

Fuzzy Stability of a Functional Equation Originating From a Telescoping Series

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Abstract

In this paper, we explore the generalized Ulam-Hyers stability of and additive functional equation of the form, $\sum_{k=1}^n T(\xi_{i+1} - \xi_i) = \sum_{k=1}^n T(\xi_{i+1}) - T(\xi_i)$ in fuzzy Banach space using Hyers method.

Key words: Additive functional equation, Telescoping series, Ulam stability, Hyers method, fuzzy Banach

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1 Introduction

The stability of functional equations is a hot topic that is dealt in the last eight decades. One of the most interesting questions concerning the stability problems of functional equations is as follows: When is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation ?

If the problem accepts a unique solution, we say the equation is stable. The first stability problem concerning group homomorphisms was raised by S.M. Ulam [42] in 1940 and affirmatively solved by D.H. Hyers [24]. Hyers theorem was generalized by T. Aoki [2] for additive mappings, Th.M. Rassias [23] and J.M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference.

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All the above stability results are further generalized by P. Gavruta [22] in 1994 considering the control function $\Upsilon(\gamma, \xi)$ as function of variables and proved the following Theorem.

Theorem 1.1 Let $(\mathcal{G}, +)$ be an Abelian group, $(\mathcal{U}, \|\cdot\|)$ be a Banach space and $\Upsilon : \mathcal{G} \times \mathcal{G} \rightarrow [0, \infty)$ be a mapping such that

$$\Upsilon(\gamma, \xi) = \sum_{k=0}^{\infty} 2^{-k} \Upsilon(2^k \gamma, 2^k \xi) < \infty.$$

If a function $T : \mathcal{G} \rightarrow \mathcal{E}$ satisfies the inequality

$$\|T(\gamma + \xi) - T(\gamma) - T(\xi)\| \leq \Upsilon(\gamma, \xi)$$

for all $\gamma, \xi \in \mathcal{G}$, then there exist a unique additive function $\mathcal{A} : \mathcal{G} \rightarrow \mathcal{E}$ such that

$$\|T(\gamma) - \mathcal{A}(\gamma)\| \leq \frac{\Upsilon(\gamma, \gamma)}{2}$$

for all $\gamma \in \mathcal{G}$. If moreover $T(t\gamma)$ is continuous in t for each fixed $\gamma \in \mathcal{G}$, then \mathcal{A} is linear.

This stability result known as generalized Hyers-Ulam-Rassias stability of functional equations.

In 2008, a special case of Gavrutas theorem for the unbounded Cauchy difference was obtained by Ravi et al., [40] by considering the summation of both the sum and the product of two norms in the sprit of Rassias approach.

The solution and stability of various additive functional equations in various Banach spaces were introduced and discussed in [4, 6, 7, 8, 11, 12, 19, 37, 36, 38, 39] and references cited there in.

Definition 1.2 [20, 43] A telescoping series is a series where each term γ_k can be written as $\gamma_k = \xi_k - \xi_{k-1}$ for some series ξ_k . It is a series whose partial sums eventually only have a finite number of terms after cancellation. The cancellation technique, with part of each term cancelling with part of the next term, is known as the method of differences that helps to determine the sum of the telescopic series.

Let ξ_k be a sequence of numbers. Then,

$$\sum_{k=1}^N (\xi_k - \xi_{k-1}) = \xi_N - \xi_0 \quad (1)$$

is called Telescoping partial sum. Many trigonometric functions also admit representation as a difference, which allows telescopic canceling between the consecutive terms.

The above sum series (1) can be turned to an additive functional equation of the form

$$\sum_{k=1}^n T(\xi_{i+1} - \xi_i) = \sum_{k=1}^n T(\xi_{i+1}) - T(\xi_i) \quad (2)$$

having solution $T(\xi) = \alpha\xi$.

Recently, the stability of the above functional equation was introduced and investigate the stability in Banach space was obtained by K. Balamurugan et.al.,[17].

