{\psi(4^{i+m}u, 0, 0, 0)}{4^{i+m}} \tag{28}$$

for all $u \in V$. Hence the sequence $\left\{ \frac{P(4^n u)}{4^n} \right\}$ is a Cauchy sequence. Since W is complete, then there exists a mapping $A : V \to W$ such that

$$A(u) = \lim_{n \to \infty} \frac{P(4^n u)}{4^n}$$

for all $u \in V$. Letting $n \to \infty$ in (27), we see that (23), for all $u \in V$.

Next to prove that A satisfies the equation (2). Replacing (u_4, u_3, u_2, u_1) by $(4^n u_4, 4^n u_3, 4^n u_2, 4^n u_1)$ in (21) divided by 4^n, we obtain that A satisfies the functional equation (2). To prove that A is unique. Let A' be the another additive mapping satisfying (2) and (23). Now,

$$\begin{aligned} \|A(u) - A'(u)\| &= \left\| \frac{P(4^n u)}{4^n} - \frac{P'(4^n u)}{4^n} \right\| \\ &= \frac{1}{4^n} \|P(4^n u) - A(4^n u) + A(4^n u) - P'(4^n u)\| \\ &\leq \frac{1}{16} \sum_{i=0}^{\infty} \frac{\psi(4^{i+n}0, 0, 0)}{4^{i+n}} \to \quad 0 \quad as \quad n \to \infty \end{aligned} \tag{29}$$

Therefore $A(u) = A'(u)$ for all $u \in V$. Hence, A is unique.

For, $r = -1$. Put $u = \frac{u}{4}$ in (24),we arrive

$$\left\| P(u) - 4P\left(\frac{u}{4}\right) \right\| \leq \frac{1}{32}\ 4\ \psi\left(\frac{u}{4}, 0, 0, 0\right) \tag{30}$$

for all $u \in V$. Again replace u by $\frac{u}{4}$ in above equation,we obtain

$$\left\| 4P\left(\frac{u}{4}\right) - 4^2 P\left(\frac{u}{4^2}\right) \right\| \leq \frac{1}{32}\ 4^2\ \psi\left(\frac{u}{4^2}, 0, 0, 0\right) \tag{31}$$

for all $u \in V$. From (30) and (31),we arrive

$$\left\| P(u) - 4^2 P\left(\frac{u}{4^2}\right) \right\| \leq \frac{1}{32}\left[4\psi\left(\frac{u}{4},0,0,0\right) + 4^2\psi\left(\frac{u}{4^2},0,0,0\right)\right] \tag{32}$$

for all $u \in V$.In general for any positive integer n such that

$$\left\| P(u) - 4^n P\left(\frac{u}{4^n}\right) \right\| \leq \frac{1}{32}\sum_{i=1}^{n} 4^i \psi\left(\frac{u}{4^i},0,0,0\right) \tag{33}$$

for all $u \in V$. The rest of the proof is similar to that of previous case. Thus the proof is complete.

Corollary 3.2 Let θ and s be nonnegative real numbers. Let a function $P : V \to W$ satisfies the inequality

$$\left\| DP(u_4,u_3,u_2,u_1) \right\| \leq \begin{cases} \theta, \\ \theta\left\{||u_4||^s + ||u_3||^s + ||u_2||^s + ||u_1||^s\right\}, s \neq 1 \end{cases} \tag{34}$$

for all $u_4, u_3, u_2, u_1 \in V$. Then there exists a unique additive function $A : V \to W$ such that

$$\|A(u) - P(u)\| \leq \begin{cases} \dfrac{\theta}{|24|}, \\ \dfrac{\theta||u||^s}{8|4-4^s|}, \end{cases} \tag{35}$$

for all $u \in V$.

3.2 Fixed point Method

Now, we will recall the fundamental results in fixed point theory.

