



## Fibonacci Sequence and Series by Second Order Difference Operator with Logarithmic Function

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### Abstract

In this paper, we introduce second order difference operator with logarithmic function and its inverse by which we obtain Fibonacci sequence and its sum. Some theorems and interesting results on the sum of the terms of advanced Fibonacci sequence are derived. Suitable examples are provided to illustrate our results and verified by using MATLAB.

**Key words:** Difference operator, Logarithmic function, Fibonacci sequence, Closed form solution.

**AMS classification:** 39A70, 39A10, 47B39, 65J10, 65Q10.

### 1. Introduction

In 2011, M.Maria Susai Manuel, et.al, [7] introduced the generalized  $\alpha$ -difference operator as  $\Delta_{\alpha(\ell)} v(k) = v(k + \ell) - \alpha v(k)$  for the real valued function  $v(k)$ . In 2014, G.Britto Antony Xavier, et.al, [2] introduced  $q$ -difference operator as  $\Delta_q v(k) = v(qk) - v(k)$ ,  $q \in (0, \infty)$  and obtained finite series solution to the corresponding generalized  $q$ -difference equation  $\Delta_q v(k) = u(k)$ . With this background, in this paper, we obtain advanced Fibonacci sequence and its sum by introducing second order difference operator with logarithmic function.

### 2. Advanced Fibonacci Sequence and its Sum Obtained by Second Order Difference Equation With Logarithmic Function

Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [6] and Vajda [10]. The  $k$ -Fibonacci sequence introduced by Falcon and Plaza [3] depends only on one integer parameter  $k$  and is defined as

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$F_{k,0} = 0, F_{k,1} = 1$  and  $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ , where  $n \geq 1, k \geq 1$ .  
 In particular, if  $k = 2$ , the Pell sequence is obtained as

$$P_0 = 0, P_1 = 1 \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1.$$

Here we introduce second order difference operator with logarithmic function  $\Delta_{\log k_\alpha} v(k) = v(k) - \alpha_1 \log(k_1)v(k-1) - \alpha_2 \log(k_2)v(k-2)$  which generates advanced Fibonacci sequence and its sum.

**Definition 2.1** A advanced Fibonacci sequence is defined as  $F_0 = 1$ ,

$$F_1 = \alpha_1 \log k_1, F_n = \alpha_1 \log(k_1 - (n-1))F_{n-1} + \alpha_2 \log(k_2 - (n-2))F_{n-2}, n \geq 2 \quad (1)$$

**Example 2.2** Taking  $k = 6, n = 2, \alpha_1 = 2, \alpha_2 = 3, k_1 = 3$  and  $k_2 = 4$  in (1), we get a Fibonacci sequence with logarithmic function  $\{1, 2.20, 7.20, 7.24, \dots\}$ .

Similarly, one can obtain Fibonacci sequences corresponding to each pair  $(\alpha_1 \log k_1, \alpha_2 \log k_2) \in \mathbb{R}^2$ .

**Definition 2.3** A second order difference operator with logarithmic function on  $v(k), k > n + 2$ , denoted as  $\Delta_{\log k} v(k)$ , is defined as

$$\Delta_{\log k_\alpha} v(k) = v(k) - \alpha_1 \log k_1 v(k-1) - \alpha_2 \log k_2 v(k-2), \quad k \in [0, \infty), \quad (2)$$

and its inverse is defined as below;

$$\text{if } \Delta_{\log k_\alpha} v(k) = u(k), \text{ then we write } v(k) = \Delta_{\log k}^{-1} u(k). \quad (3)$$

**Lemma 2.4** Let  $v(k), k > n + 2$  be a functions of  $k \in (-\infty, \infty)$ . Then we obtain

$$\Delta_{\log k_\alpha}^{-1} a^{sk} \left[ 1 - \frac{\alpha_1 \log k_1}{a^s} - \frac{\alpha_2 \log k_2}{a^{2s}} \right] = a^{sk}. \quad (4)$$