In this paper, we explore the generalized Ulam-Hyers stability of and additive functional equation (2) in fuzzy Banach space using Hyers method.

Hereafter throughout this paper, assume that (\mathcal{F}, μ) be fuzzy normed linear space and (\mathcal{W}, μ) be fuzzy Banach space. We use the following abbreviation for a given function $T : \mathcal{F} \rightarrow \mathcal{W}$ by

$$DT(\xi_{i+1}, \xi_i) = \sum_{k=1}^n T(\xi_{i+1} - \xi_i) - \left\{ \sum_{k=1}^n T(\xi_{i+1}) - T(\xi_i) \right\}$$

or all $\xi_i \in \mathcal{F}$, $(i = 1, 2, \dots, n)$.

2 Preliminaries of Fuzzy Normed Spaces

Now, we give some definitions in fuzzy Banach space which was given in [15, 16, 21, 27, 28, 29, 30, 31, 32, 44].

Definition 2.1 Let \mathcal{F} be a linear. A function $\mu : \mathcal{F} \times \mathbb{R} \rightarrow [0, 1]$ (so-called Fuzzy subset) is said to be fuzzy norm on \mathcal{F} if for all $\gamma, \xi \in \mathcal{F}$ and all $\alpha, \beta \in \mathbb{R}$,

- (F₁) $\mu(\xi, \alpha) = 0$ if $\alpha \leq 0$;
 - (F₂) $\xi = 0$ if and only if $\mu(\xi, \alpha) = 1$ for all $\alpha > 0$;
 - (F₃) $\mu(\alpha\xi, \beta) = \mu(\xi, \frac{\beta}{|\alpha|})$ if $\alpha \neq 0$;
 - (F₄) $\mu(\xi + \gamma, \beta + \delta) \geq \min \{\mu(\xi, \beta), \mu(\gamma, \delta)\}$;
 - (F₅) $\mu(\xi, \cdot)$ is a non-decreasing function on \mathbb{R} and $\lim_{\delta \rightarrow \infty} \mu(\xi, \delta) = 1$;
 - (F₆) for $\xi \neq 0$, $\mu(\xi, \cdot)$ is (upper semi) continuous on \mathbb{R} .
- The pair (\mathcal{F}, μ) is called fuzzy normed linear space.

Example 2.2 Let $(\mathcal{F}, \|\cdot\|)$ be a normed linear space. Then

$$\mu(\xi, \alpha) = \begin{cases} \frac{\alpha}{\alpha + \|\xi\|}, & \text{if } \alpha > 0, \xi \in \mathcal{F} \\ 0, & \text{if } \alpha \leq 0, \xi \in \mathcal{F} \end{cases}$$

is a fuzzy norm on \mathcal{F} .

Definition 2.3 Let (\mathcal{F}, μ) be a fuzzy normed linear space. Let $\{\xi_n\}$ be a sequence in \mathcal{F} . Then $\{\xi_n\}$ is said to convergent if there exist $\xi \in \mathcal{F}$ such that $\lim_{n \rightarrow \infty} \mu(\xi_{n+m} - \xi, \delta) = 1$, for all $\delta > 0$. In that case, ξ is called the limit of the sequence $\{\xi_n\}$ and we denote it by $\mu - \lim_{n \rightarrow \infty} \xi_n = \xi$.

Definition 2.4 A sequence $\{\xi_n\}$ is called Cauchy Sequence if for each $\epsilon > 0$ there exist n_0 such that for all $n \geq n_0$ and all $m > 0$, we have $\lim_{n \rightarrow \infty} \mu(\xi_{n+m} - \xi, \delta) > 1 - \epsilon$.

Definition 2.5 Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to complete and fuzzy normed space is called a fuzzy Banach space.

Definition 2.6 A mapping $T : \mathcal{F} \rightarrow \mathcal{W}$ between fuzzy normed space \mathcal{F} and \mathcal{W} is continuous at a point ξ_0 if for each sequence $\{\xi_n\}$ converges to ξ_0 in \mathcal{F} , the sequence $\{T(\xi_n)\}$ converges to $T(\xi_0)$. If T is continuous at each point $\xi_0 \in \mathcal{F}$ then T is said to be continuous on \mathcal{F} .