Theorem 3.3 (Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

$d(T^nx, x^*) \leq \frac{1}{1-L}\ \ d(T^nx, T^{n+1}x), \forall\ \ n \geq 0, \forall\ \ x \in X;$
$d(x, x^*) \leq \frac{1}{1-L}\ \ d(x, x^*), \forall\ \ x \in X.$

Theorem 3.4 (The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either

$(B1) \qquad d(T^nx, T^{n+1}x) = \infty \quad \forall n \geq 0,$

or

$(B2)$ there exists a natural number n_0 such that:
$[FP1] d(T^nx, T^{n+1}x) < \infty$ for all $n \geq n_0$;
$[FP2]$ The sequence (T^nx) is convergent to a fixed point y^* of T
$[FP3] y^*$ is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
$[FP4]$ $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

Theorem 3.5 Let $P : V \to W$ be a mapping for which there exists a function $\psi : V^4 \to [0, \infty)$ with condition

$$\lim_{n\to\infty} \frac{\psi(\mu_i^n u_4, \mu_i^n u_3, \mu_i^n u_2 \mu_i^n u_1)}{\mu_i^n} = 0 \tag{36}$$

where $\mu_i = 4$ if $i = 0$ and $\mu_i = \frac{1}{4}$ if $i = 1$ such that the functional inequality

$$\Big\| DP(u_4, u_3, u_2, u_1) \Big\| \leq \psi(u_4, u_3, u_2, u_1) \tag{37}$$

for all $u_4, u_3, u_2, u_1 \in V$. If there exists $L = L(i) < 1$ such that the function

$$\beta(u) = \frac{1}{8}\psi\left(\frac{u}{4}, 0, 0, 0\right) \tag{38}$$

has the property

$$\beta(u) = L\mu_i\beta\left(\frac{u}{\mu_i}\right) \tag{39}$$

for all $u \in V$. Then there exists unique additive function $A : V \to W$ satisfying the functional equation (2) and

$$\|P(u) - A(u)\| \leq \frac{L^{1-i}}{1-L}\beta(u) \tag{40}$$

holds for all $u \in V$.

Proof. Consider the set $d(g,h) = \inf\{k \in [o,\infty) : \|g(u)-h(u)\| \leq K\beta(u), u \in V\}$. It is easy to see that (V,d) is a complete. Define $T : V \to V$ by $Tg(u) = \frac{1}{\mu_i} g(\mu_i u)$, for all $u \in V$. Now

$$\begin{aligned}
&d(g,h) \leq K \\
&\Rightarrow \|g(u)-h(u)\| \leq K\beta(u), u \in V \\
&\Rightarrow \left\|\frac{1}{\mu_i} g(\mu_i u) - \frac{1}{\mu_i} h(\mu_i u)\right\| \leq \frac{1}{\mu_i} K\beta(\mu_i u), u \in V \\
&\Rightarrow \left\|\frac{1}{\mu_i} g(\mu_i u) - \frac{1}{\mu_i} h(\mu_i u)\right\| \leq LK\beta(u), u \in V \\
&\Rightarrow \|Tg(u) - Th(u)\| \leq LK\beta(u), u \in V \\
&\Rightarrow d(Tg,Th) \leq LK.
\end{aligned}$$

This implies that $d(Tg,Th) \leq Ld(g,h)$. (i.e) T is a strictly contractive mapping on V with the Lipschitz constant L.
Using (39) in (24) for the case $i=0$, it reduces to obtain

$$\left\|P(u) - \frac{1}{4}P(4u)\right\| \leq \frac{1}{4}\beta(u) \Rightarrow d(P,TP) \leq \frac{1}{4} = L = L^1 \leq \infty \tag{41}$$

for all $u \in V$. Using(39) in (30) for the case $i=1$,it reduces to arrive

$$\left\|P(u) - 4P(\frac{u}{4})\right\| \leq \beta(u) \Rightarrow d(P,TP) \leq 1 = L^0 \leq \infty \tag{42}$$

for all $u \in V$. In the above cases,we arrive

$$d(P,TP) \leq L^{1-i}.$$

Therefore$[FP1]$ holds. By the Alternative fixed point theorem $[FP2]$,we have

$$d(T^n x, T^{n+1}x) \leq \infty foralln \geq n_o.$$

It follows that there exists a fixed point A of T in V, such that

$$A(u) = \lim_{n\to\infty} \frac{p(\mu_i^n u)}{\mu_i^n}$$

for all $u \in V$. In order to prove that $A : V \to W$ is additive. Replacing (u_4, u_3, u_2, u_1) by $(\mu_i^n u_4, \mu_i^n u_3, \mu_i^n u_2, \mu_i^n u_1)$ in (37) and dividing by μ_i^k, A satisfies the functional equation (2). By Theorem $[FP3]$ A is the unique fixed point of T in the set

$$Y = \{P \in X : d(TP, A) < \infty\}$$

and A is the unique function such that

$$\|P(u) - A(u)\| \leq K\beta(u)$$

for all $u \in V$. Finally, By theorem $[FP4]$, we obtain

$$d(P, A) \leq \frac{1}{1-L} d(P, Tp).$$

This implies and conclude that

$$\|P(u) - A(u)\| \leq \frac{L^{1-i}}{1-L}\beta(u)$$

for all $u \in V$. This completes the proof of the theorem.

