



## Enumeration of 5-Uniform Hypergraph Designs with Edge 6-Distant Condition

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### Abstract

A hypergraph  $\mathcal{H}$  is 6-distant, if for edges  $e_1, e_2 \in \mathcal{H}$ , the Hamming distance  $d(e_1, e_2) \geq 6$ , which is equivalent to  $|e_1 \cap e_2| \leq 2$  on the 5-uniform hypergraph. A 5-uniform hypergraph  $\mathcal{H}$  is (3,2)-bipartite, if  $V(\mathcal{H}) = S \cup T$ , and  $\forall e \in E, |e \cap S| = 3, |e \cap T| = 2$ . In this paper, we'll research on the 6-distant (3,2)-bipartite hypergraph on  $S \cup T$ , that has a complete 3-uniform induced sub-hypergraph on  $S = 1, 2, \dots, n = [n]$ . First we'll determine number  $n$  which exists a hypergraph on the minimum size of  $|T| = \frac{2\binom{n}{3}}{A(n,4,3)}$ , then compute the number of (3,2)-bipartite hypergraphs, which satisfies the 6-distant condition, for a few entries of  $n$ .

**Key words:** 5-Uniform Hypergraph; Design Enumeration; 6-Distant; Bipartite Hypergraph.

**AMS classification:** 05C65, 05C30; Secondary: 03E05, 94B25.

## 1. Introduction

**Problem 1.1** Let  $n \geq 3$  be integer, set  $S = [n]$ ,  $T = \{n + 1, n + 2, \dots, n + \lambda\}$ .

For integer  $\lambda$ , there exists a 5-uniform (3,2)-bipartite hypergraph  $\mathcal{H} = (S \cup T, E)$ , which satisfies that:

1.  $|E| = \binom{n}{3}$ ;
2.  $\forall e \in E, |e \cap S| = 3, |e \cap T| = 2$
3.  $\forall e_1, e_2 \in E, |e_1 \cap e_2| \leq 2$

For a given integer  $n$ , determine the minimum of  $\lambda$  that  $\mathcal{H}$  exists. Let  $\lambda(n)$  be the minimum.

Here  $S$  is the origin vertex set of the hypergraph,  $T$  is the attachment set.

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Since for  $e_i, e_j \in E(\mathcal{H})$ ,  $|e_i \setminus e_j| = |e_i| - |e_i \cap e_j| \geq 5 - 2 = 3$ . It means that these 5-weight binary codes which are generated by edge  $e_i$ 's whose pairwise Hamming distances are at least 6. So for  $(3, 2)$ -bipartite hypergraph  $\mathcal{H}$ , we say it 6-distant if  $\mathcal{H}$  satisfies the three conditions above. Obviously for  $\lambda \geq 2 \cdot \binom{n}{3}$  with distinct  $e_1 \cap S$  and  $e_1 \cap e_2 \cap T = \emptyset$ ,  $\mathcal{H}$  is 6-distant. So  $\exists \lambda = 2 \cdot \binom{n}{3}$ . We concentrate on the minimum of  $\lambda$ .

For  $n \geq 3$ , let  $A(n, 4, 3)$  be the coding-theoretic function, i.e., the maximum size of binary code of length  $n$ , constant weight 3 and minimal distance 4. It equals to the maximum  $m$  that exists a set of triples  $\{A_1, A_2, \dots, A_m\} \subset \binom{[n]}{3}$ , such that  $\forall i \neq j, |A_i \cap A_j| \leq 1$ . The sequence is listed on the OEIS. [3]

**Theorem 1.2** A. E. Brouwer, J. B. Shearer, N. J. A. Sloane and W. D. Smith.[1]

$$A(n, 4, 3) = \begin{cases} \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor & n \not\equiv 5 \pmod{6}, \\ \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor - 1 & n \equiv 5 \pmod{6}. \end{cases}$$

**Problem 1.3** (1) For which  $n$ ,  $\lambda(n) = \frac{2\binom{n}{3}}{A(n,4,3)}$ , and for which  $n$ ,  $\lambda(n) > \frac{2\binom{n}{3}}{A(n,4,3)}$ ? For the first kind, we call it 6-distant 2-regular.

(2) For  $n \geq 3$  which is 6-distant 2-regular, fix  $\lambda = \lambda(n)$ , determine the number  $N(n)$  of 6-distant designs  $\mathcal{H}$ .

**Theorem 1.4**  $\lambda(n) \geq \frac{2\binom{n}{3}}{A(n,4,3)}$ . Proof: Let  $S = \{1, 2, \dots, n\} = [n]$ ,  $T = \{n+1, n+2, \dots, n+\lambda(n)\}$ , and  $\mathcal{H} = (S \cup T, E)$  is 6-distant. For each  $x \in T$ , we consider the subgraph  $\mathcal{H}_1$  of  $\mathcal{H}$  whose every edge contains  $x$ . For each two triples  $A_i, A_j \in E(\mathcal{H}_1)$ ,  $|A_i \cap A_j| \leq 2 - |\{x\}| = 1$ . Thus as definition of  $A(n, 4, 3)$ , for each  $x \in T$ ,  $x$  is contained at most  $A(n, 4, 3)$  edges in  $T$ . Then we get  $2\binom{n}{3} = 2|E| = \sum_{e \in E} |e \cap T| = \sum_{j=1}^{\lambda(n)} |\{e \in E | n+j \in e\}| \leq \sum_{j=1}^{\lambda(n)} A(n, 4, 3) = \lambda(n)A(n, 4, 3)$ .

It is easy to observe that the equality holds, for every  $j \in T$ , there are exactly  $A(n, 4, 3)$  edges contains  $j$ . For  $m, n \in \mathbb{N}$ , we'll write  $\binom{[n]}{m} = \{A \subset [n] \mid |A| = m\}$  for the family of all  $m$ -subsets of  $[n]$ .

## 2. Cases of $n \leq 4$

**Theorem 2.1**  $\lambda(3) = 2, N(3) = 1$ . Proof: It is trivial that  $|E| = 1$ , so that  $T \geq 2$ . The design  $E = \{\{1, 2, 3; 4, 5\}\}$  ensures that  $|T| = 2$  is possible. Here edge

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$e = \{a_1, a_2, a_3; a_4, a_5\}$  represents that  $e \cap S = \{a_1, a_2, a_3\}$  and  $e \cap T = \{a_4, a_5\}$ . As  $A(3, 4, 3) = 1$ , it means  $n = 3$  is 6-distant 2-regular. And since  $|V| = 5$ , the design  $E$  above is the only 5-uniform hypergraph. By definition,  $N(3) = 1$ .

**Theorem 2.2**  $\lambda(4) = 8, N(4) = 2520$ . Proof: Since  $|E| = \binom{4}{3} = 4$  and  $\forall e_1, e_2 \in E, |e_1 \cap e_2 \cap S| = 2$ . So  $\forall e_1, e_2 \in E, e_1 \cap e_2 \cap T = \emptyset$ , we get  $|T| \geq \sum_{e \in E} |E \cap T| = 2 \times 4 = 8$ . The design  $E = \{\{1, 2, 3; 5, 6\}, \{1, 2, 4; 7, 8\}, \{1, 3, 4; 9, 10\}, \{2, 3, 4; 11, 12\}\}$  ensures that  $|T| = 8$  is possible. As  $A(4, 4, 3) = 1$ , it means  $n = 4$  is 6-distant 2-regular. And for  $N(4)$ , as  $|E| = 4$  and  $\forall e_1, e_2 \in E, e_1 \cap e_2 \cap T = \emptyset$ , thus  $N(4) = \binom{8}{2_2 2_2} = 2520$ .

As we mentioned in the proof,  $n = 3, 4$  are 6-distant 2-regular.

### 3. Case of $n = 5$

**Theorem 3.1**  $\lambda(5) = 10$ , then  $n = 5$  is 6-distant 2-regular.

By Theorem (1.2),  $A(5, 4, 3) = 2$  By Theorem (1.4),  $\lambda(n) \geq \frac{2 \binom{5}{3}}{A(5, 4, 3)} = 10$ .

For  $\lambda = 10$ , this design of  $E$  is determined as listed in Appendix A(1). Hence we get  $\lambda(5) = 10$ .