3 Hyers-Ulam Method Stability Result: Direct method

Theorem 3.1 Let $d \in \{-1, 1\}$ be fixed. Also let $\zeta : \mathcal{F}^2 \rightarrow \mathcal{W}$ be a function such that for some α with $0 < \left(\frac{\alpha}{2}\right)^d < 1$ fulfilling the inequality

$$\mu'(\zeta(2^{nd}\xi_{i+1}, 2^{nd}\xi_i), \delta) \geq \mu'(\alpha^{nd}\zeta(\xi_{i+1}, \xi_i), \delta) \quad (3)$$

for all $\xi_i \in \mathcal{F}$, $(i = 1, 2, \dots, n)$ and all $\delta > 0$, and

$$\lim_{n \rightarrow \infty} \mu'(\zeta(2^{nd}\xi_{i+1}, 2^{nd}\xi_i), 2^{nd}\delta) = 1 \quad (4)$$

for all $\xi_i \in \mathcal{F}$, $(i = 1, 2, \dots, n)$ and all $\delta > 0$. Suppose that a function $T : \mathcal{F} \rightarrow \mathcal{W}$ satisfies the following inequalities

$$\mu(DT(\xi_{i+1}, \xi_i), \delta) \geq \mu'(\zeta(\xi_{i+1}, \xi_i), \delta) \quad (5)$$

for all $\xi_i \in \mathcal{F}$, $(i = 1, 2, \dots, n)$ and all $\delta > 0$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{F} \rightarrow \mathcal{W}$ which satisfies (2) and the inequality

$$\mu(\mathcal{A}(\xi) - T(\xi), \delta) \geq \mu'(2\psi(\xi), \delta|2 - \alpha|) \quad (6)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. The mapping $\mathcal{A}(\xi)$ is defined by

$$\mathcal{A}(\xi) = \mu - \lim_{n \rightarrow \infty} \frac{T(2^n \xi)}{2^n} \quad (7)$$

Proof: Assume $d = 1$. Replacing (ξ_{i+1}, ξ_i) by $(\xi, -\xi)$ in (5), we get

$$\mu(nT(2\xi) - 2nT(\xi), \delta) \geq \mu'(\zeta(\xi, -\xi), \delta) \quad (8)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Using (F_3) in (8), we land

$$\mu\left(\frac{T(2\xi)}{2} - T(\xi), \delta\right) \geq \mu'\left(\frac{\zeta(\xi, -\xi)}{2n}, \delta\right) \quad (9)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Taking $\frac{\zeta(\xi, -\xi)}{2n} = \psi(\xi)$ into (9), we obtain

$$\mu\left(\frac{T(2\xi)}{2} - T(\xi), \delta\right) \geq \mu'(\psi(\xi), \delta) \quad (10)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Replacing ξ by $2^n\xi$ in (10) and using (3) and (F_3) in (10), we get

$$\mu\left(\frac{T(2^{n+1}\xi)}{2} - T(2^n\xi), \delta\right) \geq \mu'\left(\psi(\xi), \frac{\delta}{\alpha^n}\right) \quad (11)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. It is easy to verify from (11), that

$$\mu\left(\frac{T(2^{n+1}\xi)}{2^{n+1}} - \frac{T(2^n\xi)}{2^n}, \delta\left(\frac{\alpha}{2}\right)^n\right) \geq \mu'(\psi(\xi), \delta) \quad (12)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. It is easy to see that

$$\frac{T(2^n\xi)}{2^n} - T(\xi) = \sum_{k=0}^{n-1} \left(\frac{T(2^{k+1}\xi)}{2^{k+1}} - \frac{T(2^k\xi)}{2^k}\right) \quad (13)$$