Proof: Taking  $u(k) = a^{sk} \left[ 1 - \frac{\alpha_1 \log k_1}{a^s} - \frac{\alpha_2 \log k_2}{a^{2s}} \right]$  in (2), we obtain

$$\Delta_{\log k_\alpha} a^{sk} = a^{sk} \left[ 1 - \frac{\alpha_1 \log k_1}{a^s} - \frac{\alpha_2 \log k_2}{a^{2s}} \right]. \text{ Now (4) follows from (3).}$$

**Lemma 2.5** Let  $e^{-sk}, k > n + 2$  be a function of  $k \in (-\infty, \infty)$ . Then we have

$$\Delta_{\log k_\alpha}^{-1} e^{-sk} \left[ 1 - \alpha_1 e^s \log k_1 - \alpha_2 e^{2s} \log k_2 \right] = e^{-sk}. \quad (5)$$

Proof: The proof follows by assuming  $a = e^{-1}$  in (4).

**Lemma 2.6** Let  $k^k$  be a function of  $k \in (-\infty, \infty)$ . Then we have

$$\Delta_{\log k_\alpha}^{-1} [k^k - \alpha_1(k-1)^{k-1} \log k_1 - \alpha_2(k-2)^{k-2} \log k_2] = k^k. \quad (6)$$

Proof: Taking  $v(k) = k^k$  in (2) and using (3).

**Lemma 2.7** If  $v(k) = \Delta_{\log k_\alpha}^{-1} u(k)$ ,  $F_0 = 1$ ,  $F_1 = \alpha_1 \log k_1$  and  $F_{n+1} = F_n \alpha_1 \log(k_1 - n) + F_{n-1} \alpha_2 \log(k_2 - (n-1))$  for  $i = 0, 1, 2, \dots$  then

$$v(k) - F_{n+1}v(k - (n+1)) - F_n \alpha_2 \log(k_2 - n)v(k - (n+2)) = \sum_{i=0}^n F_i u(k - i). \quad (7)$$

Proof: From (2) and (3), we arrive

$$v(k) = u(k) + \alpha_1 \log k_1 v(k-1) + \alpha_2 \log k_2 v(k-2). \quad (8)$$

Replacing  $k$  by  $k-1$  and then substituting the value of  $v(k-1)$  in (8), we get  $v(k) = u(k) + F_1 u(k-1) + (F_1 \alpha_1 \log(k_1 - 1) + \alpha_2 \log k_2)v(k-2) + F_1 \alpha_2 \log(k_2 - 1)v(k-3)$  which gives

$$v(k) = F_0 u(k) + F_1 u(k-1) + F_2 v(k-2) + F_1 \alpha_2 \log(k_2 - 1)v(k-3), \quad (9)$$

where  $F_0$ ,  $F_1$  and  $F_2$  are given in (1).

Replacing  $k$  by  $k-2$  in (8) and then substituting  $v(k-2)$  in (9), we obtain  $v(k) = F_0 u(k) + F_1 u(k-1) + F_2 u(k-2) + F_3 v(k-3) + F_2 \alpha_2 \log(k_2 - 2)v(k-4)$ , where  $F_3$  is given in (1).

Repeating this process again and again, we get (7).

**Corollary 2.8** If  $v(k)$ ,  $k > n+2$  is a closed form solution of second order difference equation with logarithmic function  $\Delta_{\log k_\alpha} v(k) = a^{sk} [1 - \frac{\alpha_1 \log k_1}{a^s} - \frac{\alpha_2 \log k_2}{a^{2s}}]$ , then we obtain  $a^{sk} - F_{n+1} a^{s(k-(n+1))} - F_n \alpha_2 \log(k_2 - n) a^{s(k-(n+2))}$

$$= \sum_{i=0}^n F_i a^{s(k-i)} \left[ 1 - \frac{\alpha_1 \log(k_1 - i)}{a^s} - \frac{\alpha_2 \log(k_2 - i)}{a^{2s}} \right]. \quad (10)$$

Proof: The proof follows by applying  $v(k) = a^{sk}$  in (7) and using (4).

