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Finite Series Solution Arising from Three-Dimensional q-Difference Equation

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Abstract

In this article, we introduce a three-dimensional q-difference operator with its equation. Also we derive certain theorems using three-dimensional q-difference operator. Suitable examples verified by MATLAB are inserted to illustrate our findings.

Key words: Three-dimensional q-difference operator, Finite series and Summation solution.
AMS classification: 39A10, 47B39, 39A70.

1. Introduction

The theory of q-derivative equations of q-calculus or quantum calculus is based on the definition of the q-derivative operator, which was introduced by Jackson [1, 2]. Several groups have intensified their research on the amazing mathematics world featuring q-calculus. However, from 1930s upto the beginning of 1980's, the theory of linear q-difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. Since 1980's, an extensive and somewhat surprising interest in the subject reappeared in many areas of mathematics, physics and applications including new difference calculus and orthogonal polynomials, q-combinatories, q-arithmetics, integrable systems and variational q-calculus.

In 1989, Miller and Ross [3] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. In 2014, Britto Antony Xavier et al. [4] introduced a q-difference operator Δ_q defined as $\Delta_q u(k) = u(qk) - u(k)$ and obtained a summation solution of the generalized q-difference equation $\Delta_q^t v(k) = u(k)$, $k \in (-\infty, \infty)$ and $q \neq 1$, in the form

$$\Delta_q^{-t}u(k)\Big|\Big|_{\frac{k}{q^m}}^{k} = \sum_{(r)_{1\to t}}^{m} u\Big(k\prod_{i=1}^{t} q^{-r_i}\Big).$$

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Then we extended this q-difference equation to generalized higher oredr q-alpha difference equation

$$\underset{(q_1)\alpha_1}{\Delta}\Big(\underset{(q_2)\alpha_2}{\Delta}\Big(\cdots\underset{(q_t)\alpha_t}{\Delta}\big(v(k)\big)\cdots\Big)\Big)=u(k), k\in(-\infty,\infty), \tag{1}$$

and obtained many results. Also we derived finite q-alpha multi-series formula and finite higher order q-alpha series formula [5]. However, finding the solution of three-dimensional q-difference equation is still in the initial stage and many aspects of this theory need to be explored.

Hence in this research paper, we derive finite solution of three-dimensional q-difference equation using three-dimensional q-difference operator.

2.Three-Dimensional q-Difference Operator and its Equation

In this section, we present some notations which will be useful for the further discussions. Also we present the definition of three-dimensional q-difference operator and its inverse.

(i) $u(k_1,k_2,k_3)$ is a real valued function; (ii) $(k_1,k_2,k_3)\in R^3$;

(iii) $\sum\limits_{\substack{r_i=1\\ i:1\to t}}^{n}=\sum\limits_{r_1=1}^{n}\sum\limits_{r_2=1}^{n}\sum\limits_{r_3=1}^{n}\cdots\sum\limits_{r_t=1}^{n}$; (iv) $\prod\limits_{j=1}^{t}q^{r_j}=q^{r_1}.q^{r_2}.q^{r_3}\cdots q^{r_t}$

(v) $q^{(r_1+r_2+r_3+\cdots+r_t)}=\Big(\prod\limits_{j=1}^{t}q^{r_j}\Big)$ and (vi) m is a positive integer.

Definition 2.1 (Three-Dimensional q-Difference Operator)
Let $u(k_1,k_2,k_3)$ be a real valued function on $[0,\infty)$ and $1\neq q$ be a fixed real number. Then the three-dimensional q-difference operator on $u(k_1,k_2,k_3)$ is defined as

$$\Delta_q u(k_1,k_2,k_3)=u(qk_1,qk_2,qk_3)-u(k_1,k_2,k_3),\ q\neq1, \tag{2}$$

and the inverse of the three-dimensional q-difference operator is defined as below:
if $\Delta_q v(k_1,k_2,k_3)=u(k_1,k_2,k_3)$, then $v(k_1,k_2,k_3)=\Delta_q^{-1}u(k_1,k_2,k_3)$

3. Three-dimensional q-summation formula

We are in a position to state and prove our main results. The purpose of this section is to obtain the three-dimensional q-summation formula of three-dimensional q-difference equation (2).