for all $\xi \in \mathcal{F}$. From equations (12) and (13), we have

$$\mu\left(\frac{T(2^n\xi)}{2^n} - T(\xi), \delta \sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k\right) \geq \min \bigcup_{k=0}^{n-1} \mu\left(\frac{T(2^{k+1}\xi)}{2^{k+1}} - \frac{T(2^k\xi)}{2^k}, \delta\left(\frac{\alpha}{2}\right)^k\right) \quad (14)$$

$$\geq \min \bigcup_{k=0}^{n-1} \left\{ \mu'(\psi(\xi), \delta) \right\} \geq \mu'(\psi(\xi), \delta) \quad (15)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Replacing ξ by $2^m\xi$ in (14) and using (3), (F_3) , we obtain

$$\mu\left(\frac{T(2^{n+m}\xi)}{2^{n+m}} - \frac{T(2^m\xi)}{2^m}, \delta \sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k\right) \geq \mu'(\psi(\xi), \frac{\delta}{\alpha^m}) \quad (16)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Replacing δ by $\alpha^m\delta$ in (16), we get

$$\mu\left(\frac{T(2^{n+m}\xi)}{2^{n+m}} - \frac{T(2^m\xi)}{2^m}, \delta \sum_{k=m}^{m+n-1} \left(\frac{\alpha}{2}\right)^k\right) \geq \mu'(\psi(\xi), \delta) \quad (17)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Using (F_3) in (17), we obtain

$$\mu\left(\frac{T(2^{n+m}\xi)}{2^{n+m}} - \frac{T(2^m\xi)}{2^m}, \delta\right) \geq \mu'\left(\psi(\xi), \frac{\delta}{\sum_{k=m}^{m+n-1} \left(\frac{\alpha}{2}\right)^k}\right) \quad (18)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$ and all $m, n \geq 0$. Since $0 < \alpha < 2$ and $\sum_{k=0}^{\infty} \left(\frac{\alpha}{2}\right)^k < \infty$, the Cauchy criterion for convergence and (F_3) implies that $\left\{\frac{T(2^n\xi)}{2^n}\right\}$ is a Cauchy sequence in (\mathcal{W}, μ) . Since (\mathcal{W}, μ) is a fuzzy Banach space, this sequence converges to some point $\mathcal{A}(\xi) \in \mathcal{W}$. So one we can define the mapping $\mathcal{A} : \mathcal{F} \rightarrow \mathcal{W}$ by

$$\mathcal{A}(\xi) = \mu - \lim_{n \rightarrow \infty} \frac{T(2^n\xi)}{2^n} \quad (19)$$

for all $\xi \in \mathcal{F}$. Letting $m = 0$ in (18), we arrive

$$\mu\left(\frac{T(2^n\xi)}{2^n} - T(\xi), \delta\right) \geq \mu'\left(\psi(\xi), \frac{\delta}{\sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k}\right) \quad (20)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Letting $n \rightarrow \infty$ in (18) and using (F_6) , we arrive

$$\mu\left(\mathcal{A}(\xi) - T(\xi), \delta\right) \geq \mu'\left(2\psi(\xi), \delta(2 - \alpha)\right) \quad (21)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Now, we need to prove \mathcal{A} satisfies the (2), replacing (ξ_{i+1}, ξ_i) by $(2^n\xi_{i+1}, 2^n\xi_i)$ in (5), respectively, we obtain

$$\mu\left(\frac{DT\left(2^n\xi_{i+1}, 2^n\xi_i\right)}{2^n}, \delta\right) \geq \mu'\left(\zeta(2^n\xi_{i+1}, 2^n\xi_i), 2^n\delta\right) \quad (22)$$

for all $\xi_i \in \mathcal{F}$, ($i = 1, 2, \dots, n$) and all $\delta > 0$. Now,