The following example is an verification of (10).

**Example 2.9** Taking  $k = 6, n = 2, s = 2, \alpha_1 = 2, \alpha_2 = 3, a = 2, k_1 = 3$  and  $k_2 = 4$  in (10), we get

$$2^{12} - F_3 2^6 - 3F_2 \log(2) 2^4 = \sum_{i=0}^2 F_i 2^{2(6-i)} \left[ 1 - \frac{2 \log(6-i)}{2^2} - \frac{4 \log(6-i)}{2^4} \right] = 524948.25$$

where  $F_0 = 1, F_1 = 2.20, F_2 = 7.20$  and  $F_3 = 7.24$ .

**Corollary 2.10** Let  $e^{-sk}, k > n + 2$  be a function of  $k \in (-\infty, \infty)$ . Then  $e^{-sk} - F_{n+1} e^{-s(k-(n+1))} - \alpha_2 \log(k_2 - n) F_n e^{-s(k-(n+2))}$

$$= \sum_{i=0}^n F_i e^{-s(k-i)} \left[ 1 - \alpha_1 \log(k_1 - i) e - \alpha_2 \log(k_2 - i) e^{2s} \right]. \quad (11)$$

Proof: Taking  $v(k) = e^{-sk}$  and applying (5) in (7), we get (11).

**Example 2.11** Taking  $k = 5, n = 2, s = 3, \alpha_1 = 3, \alpha_2 = 2, a = 1.8, k_1 = 11$  and  $k_2 = 21$  in (11), then we obtain

$$e^{-15} - F_3 e^{-6} - 2F_2 \log(19) e^{-3} = \sum_{i=0}^2 F_i e^{-3(5-i)} \left[ 1 - 3 \log(11 - i) e^3 - 2 \log(21 - i) e^6 \right] = -66.5$$

where  $F_0 = 1, F_1 = 9.13, F_2 = 86.88$  and  $F_3 = 809.51$ .

**Lemma 2.12** Let  $t \in \mathbb{N}(0), k > n + 2$ . Then a closed form solution of the second order difference equation logarithmic function

$$v(k) - \sum_{p=1}^2 \alpha_p \log k_p v(k-p) = \left[ k^t - \sum_{p=1}^2 \alpha_p \log k_p (k-p)^t \right] \text{ is}$$

$$\Delta_{\log k_\alpha}^{-1} \left[ k^t - \alpha_1 \log k_1 (k-1)^t - \alpha_2 \log k_2 (k-2)^t \right] = k^t \quad (12)$$

Proof: Taking  $v(k) = k^t$  in (2) and using (3), we get (12).

**Corollary 2.13** If  $v(k) = \Delta_{\log k_\alpha}^{-1} \left[ k^t - \sum_{p=1}^2 \alpha_p \log k_p (k-p)^t \right], k > n + 2$  is the closed form solution given in (12), then  $k^t - F_{n+1} (k-n-1)^t - F_n \alpha_2 \log(k_2 - n) (k-n-2)^t$

$$= \sum_{i=0}^n F_i \left[ (k-i)^t - \alpha_1 \log(k_1 - i) [k - (i+1)]^t - \alpha_2 \log(k_2 - i) [k - (i+2)]^t \right]. \quad (13)$$

Proof: The proof follows by taking  $u(k) = k^t - \sum_{p=1}^2 \alpha_p \log k_p (k-p)^t$  in (7).

**Example 2.14** Let  $k = 1011$ ,  $n = 3$ ,  $t = 4$ ,  $k_1 = 21$ ,  $k_2 = 9$ ,  $\alpha_1 = 5$ ,  $\alpha_2 = 3$  in Corollary (2.13). Then

$\sum_{i=0}^3 F_i u(1011 - i) = v(1011) - F_4 v(1007) - 3 \log(6) F_3 v(1006) = -10438211162429716$ ,  
 where  $u(k) = k^t - \alpha_1 \log k_1 (k-1)^t - \alpha_2 \log k_2 (k-2)^t$ ,  $F_0 = 1$ ,  $F_1 = 9.13$ ,  $F_2 = 86.48$ ,  
 $F_3 = 801.89$  and  $F_4 = 7289.82$ .