**Theorem 3.2**  $N(5) = 6 \times \lambda(5)! = 21\,772\,800$ . Proof: On case of  $n = 5$ , a lemma maps a 6-distant hypergraph to a subgraph of Petersen Graph[2]:

**Lemma 3.3** For a 6-distant hypergraph  $\mathcal{H}$ , let

$$\mathcal{G}(\mathcal{H}) = \left( \binom{[5]}{3}, \{\{e \cap S \mid i \in e, e \in E(\mathcal{H})\} \mid i \in T\} \right).$$

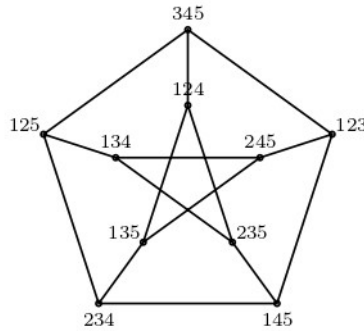
Then  $\mathcal{G}(\mathcal{H})$  is a 2-regular subgraph of the Petersen Graph. Proof: Construct a graph  $\mathcal{G}$  with vertex set  $\binom{[5]}{3}$ , and for pair of triples  $v_1, v_2 \in \binom{[5]}{3}$ ,  $(v_1, v_2) \in E(\mathcal{G}) \Leftrightarrow |v_1 \cap v_2| \leq 1$ . Then  $\mathcal{G}$  is a Petersen Graph.

For a hypergraph  $\mathcal{H}$ , let  $F_i = \{e \cap S \mid i \in e, e \in E(\mathcal{H})\}$ , then  $\mathcal{G}(\mathcal{H}) = \left( \binom{[5]}{3}, \{F_i \mid i \in T\} \right)$ .

For any  $i \in T$  and  $e_1, e_2 \ni i$ , it holds that  $|e_1 \cap e_2| \leq 2$ . And we observe that  $i \in T \cap e_1 \cap e_2$  and  $e_j \in S \cup T$ , so  $|(e_1 \cap S) \cap (e_2 \cap S)| \leq 2 - 1 = 1$ , As the same discussion in 3.1, we know that  $|F_i| = |\{e \cap S \mid i \in e\}| \leq 2$ . Combining with

$\sum_{i=6}^{15} |F_i| = 2|E(\mathcal{H})| = 20$ , so that for each  $i$ ,  $|F_i| = 2$ . Let  $F_i = \{A_1, A_2\}$ ,

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so  $|A_1 \cap A_2| \leq 2 - 1 = 1$ , therefore  $F_i \in E(\mathcal{G})$ . And since all the  $F_i$  are distinct, (otherwise  $F_i = F_j = \{e_1 \cap S, e_2 \cap S\}$ , then  $|e_1 \cap e_2| \geq 1 + |\{i, j\}| = 3$ ) we get that  $\mathcal{G}(\mathcal{H})$  is a subgraph of  $\mathcal{G}$ . For any  $A \in \binom{[5]}{3}$ ,  $\exists |e \in E(\mathcal{H}), A \in e$ . As  $|e \setminus A| = 5 - 3 = 2$ , so  $A$  is exactly on 2 edges  $F_i$  of  $(G)$ . Hence  $\mathcal{G}(\mathcal{H})$  is 2-regular.

**Lemma 3.4**  $\mathcal{G}$  have 6 distinct subgraphs with its form represented as  $\mathcal{G}(\mathcal{H})$ . Proof: For any  $\mathcal{H}$ ,  $\mathcal{G}(\mathcal{H})$  is 2-regular. Then  $\mathcal{G}(\mathcal{H})$  is union of some cycles of  $\mathcal{G}$ . Since the girth of Petersen Graph  $\mathcal{G}$  is 5, and Petersen Graph does not have a Hamilton Cycle, Thus  $\mathcal{G}(\mathcal{H})$  must be the union of two disjoint 5-cycles. Because  $\mathcal{G}$  has 12 5-cycles, and they are grouped into 6 pairs of two disjoint 5-cycles. Thus the number of subgraphs is 6.

**Lemma 3.5** Let  $\mathcal{G}'$  be a subgraph of Petersen Graph with form as union of two disjoint 5-cycles, then equation  $\mathcal{G}(\mathcal{H}) = \mathcal{G}'$  has  $10!$  solutions  $\mathcal{H}$ . Proof: Let  $E(\mathcal{G}') = \{l_1, l_2, \dots, l_{10}\}$ . Since  $F_i = \{e \cap S | i \in e, e \in E(\mathcal{H})\}$  uniquely determine  $\mathcal{H}$ , (Here because  $E(\mathcal{H}) = \{A \cup \{i | A \in F_i\} | A \in \binom{[5]}{3}\}$ ). So  $\mathcal{G}(\mathcal{H}) = \mathcal{G}' \Leftrightarrow E(\mathcal{G}(\mathcal{H})) = E(\mathcal{G}')$

$$\Leftrightarrow \{F_i | i \in T\} = \{l_1, l_2, \dots, l_{10}\}$$

$$\Leftrightarrow F_6, F_7, \dots, F_{15} \text{ is a permutation of } l_1, l_2, \dots, l_{10}.$$

That causes the number of solutions of  $\mathcal{G}(\mathcal{H}) = \mathcal{G}'$  is  $10!$ . By the Corollary and the lemma above,  $N(5) = 6 \times 10! = 21\,772\,800$ .

#### 4. Result of $\lambda(6)$

**Theorem 4.1**  $\lambda(6) = 10$ , the  $n = 6$  is 6-distant 2-regular.

The problem of deciding  $\lambda(6)$  appeared as a question in the 2002 IMO team selection test of China. This interesting problem is designed by Zonghu, Qiu, a Professor of Chinese Academy of Sciences. Proof: By Theorem (1.2),

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$A(6, 4, 3) = 4$ . By Theorem (1.4),  $\lambda(n) \geq \frac{2^{\binom{6}{3}}}{A(6,4,3)} = 10$ .

For  $\lambda = 10$ , this design of  $E$  is determined as one of the 5 listed in Appendix A(2). Hence we get  $\lambda(6) = 10$ .

## 5.Result of $N(6)$

**Claim 5.1**  $N(6) = 6 \times \lambda(6)! = 21\,772\,800$ .

Now we'll prove Claim 1.

Since the design Appendix A(2) is not easy to constuct, we will show how we get this design by analyzing the edges containing arbitrary element of  $T$ . That is to analyze the set (or say FHP)  $\{e \cap S | i \in e\}$ , for every  $i \in T$ . Then we can solve the problem by counting the designs of sets of FHPs.

We will prove that there are  $6 \times 10!$  6-distant designs to be constructed by this method in part 2, and the uniqueness of these designs in part 4.

Firstly we have to introduce some definition and some facts. They will be used throughout the proof.

### Definitions and Facts

**Definition 5.2** As we are discussing on  $[6] = \{1, 2, 3, 4, 5, 6\}$ , we define the conjugate operation on  $[6]$ ,  $2^{[6]}$ ,  $2^{2^{[6]}}$  as below:

1. A subset of  $[6]$ , usually written as letter  $A \subset [6]$ , has its conjugate (or say complement)  $\bar{A} = [6] \setminus A$ .
2. A set of subsets of  $[6]$ , usually written as letter  $F \subset 2^{[6]}$  has its conjugate  $\bar{F} = \{\bar{A} | A \in F\} = \{[6] \setminus A | A \in F\}$ .
3. A set of sets of subsets of  $[6]$ , usually written as letter  $\mathcal{F} \subset 2^{2^{[6]}}$  has its conjugate  $\bar{\mathcal{F}} = \{\bar{F} | F \in \mathcal{F}\} = \{\{[6] \setminus A | A \in F\} | F \in \mathcal{F}\}$ .

**Definition 5.3** We call  $F \subset \binom{[6]}{3}$  a Fano Half Plane (hereinafter. FHP) on  $[6]$ , iff  $F = \{A_1, A_2, A_3, A_4\}$ , and for every  $i \neq j$ ,  $|A_i| = 3$ ,  $|A_i \cap A_j| \leq 1$ .

We will prove the claim by analyzing the FHPs and its properties.

**Fact 5.4** For any 2 triples  $A, A'$  in  $F$ ,  $|A \cap A'| = 1$ .

Obviously,  $F$  is FHP  $\Leftrightarrow \bar{F}$  is FHP. We can easily find that for any  $x \in [6]$ ,  $x$  is

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exactly contained in 4 sets in  $F$ . For FHP  $F$ , we will write  $F$  in 4 number triples instead of 4 sets.