Theorem 3.1 Let $1\neq q\in(0,\infty)$ and $(k_1,k_2,k_3)\in R^3$. Then the summation

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solution of (2) is

$$\Delta_q^{-1}u(k_1,k_2,k_3)\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{r=1}^{m}u\Big(\frac{k_1}{q^r},\frac{k_2}{q^r},\frac{k_3}{q^r}\Big) \tag{3}$$

and hence

$$\Delta_q^{-t}u(k_1,k_2,k_3)\Big|\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{\substack{r_i=1\\ i:1\to t}}^{m}u\Big(\frac{k_1}{\prod\limits_{j=1}^{t}q^{r_j}},\frac{k_2}{\prod\limits_{j=1}^{t}q^{r_j}},\frac{k_3}{\prod\limits_{j=1}^{t}q^{r_j}}\Big) \tag{4}$$

Proof: From definition (2.1), we have

$$\Delta_q v(k_1,k_2,k_3)=v(qk_1,qk_2,qk_3)-v(k_1,k_2,k_3)$$

By taking $\Delta_q v(k_1,k_2,k_3)=u(k_1,k_2,k_3)$, we obtain

$$v(qk_1,qk_2,qk_3)=u(k_1,k_2,k_3)+v(k_1,k_2,k_3)$$

Replacing k_1 by $\frac{k_1}{q}$, k_2 by $\frac{k_2}{q}$ and k_3 by $\frac{k_3}{q}$, we get

$$v(k_1,k_2,k_3)=u\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big)+v\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big) \tag{5}$$

Again replacing k_1 by $\frac{k_1}{q}$, k_2 by $\frac{k_2}{q}$ and k_3 by $\frac{k_3}{q}$, we get

$$v\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big)=u\Big(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\Big)+v\Big(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\Big) \tag{6}$$

Substituting (6) in (5) and using $v(k_1,k_2,k_3)=\Delta_q^{-1}u(k_1,k_2,k_3)$, we find

$$\Delta_q^{-1}u(k_1,k_2,k_3)=u\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big)+u\Big(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\Big)+v\Big(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\Big) \tag{7}$$

Again replacing k_1 by $\frac{k_1}{q}$, k_2 by $\frac{k_2}{q}$, k_3 by $\frac{k_3}{q}$ in (6) and putting in (7),

$$\Delta_q^{-1}u(k_1,k_2,k_3)=u\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big)+u\Big(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\Big)+u\Big(\frac{k_1}{q^3},\frac{k_2}{q^3},\frac{k_3}{q^3}\Big)+v\Big(\frac{k_1}{q^3},\frac{k_2}{q^3},\frac{k_3}{q^3}\Big)$$

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Continuing this process, we get

$$\Delta_q^{-1}u(k_1,k_2,k_3)=u\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big)+\cdots+u\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big)+v\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big)$$

Putting $\Delta_q^{-1}u\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big)=v\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big)$, we get

$$\Delta_q^{-1}u(k_1,k_3,k_3)-\Delta_q^{-1}u\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big)=u\Big(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\Big)+\cdots+u\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big),$$

which gives

$$\Delta_q^{-1}u(k_1,k_2,k_3)\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{r=1}^{m}u\Big(\frac{k_1}{q^r},\frac{k_2}{q^r},\frac{k_3}{q^r}\Big)$$

Operating Δ_q^{-1} on the above equation , we get

$$\Delta_q^{-2}u(k_1,k_2,k_3)\Big\|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{r_1=1}^{n}\sum_{r_2=1}^{n}u\Big(\frac{k_1}{q^{r_1+r_2}},\frac{k_2}{q^{r_1+r_2}},\frac{k_3}{q^{r_1+r_2}}\Big)\tag{8}$$