**Corollary 2.15** Let  $v(k)$ ,  $k > n+2$  be a solution of second order difference equation with logarithmic functions

$$v(k) - \sum_{p=1}^2 \alpha_p \log k_p v(k-p) = \left[ \frac{k^t}{\sqrt{\log(bk)}} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \frac{(k-p)^t}{\sqrt{\log(b(k-p))}} \right].$$

Then we have

$$\begin{aligned} & \frac{k^t}{\sqrt{\log(bk)}} - F_{n+1} \frac{(k-n-1)^t}{\sqrt{\log(b(k-n-1))}} - \alpha_2 \log(k_2 - n) \frac{(k-n-2)^t}{\sqrt{\log(b(k-n-2))}} \\ & = \sum_{i=0}^n F_i \left[ \frac{(k-i)^t}{\sqrt{\log(b(k-i))}} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \frac{(k-i-p)^t}{\sqrt{\log(b(k-i-p))}} \right]. \end{aligned} \quad (14)$$

Proof: The proof follows by taking  $v(k) = \frac{k^t}{\sqrt{\log(bk)}}$  in the Theorem 2.7.

**Example 2.16** Let  $k = 14$ ,  $n = 3$ ,  $b = 10$ ,  $t = 7$ ,  $\alpha_1 = 15$ ,  $\alpha_2 = 19$ ,  $k_1 = 29$ ,  $k_2 = 32$  in Corollary 2.28. Then we obtain

$$\begin{aligned} & \frac{21^4}{\sqrt{\log(231)}} - F_4 \frac{(17)^4}{\sqrt{\log(187)}} - 10.5 \log(k_2 - 3) \frac{(16)^4}{\sqrt{\log(176)}} \\ & = \sum_{i=0}^3 F_i \left[ \frac{(21-i)^4}{\sqrt{\log(11(21-i))}} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \frac{(21-i-p)^4}{\sqrt{\log(11(21-i-p))}} \right] \\ & = -226630264497941.12, \end{aligned}$$

where  $F_0 = 1$ ,  $F_1 = 50.51$ ,  $F_2 = 2590.47$ ,  $F_3 = 131361.81$  and  $F_4 = 6587244.76$ .

**Lemma 2.17** If  $v(k)$ ,  $k > n+2$  is a closed form solution of the second order difference equation with logarithmic functions

$$v(k) - \sum_{p=1}^2 \alpha_p \log k_p v(k-p) = k^t \sin(sk) - \sum_{p=1}^2 \alpha_p \log(k_p - p) (k-p)^t \sin(s(k-p)),$$

then

$$k^t \sin(sk) - F_{n+1}(k-n-1)^t \sin(s(k-n-1)) - \alpha_2 \log(k_2-n)(k-n-2)^t \sin(s(k-n-2)) = \sum_{i=0}^n F_i \left[ (k-i)^t \sin(s(k-i)) - \sum_{p=1}^2 \alpha_p \log(k_p-p)(k-i-p)^t \sin(s(k-i-p)) \right]. \quad (15)$$

Proof: Taking  $u(k) = \left[ k^t \sin(sk) - \sum_{p=1}^2 \alpha_p \log(k_p-1)(k-p)^t \sin(s(k-p)) \right]$  in Theorem 2.7 and using (4), we get 15.