**Example 5.5**  $F = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\}$ , written in short form as  $F = \{123, 145, 246, 356\}$ .

We call  $P = \{\{a, b\}, \{c, d\}, \{e, f\}\}$  is a matching of  $[6]$ , if  $\{a, b, c, d, e, f\} = \{1, 2, 3, 4, 5, 6\}$ .

The set FHP represents strong connection to the pairing of  $[6]$ , with these facts below:

**Definition 5.6** For FHP  $F$ , an Absent Matching  $P_F \subset \binom{[6]}{2}$  of  $F$  is defined as all the binary subset which is not contained in any of the set in  $F$ . That is,

$$P_F = \{\{i, j\} \in \binom{[6]}{2} \mid \{i, j\} \not\subset A, \forall A \in F\}.$$

Samely, for a matching  $P$ , we shall write  $P$ 's 2-sets in number pairs instead of sets, or write directly just 6 numbers for 3 pairs. Obviously,  $|P_F| = \binom{6}{2} - |F| \cdot \binom{3}{2} = 3$ , and similarly for any  $x \in [6]$ ,  $x$  is exactly contained in 1 pair in  $P_F$ , Thus an absent matching of any  $F$  is a matching of  $[6]$ , ensures the definition well-defined.

**Example 5.7** For FHP  $F = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\}$  as in example 1, the absent matching of  $F$  is  $P_F = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ , written in short form as  $P_F = \{16, 25, 34\} = 162534$ .

**Fact 5.8** For any FHP  $F$ , a complement  $F \cup \{\{0, i, j\} \mid \{i, j\} \in P_F\}$  is a Fano plane $[4]$  on vertices  $\{0, 1, 2, 3, 4, 5, 6\}$ . This is how FHP named.

**Corollary 5.9** There are exactly 30 different FHPs.

Proof: This is because that the map  $\pi(F) = F \cup \{\{0, i, j\} \mid \{i, j\} \in P_F\}$  is a bijection between the set of FHPs and set of Fano planes on vertices  $\{0, 1, 2, 3, 4, 5, 6\}$ .

It is well known, and easy to compute that  $\text{Aut}(\text{Fano Plane}) \cong PGL(3, 2)$ , and the number of Fano planes equals to the automorphism classes of a Fano plane  $|S_7 / \sim| = \frac{|S_7|}{|PGL(3,2)|} = 30$ , thus the corollary follows.

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**Definition 5.10** For a matching  $P = \{ab, cd, ef\}$ , let  $a < c < e$ ,  $a < b$ ,  $c < d$ ,  $e < f$ . A Pullback FHP from P is defined by  $F_P = \{ace, adf, bcf, bde\}$ ,

**Fact 5.11** (1)  $P_F = P_{\overline{F}}$ .

(2)  $P_{F_P} = P$ .

(3)  $F_{P_F} = F$  or  $\overline{F}$ .

**Definition 5.12** Let  $\mathcal{F}$  be a set of ten FHPs.  $\mathcal{F}$  is Balanced FHP Set iff,

1. Every triple of  $[6]$  is exactly contained in 2 of the FHPs in  $\mathcal{F}$ ;
2. For any two FHPs  $F_1, F_2 \in \mathcal{F}$ ,  $|F_1 \cap F_2| \leq 1$

**Definition 5.13** Let  $P_1, P_2, P_3, P_4, P_5$  be 5 matchings of  $[6]$ . If for any  $i \neq j$ ,  $P_i \cap P_j = \emptyset$ , we call  $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5\}$  a Perfect Collection of matchings.

As for any matching  $P$ ,  $|P| = 3$ , so  $\bigcup_{P \in \mathcal{P}} P = \binom{[6]}{2}$ , contains all 15 pairs in  $[6]$ , so any pair  $\{a, b\} \subset [6]$  is contained in exactly 1 matching in  $\mathcal{P}$ .

**Fact 5.14** For FHP  $F$ , FHP set  $\mathcal{F}$  (not necessarily balanced), and matching  $P$ , collection of matchings  $\mathcal{P}$  ;(not necessarily perfect), it holds that

1.  $F \in \mathcal{F} \Rightarrow P_F \in \mathcal{P}_{\mathcal{F}} = \{P_F | F \in \mathcal{F}\}$
2.  $P \in \mathcal{P} \Rightarrow F_P \in \mathcal{F}_{\mathcal{P}} = \{F_P, \overline{F_P} | P \in \mathcal{P}\}$

**Proof of existence of  $6 \times 10!$  designs**

We'll discover the connections among the perfect collections, balanced FHP sets, and 6-distant designs, which is needed in the prove of the uniqueness.

**Lemma 5.15** There exist exactly 6 perfect collections of matchings. Proof: As there are exactly  $\binom{5}{1} \cdot \binom{4}{2} / 2 = 15$  matchings in  $[6]$ , let  $\mathcal{P} = \{\{P_1, P_2\} | P_1 \cap P_2 = \emptyset\}$ , here  $P_i$  are matchings in  $[6]$ . It is easy to see that  $|\mathcal{P}| = 15 \times 8 / 2 = 60$ . For each of  $\mathcal{P}$ , let  $P_1 = \{ab, cd, ef\}$ ,  $P_2 = \{af, bc, de\}$  and a perfect matching  $\mathcal{P}$  ;contains  $P_1, P_2$ , Let  $P_3 \in \mathcal{P}$  with  $\{a, d\} \in P_3$ ,  $P_3$  must be  $\{ad, bf, ce\}$ , otherwise we cannot construct  $P_4$ . So  $P_4 = \{ac, be, df\}$ , i.e.,  $\mathcal{P}$ ;uniquely determined.

Then there exists a mapping  $\tilde{f}$  from  $\mathcal{P}$  to  $\{\mathcal{P} | \mathcal{P}; \text{is perfect collection of matchings}\}$ .

As for any perfect collection  $\mathcal{P}$ ,  $|\tilde{f}^{-1}(\mathcal{P})| = \binom{5}{2} = 10$ , hence the number of perfect collections is  $\frac{|\mathcal{P}|}{|\tilde{f}^{-1}(\mathcal{P})|} = \frac{60}{10} = 6$ . Here we list all 6 perfect collections in appendix B.

**Corollary 5.16**

1. Each matching appears in exactly 2 Perfect collections.
2. Each 2 Perfect collections have exactly 1 common matching.

We can know this corollary immediately by observing these 6 perfect collections.

**Lemma 5.17** A design  $\mathcal{H}=(S \cup T, E)$  satisfies the condition (1)(2) in requirements of 6-distant, then  $\mathcal{H}$  satisfies (3)  $\Leftrightarrow \mathcal{F} = \{F_i = \{e \cap S | i \in e, e \in E\} | 7 \leq i \leq 16\}$  is a balanced FHP set.

Proof:[Proof of  $\Rightarrow$  in Lemma 5.17]

For a 6-distant design  $\mathcal{H} = (S \cup T, E)$ ,

- i> For any  $i \in T$  and  $e_1, e_2 \ni i$ ,  $|e_1 \cap e_2| \leq 2$ . And we observe that  $i \in T \cap e_1 \cap e_2$  and  $e_j \in S \cup T$ , so  $|(e_1 \cap S) \cap (e_2 \cap S)| \leq 2 - 1 = 1$ , As the same discussion in 4.1, we know that  $|F_i| = |\{e \cap S | i \in e\}| \leq 4$ . Combining with  $\sum_{i=7}^{16} |F_i| = 2|E| = 40$ ,  $|F_i| = 4$ ,  $F_i$  is FHP.
- ii> Thus  $E = \{F_7, F_8, \dots, F_{16}\}$ , here  $F_i$ 's are FHPs. Since  $|E| = |\binom{S}{3}|$ , and any two edges in  $E$  must have different triples in  $S = [6]$ , so there is a bijection from  $E$  to  $S$ 's triples  $\binom{S}{3}$ :  $\pi : e \mapsto e \cap S$ . Obviously,  $\forall e \in E, |e \cap T| = 2$ , let  $e \cap T = \{u, v\}$ . Consider the definition of  $F_i$ 's, we know  $e \cap S$  is contained in exactly 2 FHPs  $F_u, F_v$ .
- iii> Suppose 2 of these FHPs have at least 2 common triples,  $|F_i \cap F_j| \geq 2$ , Let  $A, A' \in F_i \cap F_j$ ,  $\pi^{-1}(A) = e$ ,  $\pi^{-1}(A') = e'$ ,  $e \neq e'$ . Since  $F_i \ni A' = e \cap S$ , so  $i \in e$ . Similarly,  $j \in e$  and  $i, j \in e'$ , that is,  $i, j \in e \cap e' \cap T$ . Therefore,  $A = e \setminus \{i, j\}$ ,  $A' = e' \setminus \{i, j\}$ . But triples A and A' is in a same FHP,  $|A \cap A'| = 1$ , Hence  $|e \cap e'| = |e \cap e' \cap S| + |e \cap e' \cap T| = 2 + 1 = 3$ , contradiction!