Again operating Δ_q^{-1} on (8), we get

$$\Delta_q^{-3}u(k_1,k_2,k_3)\Big\|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{r_1=1}^{n}\sum_{r_2=1}^{n}\sum_{r_3=1}^{n}u\Big(\frac{k_1}{q^{r_1+r_2+r_3}},\frac{k_2}{q^{r_1+r_2+r_3}},\frac{k_3}{q^{r_1+r_2+r_3}}\Big)$$

Continuing this process we get,

$$\Delta_q^{-t}u(k_1,k_2,k_3)\Big\|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{\substack{r_i=1\\ i:1\to t}}^{n}u\Big(\frac{k_1}{\prod_{j=1}^{t}q^{r_j}},\frac{k_2}{\prod_{j=1}^{t}q^{r_j}},\frac{k_3}{\prod_{j=1}^{t}q^{r_j}}\Big)$$

Corollary 3.2 Let $t\in N(1), u(k_1,k_2,k_3)=k_1{\cdot}k_2{\cdot}k_3$ and $(k_1,k_2,k_3)\in R^3$.Then

$$\Delta_q^{-t}(k_1{\cdot}k_2{\cdot}k_3)\Big\|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{\substack{r_i=1\\ i:1\to t}}^{n}\Big(\frac{k_1{\cdot}k_2{\cdot}k_3}{\prod_{j=1}^{t}q^{3r_j}}\Big)\tag{9}$$

Proof: The proof follows by replacing $u(k_1,k_2,k_3)$ by $k_1{\cdot}k_2{\cdot}k_3$ in equation (4).

Theorem 3.3 Let $n \in N(1)$ and $u(k_1, k_2, k_3) = k_1{\cdot}k_2{\cdot}k_3$.Then we have

$$\Delta_q^{-t}k_1{\cdot}k_2{\cdot}k_3\Big|\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} = \frac{1}{(q^3-1)^t}\Big(1-\frac{1}{q^{3m}}\Big)^{t-1}k_1{\cdot}k_2{\cdot}k_3\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} \tag{10}$$

By definition (2), we write

$$\Delta_q(k_1{\cdot}k_2{\cdot}k_3) = (qk_1{\cdot}qk_2{\cdot}qk_3) - (k_1{\cdot}k_2{\cdot}k_3),$$

which yields

$$\Delta_q^{-1}(k_1{\cdot}k_2{\cdot}k_3) = \frac{1}{(q^3-1)}k_1{\cdot}k_2{\cdot}k_3$$

By applying limits for the above equation, we find

$$\begin{aligned}\Delta_q^{-1}(k_1{\cdot}k_2{\cdot}k_3)\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} &= \frac{1}{(q^3-1)}k_1{\cdot}k_2{\cdot}k_3\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}\\ &= \frac{1}{(q^3-1)}\Big(1-\frac{1}{q^{3m}}\Big)k_1{\cdot}k_2{\cdot}k_3\end{aligned}$$

Operating Δ_q^{-1} on both sides, we obtain

$$\begin{aligned}\Delta_q^{-2}(k_1{\cdot}k_2{\cdot}k_3)\Big|\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} &= \Delta_q^{-1}\Big[\frac{1}{(q^3-1)}\Big(1-\frac{1}{q^{3m}}\Big)k_1{\cdot}k_2{\cdot}k_3\Big]\\ &= \frac{1}{(q^3-1)^2}\Big(1-\frac{1}{q^{3m}}\Big)^2k_1{\cdot}k_2{\cdot}k_3\end{aligned}$$

Continuing this process, we get

$$\Delta_q^{-t}k_1{\cdot}k_2{\cdot}k_3\Big|\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} = \frac{1}{(q^3-1)^t}\Big(1-\frac{1}{q^{3m}}\Big)^{t-1}k_1{\cdot}k_2{\cdot}k_3\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}$$

Theorem 3.4 If $u(k_1, k_2, k_3) = k_1{\cdot}k_2{\cdot}k_3$, $(k_1, k_2, k_3){\in}R^3$ and $q \neq 1$, then the numerical and complete solution of the q-difference equation (1) is given by

$$\sum_{\substack{r_i=1\\ i:1\to t}}^{n}\left(\frac{k_1{\cdot}k_2{\cdot}k_3}{\prod\limits_{j=1}^{t} q^{3r_j}}\right) = \frac{1}{(q^3-1)^t}\Big(1-\frac{1}{q^{3m}}\Big)^{t-1}k_1{\cdot}k_2{\cdot}k_3\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} \tag{11}$$

Proof: The proof follows by equating (9) and (10). The following example illustrates the above theorem.