**Corollary 2.18** A closed form solution of the logarithmic second order difference equation logarithmic function  $\Delta_{\log k_\alpha} v(k) = k^4 \sin(sk) - \sum_{p=1}^2 \alpha_p \log k_p (k-p)^4 \sin(s(k-p))$  and hence we have

$$k^4 \sin sk - F_{n+1}(k-(n+1))^4 \sin(s(k-(n+1))) - F_n \alpha_2 \log(k_2-n)(k-(n+2))^4 \times \sin(s(k-(n+2))) = \sum_{i=0}^n F_i \left[ (k-i)^4 - \sum_{p=1}^2 \alpha_p \log(k_p-p)(k-(i+p))^4 \right] \sin s(k-p-i). \quad (16)$$

Proof: The proof follows by taking  $t = 4$  in Theorem 2.17.

**Example 2.19** Let  $k = 9, a = 3, n = 3, \alpha_1 = 2, s = 2, \alpha_2 = 3, k_1 = 29.55, k_2 = 15.9$  in Corollary (2.18). We have

$$\begin{aligned} 9^4 \sin 18 - F_4(5)^4 \sin(10) - 3 \log(12.9)(4)^4 \sin(8) \\ = \sum_{i=0}^3 F_i \left[ (9-i)^4 - \sum_{p=1}^2 \alpha_p \log(k_p-p)(9-(i+p))^4 \right] \sin s(9-p-i) \\ = 2421452716882.48, \end{aligned}$$

where  $F_0 = 1, F_1 = 6.77, F_2 = 53.69, F_3 = 410.99$  and  $F_4 = 3119.24$ .

**Corollary 2.20** A closed form solution of the second order difference equation with logarithmic functions

$v(k) - \sum_{p=1}^2 \alpha_p \log(k_p-p)v(k-p) = k^t \tan(bk) - \sum_{p=1}^2 \alpha_p \log(k_p-p)(k-p)^t \tan b(k-p)$  is given by

$$\begin{aligned} k^t \tan(bk) - F_{n+1}(k-(n+1))^t \tan(b(k-(n+1))) - \\ \alpha_2 F_n \log(k_2-n)(k-(n+2))^t \tan(b(k-(n+2))) \\ = \sum_{i=0}^n F_i \left[ (k-i)^t \tan(b(k-i)) - \sum_{p=1}^2 \alpha_p \log(k_p-p)(k-i-p)^t \tan(b(k-i-p)) \right]. \quad (17) \end{aligned}$$

Proof: Taking  $v(k) = k^t \tan(bk)$  in Theorem 2.7, we get (17).

**Example 2.21** Let  $k = 2.0091$ ,  $n = 3$ ,  $t = 5$ ,  $a = 4$ ,  $b = 5$ ,  $\alpha_1 = 3$ ,  $\alpha_2 = 2$ ,  $k_1 = 34.85$  and  $k_2 = 12.5$  in Corollary (2.18). We have

$$(2.0091)^5 \tan(10.0455) - F_4(-1.9909)^5 \tan(-9.955) - 2F_3 \log(9.5)(-2.9909)^5 \tan(-14.9545) \\ = \sum_{i=0}^3 F_i [(2.0091 - i)^5 \tan(5(2.0091 - i)) - \sum_{p=1}^2 \alpha_p \log(k_p - p)(2.0091 - i - p)^5 \times \\ \times \tan(5(2.0091 - i - p))] = 1043823.96,$$

where  $F_0 = 1$ ,  $F_1 = 10.65$ ,  $F_2 = 117.61$ ,  $F_3 = 1284.11$  and  $F_4 = 13886.16$ .

**Corollary 2.22** A closed form solution of the second order difference equation with logarithmic functions

$$v(k) - \sum_{p=1}^2 \alpha_p \log(k_p - p)v(k - p) = e^{-sk} \log(bk) - \sum_{p=1}^2 \alpha_p \log(k_p - p)e^{s(k-p)} \log(b(k - p))$$

is given by

$$e^{sk} \log(bk) - F_{n+1} e^{s(k-n-1)} \log(b(k - n - 1)) - \alpha_2 F_n \log(k_2 - n) e^{s(k-n-2)} \log(b(k - n - 2)) \\ = \sum_{i=0}^n F_i e^{s(k-i)} [\log(b(k - i)) - \sum_{p=1}^2 \alpha_p \log(k_p - p) e^{-sp} \log(b(k - i - p))]. \quad (18)$$

Proof: Taking  $v(k) = e^{sk} \log(bk)$  in Theorem 2.7, we get (18).