$$\begin{aligned} & \mu\left(\sum_{k=1}^n \mathcal{A}(\xi_{i+1} - \xi_i) - \sum_{k=1}^n \{\mathcal{A}(\xi_{i+1}) - \mathcal{A}(\xi_i)\}, \delta\right) \\ & \geq \min \left\{ \mu\left(\sum_{k=1}^n \mathcal{A}(\xi_{i+1} - \xi_i) - \sum_{k=1}^n \frac{\mathcal{T}(2^n(\xi_{i+1} - \xi_i))}{2^n}, \frac{\delta}{4}\right), \right. \\ & \quad \mu\left(\sum_{k=1}^n \frac{\mathcal{T}(2^n(\xi_{i+1}))}{2^n} - \sum_{k=1}^n \mathcal{A}(\xi_{i+1}), \frac{\delta}{4}\right), \mu\left(\sum_{k=1}^n \mathcal{A}(\xi_i) - \sum_{k=1}^n \frac{\mathcal{T}(2^n(\xi_i))}{2^n}, \frac{\delta}{4}\right), \\ & \quad \left. \mu\left(\sum_{k=1}^n \frac{\mathcal{T}(2^n(\xi_{i+1} - \xi_i))}{2^n} - \sum_{k=1}^n \left\{ \frac{\mathcal{T}(2^n \xi_{i+1})}{2^n} - \frac{\mathcal{T}(2^n \xi_i)}{2^n} \right\}, \frac{\delta}{4}\right) \right\} \quad (23) \end{aligned}$$

for all $\xi_i \in \mathcal{F}$, ($i = 1, 2, \dots, n$) and all $\delta > 0$. Using (22) and (F_5) in (25), we arrive

$$\mu\left(\sum_{k=1}^n \mathcal{A}(\xi_{i+1} - \xi_i) - \sum_{k=1}^n \{\mathcal{A}(\xi_{i+1}) - \mathcal{A}(\xi_i)\}, \delta\right) \geq \left\{ 1, 1, 1, \mu'\left(\zeta(2^n \xi_{i+1}, 2^n \xi_i), 2^n \delta\right) \right\} \quad (24)$$

for all $\xi_i \in \mathcal{F}$, ($i = 1, 2, \dots, n$) and all $\delta > 0$. Letting $n \rightarrow \infty$ in (26) and using (4), we see that

$$\mu\left(\sum_{k=1}^n \mathcal{A}(\xi_{i+1} - \xi_i) - \sum_{k=1}^n \{\mathcal{A}(\xi_{i+1}) - \mathcal{A}(\xi_i)\}, \delta\right) = 1 \quad (25)$$

for all $\xi_i \in \mathcal{F}$, ($i = 1, 2, \dots, n$) and all $\delta > 0$. Using (F_2) in (25), we arrive

$$\sum_{k=1}^n \mathcal{A}(\xi_{i+1} - \xi_i) = \sum_{k=1}^n \mathcal{A}(\xi_{i+1}) - \mathcal{A}(\xi_i) \quad (26)$$

for all $\xi_i \in \mathcal{F}$, ($i = 1, 2, \dots, n$). Hence \mathcal{A} satisfies the functional equation (2). In order to prove $\mathcal{A}(\xi)$ is unique, let $\mathcal{A}'(\xi)$ be another additive mapping satisfying (2)

and (6). Hence,

$$\begin{aligned} \mu\left(\mathcal{A}(\xi) - \mathcal{A}'(\xi), \delta\right) &= \mu\left(\frac{\mathcal{A}(2^n \xi)}{2^n} - \frac{\mathcal{A}'(2^n \xi)}{2^n}, \delta\right) \\ &\geq \min \left\{ \mu\left(\frac{\mathcal{A}(2^n \xi)}{2^n} - \frac{T(2^n \xi)}{2^n}, \frac{\delta}{2}\right), \mu\left(\frac{T(2^n \xi)}{2^n} - \frac{\mathcal{A}(2^n \xi)}{2^n}, \frac{\delta}{2}\right) \right\} \\ &\geq \mu\left(\psi(\xi), \frac{\delta(2-\alpha)}{2} \left(\frac{2}{\alpha}\right)^n\right) \end{aligned}$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Since