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Example 3.5 Taking $t = 2$, $k_1 = \sin 2$, $k_2 = \sin 3$, $k_3 = \sin 4$, $q = 3$, $m = 2, n = 2$ in equation (11), we have

$$\sum_{\substack{r_i=1\\ i:1\to 2}}^{2} \left(\frac{k_1 \cdot k_2 \cdot k_3}{\prod\limits_{j=1}^{2} q^{3r_j}} \right) = \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} \Big[\frac{k_1}{q^{3(r_1+r_2)}} . \frac{k_2}{q^{3(r_1+r_2)}} . \frac{k_3}{q^{3(r_1+r_2)}} \Big]$$

$$= (\sin 2 \cdot \sin 3 \cdot \sin 4) \Big[\frac{1}{3^6} + \frac{1}{3^9} + \frac{1}{3^9} + \frac{1}{3^{12}} \Big] = 1.370610004 \times 10^{-07}$$

and

$$\frac{1}{(q^{3-1})^2} (1 - \frac{1}{q^{3m}})^2 k_1 k_2 k_3 = \frac{1}{(3^3-1)^2} \Big(1 - \frac{1}{3^6} \Big)^2 \times \sin 2 \cdot \sin 3 \cdot \sin 4$$
$$= 1.370610004 \times 10^{-07}$$

4. Product formula for three-dimensional q-difference equation

In this section, we present product formula for three-dimensional q-difference equation and suitable examples are provided to illustrate the result.

Theorem 4.1 Let $u(k_1, k_2, k_3)$ and $v(k_1, k_2, k_3)$ be two real valued functions defined on $[0, \infty)$. Then

$$\Delta_q^{-1}[u(k_1,k_2,k_3)v(k_1,k_2,k_3)] \Big|_{\frac{k_1}{q^m}, \frac{k_2}{q^m}, \frac{k_3}{q^m}}^{k_1,k_2,k_3} = [u(k_1,k_2,k_3)\Delta_q^{-1} v(k_1,k_2,k_3)$$
$$- \Delta_q^{-1}(\Delta_q u(k_1,k_2,k_3) \Delta_q^{-1} v(k_1,k_2,k_3))] \Big|_{\frac{k_1}{q^m}, \frac{k_2}{q^m}, \frac{k_3}{q^m}}^{k_1,k_2,k_3} \qquad (12)$$

Proof: By definition, we have

$$\Delta_q[u(k_1,k_2,k_3)w(k_1,k_2,k_3)] = \Delta_q u(k_1,k_2,k_3) w(qk_1,qk_2,qk_3)$$
$$+ \Delta_q w(k_1,k_2,k_3) u(k_1,k_2,k_3) \qquad (13)$$

Taking $\Delta_q w(k_1,k_2,k_3) = v(k_1,k_2,k_3)$, (13) becomes

$$\Delta_q^{-1}[u(k_1,k_2,k_3)v(k_1,k_2,k_3)] \Big|_{\frac{k_1}{q^m}, \frac{k_2}{q^m}, \frac{k_3}{q^m}}^{k_1,k_2,k_3} = \big\{ [u(k_1,k_2,k_3)\Delta_q^{-1} v(k_1,k_2,k_3)]$$
$$- \Delta_q^{-1}[\Delta_q u(k_1,k_2,k_3) \Delta_q^{-1} v(k_1,k_2,k_3)] \big\} \Big|_{\frac{k_1}{q^m}, \frac{k_2}{q^m}, \frac{k_3}{q^m}}^{k_1,k_2,k_3}$$