**Example 2.23** Let  $k = 7$ ,  $n = 3$ ,  $s = 1.5$ ,  $b = 2$ ,  $\alpha_1 = 15$ ,  $\alpha_2 = 10$ ,  $k_1 = 18$  and  $k_2 = 15$  in Corollary (2.18). We have

$$e^{10.5} \log(14) - F_4 e^{4.5} \log(6) - 10F_3 \log(12) e^3 \log(4) \\ = \sum_{i=0}^3 F_i e^{1.5(7-i)} [\log(2(7 - i)) - \sum_{p=1}^2 \alpha_p \log(k_p - p) e^{-1.5p} \log(2(7 - i - p))] \\ = -579152724.29, \text{ where } F_0 = 1, F_1 = 43.36, F_2 = 1869.61, F_3 = 78899.26 \\ \text{and } F_4 = 3252901.95.$$

**Lemma 2.24** Let  $v(k)$ ,  $k > n+2$  be a solution of the second order difference equation with logarithmic functions. Then we have,

$$v(k) - \sum_{i=1}^2 \alpha_i \log(k_i - i)v(k - i) = k^{(t)} \operatorname{cosech}(k) - \sum_{i=1}^2 \alpha_i \log k_i (k - i)^{(t)} \operatorname{cosech}(k - i),$$

then we have,  $k^{(t)} \operatorname{cosech}(k) - F_{n+1} (k - n - 1)^t \operatorname{cosech}(k - n - 1) -$

$$\alpha_2 \log(k_2 - n)(k - n - 2)^t \operatorname{cosech}(k - n - 2) = \sum_{i=0}^n F_i \times$$

$$\left[ (k - i)^{(t)} \operatorname{cosech}(k - i) - \sum_{p=1}^2 \alpha_p \log(k_p - p)(k - i - p)^{(t)} \operatorname{cosech}(k - i - p) \right]. \quad (19)$$

Proof: Taking  $v(k) = k^{(t)} \operatorname{cosech}(k)$  in Theorem 2.7 and using (4), we get 19.

**Example 2.25** Let  $k = 1.5$ ,  $b = 5$ ,  $n = 3$ ,  $t = 4$ ,  $\alpha_1 = 12$ ,  $\alpha_2 = 20$ ,  $k_1 = 50$ ,  $k_2 = 34$  in Theorem 2.24. We have,

$$(1.5)^{(4)} \operatorname{cosech}(1.5) - F_4 (-2.5)^{(4)} \operatorname{cosech}(-2.5) - 20 \log(31) (-3.5)^{(4)} \operatorname{cosech}(3.5)$$

$$= \sum_{i=0}^3 F_i [(1.5 - i)^{(4)} \operatorname{cosech}(1.5 - i) - \sum_{p=1}^2 \alpha_p \log(k_p - p)(1.5 - i - p)^{(4)} \operatorname{cosech}(1.5 - i - p)] = 8547.97,$$

where  $F_0 = 1$ ,  $F_1 = 46.94$ ,  $F_2 = 2262.91$ ,  $F_3 = 108405.04$  and  $F_4 = 5165357.99$ .

**Corollary 2.26** Let  $v(k)$ ,  $k > n + 2$  be a solution of second order difference equation with logarithmic functions

$$v(k) - \sum_{p=1}^2 \alpha_p \log k_p v(k - p) = [\log(\log(bk))k^{(t)} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \log(\log(b(k - p)))(k - p)^{(t)}].$$

Then we have

$$\log(\log(bk))k^{(t)} - F_{n+1} \log(\log(b(k - n - 1)))(k - n - 1)^{(t)} -$$

$$\alpha_2 \log(k_2 - n) \log(\log(b(k - n - 2)))(k - n - 2)^{(t)}$$

$$= \sum_{i=0}^n F_i [\log(\log(b(k - i)))(k - i)^{(t)} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \log(\log(b(k - i - p)))(k - i - p)^{(t)}]. \quad (20)$$

Proof: Taking  $u(k) = \log(\log(bk))k^{(t)} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \log(\log(b(k - p)))(k - p)^{(t)}$  in Theorem 2.7 and using (4), we get (20).