Thus for any  $i, j$ ,  $|F_i \cap F_j| \leq 1$ , that is,  $\mathcal{F}$ ;is a balanced FHP set.

Proof:[Proof of  $\Leftarrow$  in Lemma (5.17)]

Let  $\mathcal{F} = \{F_i = \{e \cap S | i \in e\} | i \in T\}$  be balanced. Suppose  $e, e' \in E$  that  $|e \cap e'| \geq 3$ . Since  $e, e' \subset S \cup T$ , so  $3 \leq |e \cap e'| = |e \cap e' \cap S| + |e \cap e' \cap T|$ .

- (1)  $|e \cap e' \cap S| \geq 3$ . Let  $e = \{a, b, c, \alpha, \beta\}$ ,  $e' = \{a, b, c, \alpha', \beta'\}$ , here  $a, b, c \in S$ . With

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- $e \neq e'$  we know  $\alpha, \beta, \alpha', \beta'$  are at least 3 different elements, as definition of  $F_i$ ,  $\{a, b, c\} \in F_\alpha, F_\beta, F_{\alpha'}, F_{\beta'}$ , contradicts to the definition of balanced FHP set.
- (2)  $|e \cap e' \cap S| = 2, |e \cap e' \cap T| \geq 1$ . Let  $e = \{a, b, c, \alpha, \beta\}$ ,  $e' = \{a, b, c', \alpha, \beta'\}$ . As the definition of  $F_i$ ,  $\{a, b, c\}, \{a, b, c'\} \in F_\alpha$ , contradicts to the definition of FHP.
- (3)  $|e \cap e' \cap S| = 1, |e \cap e' \cap T| = 2$ . Let  $e = \{a, b, c, \alpha, \beta\}$ ,  $e' = \{a, b', c', \alpha, \beta\}$ . As the definition of  $F_i$ ,  $\{a, b, c\}, \{a, b', c'\} \in F_\alpha \cap F_\beta$ , so  $|F_\alpha \cap F_\beta| \geq 2$ , contradicts to the definition of balanced FHP set.
- (4)  $|e \cap e' \cap T| \geq 3$ . This is impossible because  $|e \cap T| = 2$ .

So there always be a contradiction. Thus  $\mathcal{H}$  satisfies the condition (3) in the requirements of 6-distant.

**Corollary 5.18** For a 6-distant design  $\mathcal{H} = (V, E)$ , Let  $\mathcal{F}(\mathcal{H}) = \{F_i = \{e \cap S | i \in e \in E\} | i \in T\}$  be the balanced FHP set induced by  $\mathcal{H}$ .

**Lemma 5.19** Let  $\mathcal{F}$  be a balanced FHP set, then equation  $\mathcal{F}(\mathcal{H}) = \mathcal{F}$  has  $10!$  solutions  $\mathcal{H}$ .

Proof: Let  $E(\mathcal{G}') = \{l_1, l_2, \dots, l_{10}\}$ . Since  $F_i = \{e \cap S | i \in e, e \in E(\mathcal{H})\}$  uniquely determine  $\mathcal{H}$ , (Here because  $E(\mathcal{H}) = \{A \cup \{i | A \in F_i\} | A \in \binom{[6]}{3}\}$ ). So

$$\begin{aligned} E(\mathcal{H}) = \mathcal{F} &\Leftrightarrow \{F_i | i \in T\} = \{l_1, l_2, \dots, l_{10}\} \\ &\Leftrightarrow F_7, F_8, \dots, F_{16} \text{ is a permutation of } l_1, l_2, \dots, l_{10}. \end{aligned}$$

Hence the number of solutions of  $E(\mathcal{H}) = \mathcal{F}$  is  $10!$ .

**Lemma 5.20** For a perfect collection  $\mathcal{P}$ ,  $\mathcal{F}_{\mathcal{P}} = \{F_P, \overline{F_P} | P \in \mathcal{P}\}$  is a balanced FHP set.

Proof: Just write down the 6 perfect collections  $\mathcal{P}_i$ , and compute each  $\mathcal{F}_{\mathcal{P}_i}$  by the definition, simply observe and verify to get the answer. These  $\mathcal{P}_i$ 's and  $\mathcal{F}_{\mathcal{P}_i}$ 's are listed in Appendix B and C.

**Theorem 5.21** The 6 perfect collections induce exactly  $6 \times 10!$  designs of  $\mathcal{H}$ .

Proof: For a perfect collection  $\mathcal{P}$ , by Lemma (5.20),  $\mathcal{F}_{\mathcal{P}}$  is a balanced FHP set. By Corollary (5.19), for  $\mathcal{F}_{\mathcal{P}}$ , it induces  $10!$  designs of  $\mathcal{H}$ . Thus these 6 perfect collections induce  $6 \times 10!$  designs of  $\mathcal{H}$ . Below we'll prove these 6 designs are all the essential designs with different balanced FHP sets, and they induce all  $6 \times 10!$  designs by Lemma (5.19).

### Intersection and distribution properties of FHPs in 6 balanced FHP sets

Before we prove the uniqueness of these designs, we'll introduce some behavior of balanced FHP sets. and we'll observe how FHP's triples are distributed in induced 6 balanced FHP sets in Appendix C. These properties are necessary for proving the uniqueness.

**Lemma 5.22** A perfect collection  $\mathcal{P}, \mathcal{F}_{\mathcal{P}} = \{F_1, F_2, F_3, F_4, F_5; \overline{F_1}, \overline{F_2}, \overline{F_3}, \overline{F_4}, \overline{F_5}\}$ , satisfies that  $\bigcup_{i=1}^5 F_i$  and  $\bigcup_{i=1}^5 \overline{F_i}$  equally divide  $\binom{[6]}{3}$ , has both 10 triples of [6] and with empty intersection.

Proof: Same as lemma (5.20), just write down the 6  $\mathcal{F}_{\mathcal{P}_i}$  calculated in lemma (5.20), simply observe and verify to get the answer. These  $\mathcal{P}_i$ 's and  $\mathcal{F}_{\mathcal{P}_i}$ 's are listed in Appendix B and C.

**Lemma 5.23** For FHP  $F_1, F_2$ , if  $|P_{F_1} \cap P_{F_2}| = 1$ , then exactly one of following holds:

1.  $|F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 0, |F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 2;$
2.  $|F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 2, |F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 0;$

Proof: W.l.o.g. let  $P_{F_1} = \{12, 34, 56\}$ ,

(1)  $P_{F_2} = \{12, 36, 45\}$ . Then

$F_1 = \{135, 146, 236, 245\}, \overline{F_1} = \{136, 145, 235, 246\}, F_2 = \{134, 156, 235, 246\},$   
 $\overline{F_2} = \{135, 146, 234, 256\}$ . So  $|F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 2, |F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 0$ .

(2)  $P_{F_2} = \{12, 35, 46\}$ .  $F_2 = \{134, 156, 236, 245\}, \overline{F_2} = \{136, 145, 234, 256\}$ . So  
 $|F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 0, |F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 2$ .

**Lemma 5.24** For FHP  $F_1, F_2$ , if  $P_{F_1} \cap P_{F_2} = \emptyset$ , then exactly one of the following exists:

1.  $|F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 1, |F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 0;$
2.  $|F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 0, |F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 1;$

Proof: W.l.o.g. let  $P_{F_1} = \{12, 34, 56\}$ . As the union any two matchings of [6] is the edge set of a Hamiltonian cycle of [6], whose all the Hamiltonian cycles of [6] are isomorphic, again w.l.o.g. let  $P_{F_2} = \{16, 23, 45\}$ . Then  $F_1 = \{135, 146, 236, 245\},$   
 $\overline{F_1} = \{136, 145, 235, 246\}, F_2 = \{124, 135, 256, 346\}, \overline{F_2} = \{125, 134, 246, 356\}$ . So  
 $|F_1 \cap F_2| = |\overline{F_1} \cap \overline{F_2}| = 1, |F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 0$ .