The following corollary gives formula for finite series involving logarithmic function:

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Corollary 4.2 For a real valued function $v(k_1,k_2,k_3)$ and $(k_1,k_2,k_3)\in R^3$,

$$\Delta_q^{-1}[\log(k_1\cdot k_2\cdot k_3)v(k_1,k_2,k_3)]\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\big\{[\log(k_1.k_2.k_3)\Delta_q^{-1}v(k_1,k_2,k_3)] -[3\log q\Delta_q^{-2}v(qk_1,qk_2,qk_3)]\big\}\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} \qquad (14)$$

$$\Delta_q^{-t}[\log(k_1\cdot k_2\cdot k_3)v(k_1,k_2,k_3)]\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\big\{[\log(k_1\cdot k_2\cdot k_3)\Delta_q^{-t}v(k_1,k_2,k_3)] -[t\times 3\log q\Delta_q^{-(t+1)}v(qk_1,qk_2,qk_3)]\big\}\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} \qquad (15)$$

and hence

$$\sum_{\substack{r_i=1\\ i:1\to t}}^{n}\left[v\Big(\frac{k_1}{\prod\limits_{j=1}^{t}q^{r_j}},\frac{k_2}{\prod\limits_{j=1}^{t}q^{r_j}},\frac{k_3}{\prod\limits_{j=1}^{t}q^{r_j}}\Big)\log\Big(\frac{k_1\cdot k_2\cdot k_3}{\prod\limits_{j=1}^{t}q^{3r_j}}\Big)\right]=\big\{[\log(k_1\cdot k_2\cdot k_3)\Delta_q^{-t}v(k_1,k_2,k_3)] -[t\times 3\log q\Delta_q^{-(t+1)}v(qk_1,qk_2,qk_3)]\big\}\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3} \qquad (16)$$

Proof: Equation (14) follows by taking $u(k_1,k_2,k_3)=\log(k_1\cdot k_2\cdot k_3)$ in (12) Now by applying Δ_q^{-1} on equation (14) and using equation (12), we get

$$\Delta_q^{-2}[\log(k_1\cdot k_2\cdot k_3)v(k_1,k_2,k_3)]\Big|\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\big\{\Delta_q^{-1}[\log(k_1\cdot k_2\cdot k_3)\Delta_q^{-1}v(k_1,k_2,k_3)] -\Delta_q^{-1}[3\log q\Delta_q^{-2}v(qk_1,qk_2,qk_3)]\big\}\Big|_{\frac{k_1}{q^m},\frac{k_2}{qm},\frac{k_3}{q^m}}^{k_1,k_2,k_3}$$

By repeating this process we get the proof of (15).
Replacing $u(k_1,k_2,k_3)$ by $\log(k_1\cdot k_2\cdot k_3)$ in (3), we get

$$\Delta_q^{-1}\log(k_1\cdot k_2\cdot k_3)\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}=\sum_{r=1}^{n}\log\Big(\frac{k_1\cdot k_2\cdot k_3}{q^{3r}}\Big)$$

Substituting these values in (14),we get

$$\sum_{r=1}^{n}\left[v\Big(\frac{k_1}{q^{3r}},\frac{k_2}{q^{3r}},\frac{k_3}{q^{3r}}\Big)\log\Big(\frac{k_1\cdot k_2\cdot k_3}{q^{3r}}\Big)\right]=\big\{[\log(k_1\cdot k_2\cdot k_3)\Delta_q^{-2}v(k_1,k_2,k_3)] -[2\times 3\log q\Delta_q^{-3}v(qk_1,qk_2,qk_3)]\big\}\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}$$

Contiuning this process

$$\sum_{\substack{r_i=1\\ i:1\to t}}^{n}\left[v\Big(\frac{k_1}{\prod\limits_{j=1}^{t}q^{r_j}},\frac{k_2}{\prod\limits_{j=1}^{t}q^{r_j}},\frac{k_3}{\prod\limits_{j=1}^{t}q^{r_j}}\Big)\log\Bigg(\frac{k_1{\cdot}k_2{\cdot}k_3}{\prod\limits_{j=1}^{t}q^{3r_j}}\Bigg)\right]=\Big\{[\log(k_1{\cdot}k_2{\cdot}k_3)\Delta_q^{-t}v(k_1,k_2,k_3)]$$

$$-[t{\times}3\log q\Delta_q^{-(t+1)}v(qk_1,qk_2,qk_3)]\Big\}\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}$$