Corollary 3.6 Let θ and s be non negative real numbers. Let a function $P : V \to W$ satisfies the inequality

$$\left\|DP(u_4, u_3, u_2, u_1)\right\| \leq \begin{cases} \theta, \\ \theta\left\{||u_4||^s + ||u_3||^s + ||u_2||^s + ||u_1||^s\right\}, s < 1 \quad or \quad s > 1; \end{cases}$$

for all $u_4, u_3, u_2, u_1 \in V$ .Then there exists a unique additive function $A : V \to W$ such that

$$\|A(u) - P(u)\| \leq \begin{cases} \dfrac{\theta}{|24|}, \\ \dfrac{\theta||u||^s}{8|4-4^s|} \end{cases} \tag{43}$$

for all $u \in V$.

4 Stability Results in RN-Space

In this segment, the author present the Basis definations and generalized Ulam - Hyers stability of functional equation (2) in RN space.

4.1 Basics of RN-Space

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [4,20,21].

Hereafter, this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : R \cup \{-\infty, +\infty\} \to [0, 1]$ such that F is leftcontinuous and nondecreasing on $R, F(0) = 0$ and $F(+\infty) = 1$ D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- \quad F(+\infty) = 1$ where $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t \to x} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in R$. The maximal element for Δ^+ in this order is the d.f. given by:

$$\epsilon_0(t) = \begin{cases} 0, & if \quad t \leq 0, \\ 1, & if \quad t > 0. \end{cases}$$

Definition 4.1 A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a, 1) = a$ for all $a \in [0, 1]$;

(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 4.2 A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous $t-$norm and μ is a mapping from X into D^+ satisfying the following conditions:

(RN1) $\mu_x(t) = \varepsilon(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha\, x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Example 4.3 Every normed spaces $(X, || \cdot ||)$ defines a random normed space

(X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + ||x||}$$

and T_M is the minimum $t-$norm. This space is called the induced random normed space.

Definition 4.4 Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.

(3) A RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 4.5 If (X, μ, T) is a RN-space and $\{xn\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Hereafter, Let us consider V to be a linear space and (W, μ, T) to be a complete RN-space. Define a mapping $DP : V \to W$ by

$$\begin{aligned} &DP(u_4, u_3, u_2, u_1) \\ &= P\left(\sum_{i=1}^{4} i\ u_i\right) + P\left(\sum_{i=1;i\neq 3}^{4} i\ u_i - 3\ u_3\right) + P\left(\sum_{i=1;i\neq 2}^{4} i\ u_i - 2\ u_2\right) \\ &+ P\left(\sum_{i=2}^{4} iu_i - 1u_1\right) + P\left(\sum_{i=1;i\neq 2,3}^{4} iu_i - \sum_{i=2}^{3} iu_i\right) + P\left(\sum_{i=2;i\neq 3}^{4} iu_i - \sum_{i=2;i\neq 2}^{3} iu_i\right) \\ &+ P\left(\sum_{i=3}^{4} iu_i - \sum_{i=1}^{2} iu_i\right) + P\left(-\sum_{i=1}^{3} iu_i + 4u_4\right) - 32P(u_4) \end{aligned}$$

for all $u_4, u_3, u_2, u_1 \in V$.