$$\lim_{n \rightarrow \infty} \frac{\delta(2-\alpha)}{2} \left(\frac{2}{\alpha}\right)^n = \infty$$

we obtain

$$\lim_{n \rightarrow \infty} \mu\left(\psi(\xi), \frac{\delta(2-\alpha)}{2} \left(\frac{2}{\alpha}\right)^n\right) = 1$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Thus

$$\mu\left(\mathcal{A}(\xi) - \mathcal{A}'(\xi), \delta\right) = 1$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Hence, we have $\mathcal{A}(\xi) = \mathcal{A}'(\xi)$. Therefore $\mathcal{A}(\xi)$ is unique. For $d = -1$. It follows from (8) that

$$\mu\left(T(2\xi) - 2T(\xi), \delta\right) \geq \mu'\left(\frac{\zeta(\xi, -\xi)}{n}, \delta\right) \tag{27}$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Replacing ξ by $\frac{\xi}{2}$ in (27), we achieve

$$\mu\left(T(\xi) - 2T\left(\frac{\xi}{2}\right), \delta\right) \geq \mu'\left(\frac{\zeta\left(\frac{\xi}{2}, -\frac{\xi}{2}\right)}{n}, \delta\right) \tag{28}$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Letting $\frac{\zeta\left(\frac{\xi}{2}, -\frac{\xi}{2}\right)}{n} = \psi(\xi)$ into (28), we obtain

$$\mu\left(T(\xi) - 2T\left(\frac{\xi}{2}\right), \delta\right) \geq \mu'\left(\psi(\xi), \delta\right) \tag{29}$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. Replacing ξ by $\frac{\xi}{2^n}$ in (29) and using (3) and (F_3) in (29), we arrive

$$\mu\left(T\left(\frac{\xi}{2^n}\right) - 2T\left(\frac{\xi}{2^{n+1}}\right), \delta\right) \geq \mu'(\psi(\xi), \delta\alpha^n) \quad (30)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. It is easy to verify from (30), that

$$\mu\left(2^n T\left(\frac{\xi}{2^n}\right) - 2^{n+1} T\left(\frac{\xi}{2^{n+1}}\right), \delta\left(\frac{2}{\alpha}\right)^n\right) \geq \mu'(\psi(\xi), \delta) \quad (31)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$. The rest of the proof is similar to that of previous case ($d = 1$). This completes the proof of the theorem.

From Theorem (3.1), we obtain the following corollary concerning the generalized Ulam-Hyers stability for the functional equation (2).

Corollary 3.2 Assume θ be positive number and λ be a real number with $\lambda \neq 1$. Let $T : \mathcal{F} \rightarrow \mathcal{W}$ be an additive function satisfies the inequality

$$\mu(DT(\xi_{i+1}, \xi_i), \delta) \geq \begin{cases} \mu'(\theta, \delta) \\ \mu'\left(\theta \sum_{i=1}^n (\|\xi_{i+1}\|^\lambda + \|\xi_i\|^\lambda), \delta\right) \\ \mu'\left(\theta \prod_{i=1}^n (\|\xi_{i+1}\|^\lambda \|\xi_i\|^\lambda), \delta\right) \end{cases} \quad (32)$$

for all $\xi_i \in \mathcal{F}$, ($i = 1, 2, \dots, n$) and all $\delta > 0$. Then there exist unique additive function $\mathcal{A} : \mathcal{F} \rightarrow \mathcal{W}$ such that

$$\mu(\mathcal{A}(\xi) - T(\xi), \delta) \geq \begin{cases} \mu'(\theta, n\delta) \\ \mu'(2\theta\|\xi\|^\lambda, \delta|2^\lambda - 2|) \\ \mu'(2\theta\|u\|^{n\lambda}, n\delta|2^\lambda - 2|) \end{cases} \quad (33)$$

for all $\xi \in \mathcal{F}$ and all $\delta > 0$.

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