Corollary 4.3 Taking $v(k_1,k_2,k_3)=k_1{\cdot}k_2{\cdot}k_3$, $1\neq q\in(0,\infty)$, we have from (10),

$$\begin{aligned}\Delta_q^{-t}k_1{\cdot}k_2{\cdot}k_3\Big\|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}&=\frac{1}{(q^3-1)^t}(1-\frac{1}{q^{3m}})^{t-1}k_1{\cdot}k_2{\cdot}k_3\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}\\ \Delta_q^{-t+1}k_1{\cdot}k_2{\cdot}k_3\Big\|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}&=\frac{1}{(q^3-1)^{t+1}}(1-\frac{1}{q^{3m}})^{t}k_1{\cdot}k_2{\cdot}k_3\Big|_{\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}}^{k_1,k_2,k_3}\end{aligned}$$

Substitute these values in (16),we get

$$\sum_{\substack{r_i=1\\ i:1\to t}}^{n}\left[\left(\frac{k_1{\cdot}k_2{\cdot}k_3}{\prod\limits_{j=1}^{t}q^{3r_j}}\right)\log\left(\frac{k_1{\cdot}k_2{\cdot}k_3}{\prod\limits_{j=1}^{t}q^{3r_j}}\right)\right]=\left\{\left[\frac{\log(k_1{\cdot}k_2{\cdot}k_3)}{(q^3-1)^t}(1-\frac{1}{q^{3m}})^{t-1}k_1{\cdot}k_2{\cdot}k_3\right]\right.$$

$$\left.-\left[\frac{t{\times}3\log q{\times}q^3}{(q^3-1)^{t+1}}(1-\frac{1}{q^{3m}})^{t}k_1{\cdot}k_2{\cdot}k_3\right]\right\}\quad(17)$$

An example verified by MATLAB is given below to illustrate Corollary (4.3):

Example 4.4 Taking $t=3, k_=26, k_2=37, k_3=66, q=4, m=3$ and $n=3$ in (17), we get

$$\sum_{\substack{r_i=1\\ i:1\to 3}}^{3}\left[\Big(\frac{26.37.66}{\prod\limits_{j=1}^{3}q^{3r_j}}\Big)\log\Big(\frac{26.37.66}{\prod\limits_{j=1}^{3}q^{3r_j}}\Big)\right]=\sum_{r_1=1}^{3}\sum_{r_2=1}^{3}\sum_{r_3=1}^{3}\Big[\Big(\frac{63492}{4^{3r_1}.4^{3r_2}.4^{3r_3}}\Big)$$

$$-\log\Big(\frac{63492}{4^{3r_1}.4^{3r_2}.4^{3r_3}}\Big)\Big]=-0.017819$$

and

$$\Big\{\Big[\frac{\log(k_1{\cdot}k_2{\cdot}k_3)}{(4^3-1)^3}\Big(1-\frac{1}{4^9}\Big)^2 k_1{\cdot}k_2{\cdot}k_3 - \frac{3{\times}4^3{\times}3\log(4)}{(4^{3-1})^4}$$

$$\Big(1-\frac{1}{4^9}\Big)^3 k_1{\cdot}k_2{\cdot}k_3\Big]\Big\}\Big|_{0.40625,0.578125,1.03125}^{26,37.66} = -0.017819$$

5. Conclusion

In this research work, we define three-dimensional q-difference operator. The closed form solution found in this work agreed very well with the numerical solution of the three-dimensional q-difference equation. Also we dreive product formula of polynomial and logarithmic function using three-dimensional q-difference operator. Moreover necessary examples are provided to illustrate our finidings.

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