**Lemma 5.25** For FHP  $F$  and a perfect collection  $\mathcal{P}$ , let

$\mathcal{F}_{\mathcal{P}} = \{F_1, F_2, F_3, F_4, F_5; \overline{F_1}, \overline{F_2}, \overline{F_3}, \overline{F_4}, \overline{F_5}\}$  as in lemma (5.22). If  $F \subset \bigcup_{i=1}^5 F_i$ , then

$F_i \in \{F_1, F_2, F_3, F_4, F_5\}$ . Proof: Again w.l.o.g. let

$\mathcal{P} = \{123456, 132546, 142635, 152436, 162345\} = \mathcal{P}_a$  as in appendix B. Then

$\{A \in F | 1 \in A\} \subset \{A \in \mathcal{F}_{\mathcal{P}} | 1 \in A\} = \{123, 124, 135, 146, 156\}$ . Since every two triples  $A, A'$  in  $F$  have 1 common number,  $\{A \in F | 1 \in A\}$  must be one of the following:

$$\{135, 146\}, \{124, 156\}, \{123, 156\}, \{123, 146\}, \{124, 135\}.$$

Whichever  $\{A \in F | 1 \in A\}$  is, let  $r$  be the number not appearing in this set.

Consider  $\{A \in F | r \in A\}$ ,  $F$  equals one of the following, corresponding to which  $\{A \in F | 1 \in A\}$  equals above:

$$\begin{aligned} &\{135, 146, 236, 245\}, \{124, 156, 236, 345\}, \{123, 156, 245, 346\}, \\ &\{123, 146, 256, 345\}, \{124, 135, 256, 346\}. \end{aligned}$$

That is a part of  $\mathcal{F}_{\mathcal{P}_a}$  as in Appendix C.

**Corollary 5.26** For a perfect collection  $\mathcal{P}$ , with  $\mathcal{F}_{\mathcal{P}}$  same notation as in lemma (5.22),  $|F_i \cap F_j| = |\overline{F_i} \cap \overline{F_j}| = 1, \forall 1 \leq i < j \leq 5$ .

As Lemma (5.22) indicates that  $|F_i \cap \overline{F_j}| = |\overline{F_i} \cap F_j| = 0, \forall 1 \leq i, j \leq 5$ , we see that Lemma (5.22) and Corollary (5.26) shows the properties of the intersections of FHPs in a balanced FHP set.

Now we'll introduce some distribution properties of FHPs in the induced balanced FHP sets listed in Appendix C.

**Lemma 5.27** For FHP  $F$  and a perfect collection  $\mathcal{P}$ ,  $P_F \notin \mathcal{P}$ , then

$\exists F_1, F_2, F_3 \in \mathcal{F}_{\mathcal{P}}$ ,

1.  $|F \cap F_1| = |F \cap F_2| = |F \cap F_3| = 2$ ,
2.  $|F \cap F_1 \cap F_2| = |F \cap F_2 \cap F_3| = |F \cap F_1 \cap F_3| = 1$ ,
3.  $|F \cap \bigcup_{i=1}^5 F_i| = 3, |F \cap \bigcup_{i=1}^5 \overline{F_i}| = 1$ .

Proof: As for any pair of matchings  $P_1, P_2$ ,  $|P_1 \cap P_2| = 0, 1$  or  $3$ , and

$\forall P_0 \in \mathcal{P}, P_F \neq P_0$ , thus  $|P_F \cap P_0| = 0, 1$ . Since  $|P_F| = 3$ , and  $P_F \subset \bigcup_{P \in \mathcal{P}} P = \binom{[6]}{2}$ ,

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then  $\exists P_1, P_2, P_3 \in \mathcal{P}$  that  $|P_F \cup P_i| = 1, i = 1, 2, 3$ . By Lemma (5.27),  $\exists F_1, F_2, F_3 \in \mathcal{F}_{\mathcal{P}}; |F \cap F_i| = 2$ .

Suppose that  $|F \cap \bigcup_{i=1}^3 F_i| = 4$ , that is,  $F \subset \bigcup_{i=1}^3 F_i$ ,

1.  $\exists i, j, F_i \cap F_j = \emptyset$ . W.l.o.g.  $i = 1, j = 2$ . By Corollary (5.26), let  $|F_3 \cap F_1| = 1$ ,  $|F_3 \cap F_2| = 0$ . Thus

$$4 = |F| \geq |F \cap (F_1 \cup F_3)| + |F \cap F_2| = \sum_{i=1}^3 |F \cap F_i| - |F \cap F_1 \cap F_3| \geq 6 - 1 = 5,$$

contradiction.

2.  $|F_1 \cap F_2| = |F_2 \cap F_3| = |F_1 \cap F_3| = 1$ . By Lemma( 5.25),  $F \in \mathcal{F}_{\mathcal{P}}$ . Thus  $P_F \in \mathcal{P}_{\mathcal{F}_{\mathcal{P}}} = \mathcal{P}$ , contradiction.

Hence  $|F \cap \bigcup_{i=1}^3 F_i| \leq 3$ . It holds that

$$\begin{aligned} 3 &\geq \left| F \cap \bigcup_{i=1}^3 F_i \right| = \sum_{i=1}^3 |F \cap F_i| - \sum_{1 \leq i < j \leq 3} |F \cap F_i \cap F_j| + |F \cap F_1 \cap F_2 \cap F_3| \\ &= 6 - \sum_{1 \leq i < j \leq 3} |F \cap F_i \cap F_j| + 0 \\ &\geq 6 - \sum_{1 \leq i < j \leq 3} |F_i \cap F_j| \geq 6 - 3 = 3. \end{aligned}$$

Then all the equalities hold, that is  $\forall i, j, |F_i \cap F_j| = |F \cap F_i \cap F_j| = 1$ , and  $F_i \cap F_j \subset F$ . And  $3 = |F \cap \bigcup_{i=1}^3 F_i| \leq |F \cap \bigcup_{i=1}^5 F_i| = 3$ , so  $|F \cap \bigcup_{i=1}^5 F_i| = 3$ ,  $|F \cap \bigcup_{i=1}^5 \overline{F_i}| = |F| - 3 = 1$ .

### Proof of uniqueness of these $6 \times 10!$ designs

**Theorem 5.28** These  $6 \times 10!$  designs are all 6-distant designs.

Proof: We'll prove that the FHPs in balanced FHP sets must be arranged as we constructed in part 2. And we'll use the intersection and distribution properties just proved in part 3 to prove it.

For a design  $\mathcal{H}$ , by Lemma (5.18), we consider  $\mathcal{F} = \mathcal{F}(\mathcal{H})$ . By Lemma (5.17),  $\mathcal{F}$  is balanced.

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Now we proof that for some perfect collection  $\mathcal{P}$ ,  $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$ . We will prove this in two cases.

**Case 1:**  $\exists F \in \mathcal{F}$  such that  $\overline{F} \in \mathcal{F}$

Let  $P = P_F = P_{\overline{F}}$ . As  $\mathcal{F}$  is balanced, any two FHPs in  $\mathcal{F}$  have at most 1 common triple. As Corollary (5.16).(1),  $P$  is in two of 6 perfect collections, let  $P \in \mathcal{P}_1 \cap \mathcal{P}_2$ . By Lemma (5.23),  $\forall F' \in \mathcal{F}$ , if  $P_{F'} \neq P$ , then  $|F \cap F'| \neq 2$ ,  $|\overline{F} \cap F'| \neq 2 \Rightarrow P_{F'} \cap P = \emptyset$ . Same as the construction in lemma(5.15),  $P_{F'}$  and  $P$  appears in 1 perfect matching, so  $P_{F'} \in \mathcal{P}_1 \cup \mathcal{P}_2$ . Then  $\mathcal{F} \subset \mathcal{F}_{\mathcal{P}_1} \cup \mathcal{F}_{\mathcal{P}_2}$ . It holds that

$$|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| + |\mathcal{F}_{\mathcal{P}_2} \cap \mathcal{F}| = |\mathcal{F}| + |\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}_{\mathcal{P}_2} \cap \mathcal{F}| = 10 + 2 = 12.$$