**Example 2.27** Let  $k = 16$ ,  $n = 3$ ,  $b = 10$ ,  $t = 4$ ,  $\alpha_1 = 15$ ,  $\alpha_2 = 19$ ,  $k_1 = 29$ ,  $k_2 = 32$  in Corollary 2.26. Then we obtain

$$\log(\log(160))16^{(4)} - F_4 \log(\log(120))(12)^{(4)} - 19 \log(k_2 - 3) \log(\log(110))(11)^{(4)}$$

$$= \sum_{i=0}^3 F_i [\log(\log(10(16 - i)))(16 - i)^{(4)} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \log(\log(10(16 - i - p)))(16 - i - p)^{(4)}]$$

$= -225566400698.78$ , where  $F_0 = 1$ ,  $F_1 = 50.51$ ,  $F_2 = 2590.47$ ,  $F_3 = 131361.81$  and  $F_4 = 6587244.76$ .

**Corollary 2.28** Let  $v(k)$ ,  $k > n + 2$  be a solution of second order difference equation with logarithmic functions

$$v(k) - \sum_{p=1}^2 \alpha_p \log k_p v(k - p) = \left[ \frac{k^{(t)}}{\log(\log(bk))} - \sum_{p=1}^2 \alpha_p \log(k_p - p) \frac{(k - p)^{(t)}}{\log(\log(b(k - p)))} \right].$$

Then we have

$$\begin{aligned} & \frac{k^{(t)}}{\log(\log(bk))} - F_{n+1} \frac{(k-n-1)^{(t)}}{\log(\log(b(k-n-1)))} - \alpha_2 \log(k_2-n) \frac{(k-n-2)^{(t)}}{\log(\log(b(k-n-2)))} \\ &= \sum_{i=0}^n F_i \left[ \frac{(k-i)^{(t)}}{\log(\log(b(k-i)))} - \sum_{p=1}^2 \alpha_p \log(k_p-p) \frac{(k-i-p)^{(t)}}{\log(\log(b(k-i-p)))} \right]. \end{aligned} \quad (21)$$

Proof: The proof follows by taking  $v(k) = \frac{k^{(t)}}{\log(\log(bk))}$  in the Theorem 2.7.

**Example 2.29** Let  $k = 21$ ,  $n = 3$ ,  $b = 11$ ,  $t = 4$ ,  $\alpha_1 = 14$ ,  $\alpha_2 = 10.5$ ,  $k_1 = 27$ ,  $k_2 = 29$  in Corollary 2.28. Then we obtain

$$\begin{aligned} & \frac{21^{(4)}}{\log(\log(231))} - F_4 \frac{(17)^{(4)}}{\log(\log(187))} - 10.5 \log(k_2-3) \frac{(16)^{(4)}}{\log(\log(176))} \\ &= \sum_{i=0}^3 F_i \left[ \frac{(21-i)^{(4)}}{\log(\log(11(21-i)))} - \sum_{p=1}^2 \alpha_p \log(k_p-p) \frac{(21-i-p)^{(4)}}{\log(\log(11(21-i-p)))} \right] \\ &= -242342850538.15, \end{aligned}$$

where  $F_0 = 1$ ,  $F_1 = 46.14$ ,  $F_2 = 2140.03$ ,  $F_3 = 98053.51$  and  $F_4 = 4436729.18$ .

### 3. Conclusion

We obtained summation formula to advanced Fibonacci sequence by introducing second order difference operator with logarithmic function and have derived certain results on closed and summation form solution of second order difference equation which will be used to our further research.

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