4.2 Direct Method

Theorem 4.6 Let $r \in \{-1, 1\}$ and $P : V \to W$ be a mapping for which there exist a function $\psi : V^4 \to D^+$ with the condition

$$\lim_{m\to\infty} T_{i=0}^{\infty} \psi_{4^{r(n+m}u,0,0,0}\left(32.4^{r(n+m)}t\right) = 1 = \lim_{n\to\infty} \psi_{4^{nr}u_4,^{nr}u_3,4^{nr}u_2,4^{nr}u_1}\left(4^{nr}t\right) \tag{44}$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $t > 0$, satisfying the functional inequality

$$\mu_{DP(u_4,u_3,u_2,u_1)}(t) \geq \psi_{u_4,u_3,u_2,u_1}(t) \tag{45}$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $t > 0$. Then there exists a unique additive mapping $A : V \to W$ satisfying the functional equation (2) such that

$$\mu_{A(u)-P(u)}(t) \geq T_{n=0}^{\infty} \psi_{4^{nr},0,0,0}\left(32.4^{nr}t\right) \tag{46}$$

for all $u \in V$ and all $t > 0$.
Proof. Assume $r = 1$. Replacing $u_4 = u$ and $u_3 = u_2 = u_1 = 0$ in (2), we get

$$\mu_{8P(4u)-32P(u)}(t) \geq \psi_{u,0,0,0}(t) \tag{47}$$

for all $u \in V$ and all $t > 0$.

$$\mu_{\frac{P(4u)}{4}-P(u)}\left(\frac{t}{32}\right) \geq \psi_{u,0,0,0}(t) \tag{48}$$

for all $u \in V$ and all $t > 0$. Replace u by $4^n u$ in (48), we have

$$\mu_{\frac{P(4^{n+1}u)}{4^{n+1}}-\frac{P(4^n u)}{4^n}}(t) \geq \psi_{4^n,0,0,0}\left(32.4^n t\right) \tag{49}$$

for all $u \in V$ and all $t > 0$. It is easy to see that

$$\frac{P(4^n u)}{4^n} - P(u) = \sum_{k=0}^{n-1}\left(\frac{P(4^{k+1}u)}{4^{k+1}} - \frac{P(4^k u)}{4^k}\right) \tag{50}$$

for all $u \in V$ and all $t > 0$. From equations (49) and (50), we have

$$\begin{aligned} \mu_{\frac{P(4^n u)}{5^n} - P(u)}(t) =& \mu_{\sum_{k=0}^{n-1} \frac{P(4^{k+1}u)}{4^{k+1}} - \frac{P(4^k u)}{4^k}} \left(32.4^{k+1} t\right), \\ \geq& T_{k=0}^{n-1} \psi_{4^k u,0,0,0} \left(32.4^k t\right), \end{aligned} \tag{51}$$

for all $u \in V$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\dfrac{P(4^n u)}{4^n}\right\}$, replace u by $4^m u$ in (51) for any $m > n > 0$, we arrive
$\mu_{\frac{P(4^{n+m}u)}{4^{n+m}} - \frac{P(4^m u)}{4^m}}(t) \geq T_{k=0}^{n-1} \psi_{4^{k+m}u,0,0,0} \left(32.4^k t\right),$

$$= T_{k=0}^{n+m-1} \psi_{4^k u,0,0,0} \left(32.4^k t\right), \to 1 \quad as \;\; n \to \infty \tag{52}$$

for all $u \in V$ and all $t > 0$. Hence the sequence $\left\{\dfrac{P(4^n u)}{4^n}\right\}$ is a Cauchy sequence. Since W is complete, there exists a mapping $A : V \to W$ such that

$$\mu_{A(u)}(t) = \lim_{n \to \infty} \mu_{\frac{P(4^n u)}{4^n}}(t) \quad \forall \;\; u \in V, t > 0.$$

Letting $n \to \infty$ we see that (46) holds for all $u \in V$ and all $t > 0$. To prove that A satisfies (2), Replacing (u_4, u_3, u_2, u_1) by $(4^n u_4, 4^n u_3, 4^n u_2, 4^n u_1)$ and dividing by 4^n in (45), we obtain

$$\frac{\mu_{DP(4^n u_4, 4^n u_3, 4^n u_2, 4^n u_1)}}{4^n}(t) \geq \psi_{4^n u_4, 4^n u_3, 4^n u_2, 4^n u_1}(4^n t) \tag{53}$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $t > 0$. Letting $n \to \infty$ in the above inequality and using the definition of $A(u)$, we see that A satisfies (2) for all $u_4, u_3, u_2, u_1 \in V$. Therefore the mapping A is Additive. Finally, to prove the uniqueness of the additive function A , let us assume that there exists a additive function A' which satisfies (2) and (46) .Since $P(4^n u) = 4^n A(u)$ and $P'(4^n u) = 4^n A'(u)$ for all $u \in V$ and all $n \in \mathbb{N}$, it follows from (46) that