W.l.o.g.  $|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| \geq 6 \geq |\mathcal{F}_{\mathcal{P}_2} \cap \mathcal{F}|$ . Suppose  $\mathcal{F}_{\mathcal{P}_1} \neq \mathcal{F}$ , let  $F' \in \mathcal{F} \setminus \mathcal{F}_{\mathcal{P}_1}$ . Then  $P_{F'} \notin \mathcal{P}_1$ . By Lemma (5.27),  $\exists F_3, F_4, F_5 \in \mathcal{F}_{\mathcal{P}_1}$ , that  $|F' \cap F_3| = |F' \cap F_4| = |F' \cap F_5| = 2$  and  $|F' \cap F_3 \cap F_4| = |F' \cap F_4 \cap F_5| = |F' \cap F_5 \cap F_3| = 1$ . Here  $F' \in \mathcal{F} \setminus \mathcal{F}_{\mathcal{P}_1} \subset \mathcal{F}_{\mathcal{P}_2}$  and  $F, \overline{F} \in \mathcal{F}_{\mathcal{P}_2}$ , and  $\mathcal{F}_{\mathcal{P}_2}$  is balanced, therefore  $\{F_3, F_4, F_5\} \cap \{F, \overline{F}\} = \emptyset$ . Let  $F_1 = F \in \mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}$ , also  $\overline{F}_1 = \overline{F} \in \mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}$ . On the other hand,  $\mathcal{F}$  is balanced,  $F_3, F_4, F_5 \notin \mathcal{F}$ , therefore  $|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| \leq 7$ . We denote  $F_2 \in \mathcal{F}_{\mathcal{P}_1}$  be the lemma (5.22) mentioned, by Corollary (5.26),  $\{F_i\}_{i=1}^5$  have pairwise nonempty intersection.

- (1)  $|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| = 7$ . Thus  $F_1, F_2, \overline{F}_1, \overline{F}_2 \in \mathcal{F}$ . For any  $F' \in \mathcal{F}$ . Same as Corollary (5.16).(1),  $P_{F_2}$  is in two of 6 perfect collections, as  $F_2 \in \mathcal{F}_{\mathcal{P}_1}$ , let  $P_{F_2} \in \mathcal{P}_1 \cap \mathcal{P}_3$ , so  $F_2 \in \mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}_{\mathcal{P}_3}$ . Then same as above  $\mathcal{F} \subset \mathcal{F}_{\mathcal{P}_1} \cup \mathcal{F}_{\mathcal{P}_3}$ . Hence

$$\mathcal{F} \subset (\mathcal{F}_{\mathcal{P}_1} \cup \mathcal{F}_{\mathcal{P}_3}) \cap \mathcal{F}_{\mathcal{P}_1} \cup \mathcal{F}_{\mathcal{P}_2} = \mathcal{F}_{\mathcal{P}_1} \cup (\mathcal{F}_{\mathcal{P}_2} \cap \mathcal{F}_{\mathcal{P}_3}).$$

So it holds that  $|\mathcal{F}| \leq |\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| + |\mathcal{F}_{\mathcal{P}_2} \cap \mathcal{F}_{\mathcal{P}_3}| = 7 + 2 = 9$ , contradiction!

- (2)  $|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| = 6 = |\mathcal{F}_{\mathcal{P}_2} \cap \mathcal{F}|$ . Same reason as (1),  $\{F_1, F_2, \overline{F}_1, \overline{F}_2\} \notin \mathcal{F}$ . As  $F_3, F_4, F_5 \notin \mathcal{F}$ , then  $F_1, \overline{F}_1, \overline{F}_3, \overline{F}_4, \overline{F}_5 \in \mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}$ . Suppose  $F_2 \in \mathcal{F}$ , consider  $F', F'' \in \mathcal{F} \setminus \mathcal{F}_{\mathcal{P}_1}$ . Then  $P_{F'}, P_{F''} \notin \mathcal{P}_1$ . By Lemma (5.27),  $\exists F^1, F^2, F^3 \in \mathcal{F}_{\mathcal{P}_1}$ ,  $|F' \cap F^1 \cap F^2| = |F' \cap F^2 \cap F^3| = |F' \cap F^1 \cap F^3| = 1$ , and  $|F'' \cap F^1 \cap F^2| = |F'' \cap F^2 \cap F^3| = |F'' \cap F^1 \cap F^3| = 1$ . As  $\mathcal{F}_{\mathcal{P}_1}$  is balanced,  $|F^1 \cap F^2| = |F^2 \cap F^3| = |F^1 \cap F^3| = 1$ , so  $(F^1 \cap F^2) \cup (F^2 \cap F^3) \cup (F^1 \cap F^3) \subset F'$  And,  $(F^1 \cap F^2) \cup (F^2 \cap F^3) \cup (F^1 \cap F^3) \subset F''$ , we get  $|F' \cap F''| \geq 3$ , contradicts to balance of  $\mathcal{F}$ .

Then  $F_2 \notin \mathcal{F}$ , so that  $\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F} = \{F_1, \overline{F}_1, \overline{F}_3, \overline{F}_4, \overline{F}_5\}$ . Let  $F' \in \mathcal{F} \setminus \mathcal{F}_{\mathcal{P}_1}$ ,

then  $P_{F'} \notin \mathcal{P}_1$ . By Lemma (5.27),  $|F' \cap \bigcup_{i=1}^5 \overline{F}_i| = 1$  or 3. Again let  $A \in F' \cap \bigcup_{i=1}^5 \overline{F}_i$ . As  $\mathcal{P}_{\mathcal{F}_1}$  balanced, A appears twice in  $F_i, \overline{F}_i$ ,  $i = 1, 2, 3, 4, 5$ . By Lemma (5.23), A appears twice in  $\overline{F}_i$ ,  $i = 1, 2, 3, 4, 5$ . But A appears in  $F' \neq \overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4, \overline{F}_5$ , and  $F', \overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4, \overline{F}_5 \in \mathcal{F}$ , therefore A appears in at least 3 sets in  $\mathcal{F}$ , contradiction!

That results  $\mathcal{F} = \mathcal{F}_{\mathcal{P}_1}$ , that is,  $\mathcal{F}$  ;is listed in the Appendix C.

**Case 2:**  $\forall F \in \mathcal{F}, \overline{F} \notin \mathcal{F}$

By Corollary (5.16), for each FHP F,  $P_F$  is in two of the perfect collections:  $P_F \in \mathcal{P}_1 \cap \mathcal{P}_2$ . So for each FHP F, F is in 2 of the 6 balanced FHP sets induced by these 6 perfect collections. It holds that

$$\sum_{\mathcal{F}' \text{ balanced}} |\mathcal{F}' \cap \mathcal{F}| = 2 \times |\mathcal{F}| = 20.$$

Then there exists a balanced FHP set  $\mathcal{F}_{\mathcal{P}_1}$ ,  $|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| \geq \lceil \frac{20}{6} \rceil = 4$ . By Lemma (5.20),  $\mathcal{F}_{\mathcal{P}_1}$  consists of 5 pairs of conjugate FHPs, so  $|\mathcal{F}_{\mathcal{P}_1} \cap \mathcal{F}| \leq 5$ . Let

$\mathcal{F}_{\mathcal{P}_1} = \{F_1, F_2, F_3, F_4, F_5; \overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4, \overline{F}_5\}$  as in lemma (5.22) so that  $\bigcup_{i=1}^5 F_i$  and  $\bigcup_{i=1}^5 \overline{F}_i$  equally divide  $\binom{[6]}{3}$ . W.l.o.g., let

$$\begin{aligned} \{F_1, F_2, F_3, F_4, F_5\} \cap \mathcal{F} &= \{F_1, F_2, \dots, F_u\}, \\ \{\overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4, \overline{F}_5\} \cap \mathcal{F} &= \{\overline{F}_5, \overline{F}_4, \dots, \overline{F}_{5-v+1}\}, \end{aligned}$$

and symmetrically,  $u \geq v$ . As  $4 \leq u + v \leq 5$ , then all possible  $(u, v) = (5, 0), (4, 0); (4, 1), (3, 1); (3, 2), (2, 2)$ .

Since  $|\mathcal{F} \setminus \mathcal{F}_{\mathcal{P}_1}| = 10 - (u + v) \geq 5$ , let  $\{F_6, F_7, F_8, F_9, F_{10}\} \subseteq \mathcal{F} \setminus \mathcal{F}_{\mathcal{P}_1}$ .

(1)  $(u, v) = (3, 2), (2, 2)$ , that is,  $F_1, F_2, \overline{F}_4, \overline{F}_5 \in \mathcal{F}$  For each  $i$ ,  $6 \leq i \leq 10$ ,  $F_i \notin \mathcal{F}_{\mathcal{P}_1}$ , so  $P_{F_i} \notin \mathcal{P}_1$ , by Lemma (5.27),  $\exists F_i^1, F_i^2, F_i^3 \in \mathcal{F}_{\mathcal{P}_1}$  s.t.  $|F_i^j \cap F_i| = 2$ . As  $\mathcal{F}$  balanced,  $F_i^1, F_i^2, F_i^3 \in \mathcal{F}_{\mathcal{P}_1} \setminus \mathcal{F} \subseteq \{F_3, F_4, F_5; \overline{F}_1, \overline{F}_2, \overline{F}_3\}$ . By the intersection property (5.27) of  $F_i^1, F_i^2, F_i^3$  we know that

$$\{F_i^1, F_i^2, F_i^3\} = \{F_3, F_4, F_5\} \text{ or } \{\overline{F}_1, \overline{F}_2, \overline{F}_3\}, i = 6, 7, 8, 9, 10.$$

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So there must be  $6 \leq i < j \leq 10$ , s.t.  $\{F_i^1, F_i^2, F_i^3\} = \{F_j^1, F_j^2, F_j^3\}$ .

As  $\mathcal{F}_{\mathcal{P}_1}$  is balanced,  $|F_i^1 \cap F_i^2| = |F_i^2 \cap F_i^3| = |F_i^1 \cap F_i^3| = 1$ , so

$(F_i^1 \cap F_i^2) \cup (F_i^2 \cap F_i^3) \cup (F_i^1 \cap F_i^3) \subset F_i$ , therefore,

$(F_i^1 \cap F_i^2) \cup (F_i^2 \cap F_i^3) \cup (F_i^1 \cap F_i^3) \subset F_j$ , we get  $|F_i \cap F_j| \geq 3$ , contradicts to balance of  $\mathcal{F}$ .

(2)  $(u, v) = (4, 1), (3, 1)$ , that is,  $F_1, F_2, F_3, \overline{F_5} \in \mathcal{F}$ . Similar as (1), for each  $i$ ,

$6 \leq i \leq 10$ ,  $F_i \notin \mathcal{F}_{\mathcal{P}_1}$ , so  $P_{F_i} \notin \mathcal{P}_1$ , by Lemma (5.27),  $\exists F_i^1, F_i^2, F_i^3 \in \mathcal{F}_{\mathcal{P}_1}$ , s.t.

$|F_i^j \cap F_i| = 2$ . As  $\mathcal{F}$  balanced,  $F_i^1, F_i^2, F_i^3 \in \mathcal{F}_{\mathcal{P}_1} \setminus \mathcal{F} \subseteq \{F_4, F_5; \overline{F_2}, \overline{F_3}, \overline{F_4}, \overline{F_5}\}$ . By the intersection property (5.27) of  $F_i^1, F_i^2, F_i^3$ , it holds that

$$\begin{aligned} \{F_i^1, F_i^2, F_i^3\} &= \{\overline{F_2}, \overline{F_3}, \overline{F_4}\} \text{ or } \{\overline{F_2}, \overline{F_3}, \overline{F_5}\} \text{ or } \{\overline{F_2}, \overline{F_4}, \overline{F_5}\} \\ &\text{ or } \{\overline{F_3}, \overline{F_4}, \overline{F_5}\}, i = 6, 7, 8, 9, 10. \end{aligned}$$

So there must be  $6 \leq i < j \leq 10$ , s.t.  $\{F_i^1, F_i^2, F_i^3\} = \{F_j^1, F_j^2, F_j^3\}$ .

As  $\mathcal{F}_{\mathcal{P}_1}$  is balanced,  $|F_i^1 \cap F_i^2| = |F_i^2 \cap F_i^3| = |F_i^1 \cap F_i^3| = 1$ , so

$(F_i^1 \cap F_i^2) \cup (F_i^2 \cap F_i^3) \cup (F_i^1 \cap F_i^3) \subset F_i$ , therefore,

$(F_i^1 \cap F_i^2) \cup (F_i^2 \cap F_i^3) \cup (F_i^1 \cap F_i^3) \subset F_j$ , we get  $|F_i \cap F_j| \geq 3$ , contradicts to balance of  $\mathcal{F}$ .

(3)  $(u, v) = (5, 0), (4, 0)$ , that is,  $F_1, F_2, F_3, F_4 \in \mathcal{F}$ . We denote  $\mathfrak{F} = \bigcup_{j=1}^5 F_j$ . We

use technique of double counting on  $M = |\{(A, F) \in \mathfrak{F} \times \mathcal{F} | A \in F\}|$ . i.e., the numbers of triples in  $\mathfrak{F}$ 's triples appear in all 10 FHPs in  $\mathcal{F}$ .

On one hand, count M with triples A,  $|\mathfrak{F}| = |\bigcup_{j=1}^5 F_j| = 10$ , for any triple A, by the

definition of balanced FHP set, A is exactly contained in 2 of the FHPs in  $\mathcal{F}$ , so

$$M = \sum_{A \in \mathfrak{F}} |\{F \in \mathcal{F} | A \in F\}| = \sum_{A \in \mathfrak{F}} 2 = 2 \times |\mathfrak{F}| = 20.$$

On the other hand, count M with FHP  $F$ . As  $F_1, F_2, F_3, F_4 \in \mathcal{F}$ , and

$F_j \subset \mathfrak{F}$  ( $j = 1, 2, 3, 4$ ), we know  $F_1, F_2, F_3, F_4$  contribute  $4 \times 4 = 16$  to M. For

$F_6, F_7, F_8, F_9, F_{10} \notin \mathcal{F}$ , for each  $i$ ,  $6 \leq i \leq 10$ ,  $F_i \notin \mathcal{F}_{\mathcal{P}_1}$ , so  $P_{F_i} \notin \mathcal{P}_1$ . By Lemma

(5.27),  $|F_i \cap \mathfrak{F}| (= 1 \text{ or } 3) \geq 1$ ,  $\forall i = 6, 7, 8, 9, 10$ , we know  $F_6, F_7, F_8, F_9, F_{10}$

contribute at least  $5 \times 1 = 5$  to M. Hence,

$$M = \sum_{F \in \mathcal{F}} |F \cap \mathfrak{F}| \geq \sum_{j=1}^4 |F_j \cap \mathfrak{F}| + \sum_{i=6}^{10} |F_i \cap \mathfrak{F}| \geq \sum_{j=1}^4 4 + \sum_{i=6}^{10} 1 = 21,$$

contradiction!

So it is not possible for all  $F \in \mathcal{F}$ , that  $\overline{F} \notin \mathcal{F}$ .

Hence for any design  $\mathcal{H}$ , there exists a perfect collection  $\mathcal{P}$ ,  $\mathcal{F}(\mathcal{H}) = \mathcal{F}_{\mathcal{P}}$ .

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By lemma (5.15) and (5.19), we know  $N(6) = 6 \times 10! = 21\,772\,800$ . Thus we proved our claim.

## 6. Case of $n = 7$

By theorem (1.2), it is well known that  $A(7, 4, 3) = 7$  with design of Fano plane, and maximality of considering 2-subsets.

**Theorem 6.1**  $n = 7$  is not 6-distant 2-regular, that is, there doesn't exist a 6-distant hypergraph  $\mathcal{H} = (S \cup T, E)$  with  $T = \{8, 9, \dots, 17\}$

Proof: Since  $n = 7$ , thus  $S = [7]$ , let  $\lambda = 10$  and  $T = \{8, 9, \dots, 17\}$ .

Suppose there exists a 6-distant hypergraph  $\mathcal{H} = (S \cup T, E)$ . Using same method as deciding  $\lambda(6)$ , it holds that  $\lambda(7) \geq 10$ . As  $\lambda(7) = 10 = \frac{2|E|}{|\text{Fano plane}|}$ , using same method as lemma (5.17).i >, it holds that  $\{e \cap S | i \in e\}$  is a Fano Plane on  $[7]$ .