$$\begin{aligned} \mu_{A(u)-A'(u)}(t) &= \mu_{P(4^n u)-P'(4^n u)}(4^n t), u \in V \\ &= \mu_{P(4^n u)-A(4^n u)+P(4^n u)-A'(4^n u)}(4^n t), \\ &\geq T\left[T_{k=0}^{\infty} \psi_{4^{n+k}u,0,0,0}\left(32.4^{n+k} t\right), T_{k=0}^{\infty} \psi_{4^{n+k}u,0,0,0}\left(4^{n+k} t\right)\right] \\ &\to 1 \quad as \;\; n \to \infty \end{aligned}$$

for all $u \in V$ and all $t > 0$. Hence A is unique.

For $r = -1$, we can prove a similar stability result. This completes the proof of the theorem.

Corollary 4.7 Let θ and s be nonnegative real numbers. Let a function $P : V \to W$ satisfies the inequality

$$\mu_{DP(u_4,u_3,u_2,u_1)}(t) \leq \begin{cases} \psi_\theta(t), \\ \psi_{\theta\{||u_4||^s+||u_3||^s+||u_2||^s+||u_1||^s\}}(t), s < 1 \quad or \quad s > 1; \end{cases} \tag{54}$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $t > 0$. Then there exists a unique additive function $A : V \to W$ such that

$$\mu_{P(u)-A(u)}(t) \leq \begin{cases} \psi_\theta(|24|t), \\ \psi_{\theta||u||^s}(8|4-4^s|t), \end{cases} \tag{55}$$

for all $u \in V$and all $t > 0$.

4.3 Fixed Point Method

Theorem 4.8 Let $P : V \to W$ be a mapping for which there exist a function $\eta : V^4 \to D^+$ Satisfying the following

$$\lim_{n\to\infty} \eta_{a_i^n u_4, a_i^n u_3, a_i^n u_2, a_i^n u_1}(a_i^n s) = 1 \tag{56}$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $s > 0$ and

$$\eta_{DP(u_4,u_3,u_2,u_1)}(s) \geq \psi_{u_4,u_3,u_2,u_1}(s) \tag{57}$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $s > 0$, where

$$a_i = \begin{cases} 4; i = 0 \\ \dfrac{1}{4}; i = 1 \end{cases} \tag{58}$$

If there exists $L = L(i)$ has the property

$$\eta_{u,0,0,0}\left(\frac{s}{a_i}\right) \geq \psi_{a_i u,0,0,0}(8LS) \quad (59)$$

for all $u \in V$. Then there exists a unique additive function $A : V \to W$ such that

$$\eta_{P(u)-A(u)}\left(\frac{L^{1-i}}{1-L}s\right) \geq \psi_{u,0,0,0}(s) \quad (60)$$

holds for all $u \in V$.
Proof Define a set $C = \{C : V \to W, C(0) = 0\}$ and introduce the generalized metric on C by

$$d(g,h) = \inf\{k \in [o,\infty) : \eta_{g(u)-h(u)}(ks) \geq \psi_{u,0,0,0}(s), u \in V, s > 0\}.$$

It is easy to see that (c,d) is complete. Define $l : C \to C$ by $lg(u) = \frac{1}{a_i}g(a_i u)$, for all $u \in V$.

Corollary 4.9 Let θ and s be nonnegative real numbers. Let a function $P : V \to W$ satisfies the inequality

$$\mu_{DP(u_4,u_3,u_2,u_1)}(t) \leq \begin{cases} ll\psi_\theta(t), \\ \psi_{\theta\{||u_4||^s+||u_3||^s+||u_2||^s+||u_1||^s\}}(t), s < 1 \quad or \quad s > 1; \end{cases} \quad (61)$$

for all $u_4, u_3, u_2, u_1 \in V$ and all $t > 0$. Then there exists a unique additive function $A : V \to W$ such that

$$\mu_{P(u)-A(u)}(t) \leq \begin{cases} \psi_\theta(|24|t), \\ \psi_{\theta||u||^s}(8|4-4^s|\ t), \end{cases} \quad (62)$$

for all $u \in V$and all $t > 0$.

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