Consider on vertice subset  $(S \setminus \{7\}) \cup T \triangleq \tilde{S} \cup T$ , the induced sub-hypergraph is  $\tilde{\mathcal{H}} = (\tilde{S} \cup T, \tilde{E})$ . (Here obviously  $\tilde{E} = \{e \in E | 7 \notin e\}$ .) Then  $\tilde{\mathcal{H}}$  is 6-distant on  $\tilde{S} \cup T$  with  $n = 6$ . By Lemma (5.17),  $\mathcal{F}(\tilde{\mathcal{H}})$  is a balanced FHP set. By Theorem (5.28),  $\mathcal{F}(\tilde{\mathcal{H}})$  is induced by some perfect collections listed in Appendix B. That is,  $\mathcal{F}(\tilde{\mathcal{H}})$  is one of the balanced FHP sets listed in Appendix C. By Lemma (5.20),  $\forall F \in \mathcal{F}(\tilde{\mathcal{H}}), \bar{F} \in \mathcal{F}(\tilde{\mathcal{H}})$ .

Let  $\tilde{F} = \{e \cap S | 8 \in e, 7 \notin e\} \in \mathcal{F}(\tilde{\mathcal{H}})$ , then  $\bar{\tilde{F}} \in \mathcal{F}(\tilde{\mathcal{H}})$ , thus let  $\bar{\tilde{F}} = \{e \cap S | k \in e, 7 \notin e\}$  for some  $k \geq 9$ . Consider  $\hat{F} \triangleq \{e \cap S | 7, 8 \in e\} = F \setminus \tilde{F}$ . As  $F$  is a Fano Plane on  $[7]$ ,  $\tilde{F}$  is an FHP on  $[6]$ , by Fact (5.8),  $\hat{F} = \{\{i, j, 7\} | \{i, j\} \in P_{\tilde{F}}\}$ . That is,  $\{\{i, j, 7\} | \{i, j\} \in P_{\tilde{F}}\} \in \{e \cap S | 8 \in e\}$ . For the same reason,  $\{\{i, j, 7\} | \{i, j\} \in P_{\bar{\tilde{F}}}\} \in \{e \cap S | k \in e\}$ . By Fact (5.11),  $P_{\tilde{F}} = P_{\bar{\tilde{F}}}$ , so that  $\{\{i, j, 7\} | \{i, j\} \in P_{\tilde{F}}\} \in \{e \cap S | 8 \in e\} \cap \{e \cap S | k \in e\}$ . As  $|P_{\tilde{F}}| = 3$ , there exist three edges  $e_1, e_2, e_3 \supset \{7, 8, k\}$ . It contradicts to the 6-distant condition of  $\mathcal{H}$ . Hence  $\lambda(7) > 10 = \frac{2|E|}{A(7,4,3)}$ , it causes that 7 is not 6-distant 2-regular.

## 7. Conclusion and Open Conjectures

Although these works are on some small scales of  $n$ , it is shown that there are some kinds of patterns in the designs. By the past sections, the pattern on  $n = 5$  is the cycles on Peterson Graph, and pattern on  $n = 6$  is the FHP and perfect matchings on  $[6]$ . After we construct these designs with regular or beautiful patterns, we must prove that there isn't any other designs. Either we analyze on

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the rules of patterns, or compute the numbers of some items is effective to solve the problem.

On the other hand, although these results are solvable by computer searching, the computers always have a limit so that you cannot solve the problem as  $n \rightarrow \infty$ . The searching can only give us the result or design, but no theorems or lemmas about the rules of the patterns. These lemmas and theorems discovered by manual work can help us to find some propositions for arbitrary  $n$ . It is helpful for solving larger size of  $n$  without exhaustive searching, just like we apply (5.17).i to the case  $n = 7$ .

These results below are proved in our article in the past sections:

$n$	$\lambda(n)$	$A(n, 4, 3)$	Is 6-distant 2-regular	$N(n)$	$N(n)/\lambda(n)!$
3	2	1	True	1	1/2
4	8	1	True	2 520	1/16
5	10	2	True	21 772 800	6
6	10	4	True	21 772 800	6
7	$\geq 11$	7	False	N/A	N/A

**Conjecture 7.1** For integer  $n \geq 4$ ,  $n$  is 6–distant 2–regular iff  $n \not\equiv 1$  or  $3 \pmod{6}$ , which is equivalent to  $A(n, 4, 3) < \frac{n(n-1)}{6}$  by theorem (1.2).

The number  $\frac{n(n-1)}{6} - A(n, 4, 3)$  is the number of pairs not in any triples in  $[n]$ , which is called absent pairs as we discuss  $N(6)$ . As the method we used that in section 5, for even number  $n$ , it can be used to construct the absent pairing of  $[n]$ .

**Conjecture 7.2** For integer  $n \equiv 5$  or  $0 \pmod{6}$ , if  $n$  is 6–distant 2–regular, then  $6 \times \lambda(n)! \mid N(n)$ .

## A. Design examples

**A design that satisfies  $\lambda(5) = 10$**

$\{1, 2, 3; 6, 7\}, \{1, 2, 4; 11, 12\}, \{1, 2, 5; 8, 9\}, \{1, 3, 4; 13, 14\}, \{1, 3, 5; 11, 15\}$   
 $\{1, 4, 5; 6, 10\}, \{2, 3, 4; 9, 10\}, \{2, 3, 5; 12, 13\}, \{2, 4, 5; 14, 15\}, \{3, 4, 5; 7, 8\}$

**A design that satisfies  $\lambda(6) = 10$**

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$$\begin{array}{cccc}
 \{1, 2, 3; 9, 10\}, & \{4, 5, 6; 10, 15\}, & \{1, 2, 4; 8, 11\}, & \{3, 5, 6; 13, 16\} \\
 \{1, 2, 5; 14, 16\}, & \{3, 4, 6; 9, 11\}, & \{1, 2, 6; 13, 15\}, & \{3, 4, 5; 8, 10\} \\
 \{1, 3, 4; 15, 16\}, & \{2, 5, 6; 10, 11\}, & \{1, 3, 5; 7, 11\}, & \{2, 4, 6; 12, 16\} \\
 \{1, 3, 6; 12, 14\}, & \{2, 4, 5; 7, 9\}, & \{1, 4, 5; 12, 13\}, & \{2, 3, 6; 7, 8\} \\
 \{1, 4, 6; 7, 10\}, & \{2, 3, 5; 12, 15\}, & \{1, 5, 6; 8, 9\}, & \{2, 3, 4; 13, 14\}
 \end{array}$$

Generated by  $\mathcal{F}_{\mathcal{P}_a}$  and permutation (7, 8, 9, 10, 11, 12, 13, 14, 15, 16).

## B. All 6 perfect collections

Here sets are written in abbreviated form,

e.g.  $P_F = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\} = \{16, 25, 34\}$

$$\mathcal{P}_a = \{\{12, 34, 56\}, \{13, 25, 46\}, \{14, 26, 35\}, \{15, 24, 36\}, \{16, 23, 45\}\}$$

$$\mathcal{P}_b = \{\{12, 34, 56\}, \{13, 26, 45\}, \{14, 25, 36\}, \{15, 23, 46\}, \{16, 24, 35\}\}$$

$$\mathcal{P}_c = \{\{12, 35, 46\}, \{13, 24, 56\}, \{14, 25, 36\}, \{15, 26, 34\}, \{16, 23, 45\}\}$$

$$\mathcal{P}_d = \{\{12, 35, 46\}, \{13, 26, 45\}, \{14, 23, 56\}, \{15, 24, 36\}, \{16, 25, 34\}\}$$

$$\mathcal{P}_e = \{\{12, 36, 45\}, \{13, 24, 56\}, \{14, 26, 35\}, \{15, 23, 46\}, \{16, 25, 34\}\}$$

$$\mathcal{P}_f = \{\{12, 36, 45\}, \{13, 25, 46\}, \{14, 23, 56\}, \{15, 26, 34\}, \{16, 24, 35\}\}$$

## C. Induced Balanced FHP sets

Here sets are written in abbreviated form,

e.g.  $F = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\} = \{123, 145, 246, 356\}$ .

For one balanced FHP set, the FHPs are arranged like

$$\mathcal{F}_{\mathcal{P}} = \{F_1, F_2, F_3, F_4, F_5; \overline{F_1}, \overline{F_2}, \overline{F_3}, \overline{F_4}, \overline{F_5}\}$$

$$\mathcal{F}_{\mathcal{P}_a} = \{\{135, 146, 236, 245\}, \{124, 156, 236, 345\}, \{123, 156, 245, 346\}, \{123, 146, 256, 345\}, \{124, 135, 256, 346\};$$

$$\{136, 145, 235, 246\}, \{126, 145, 234, 356\}, \{125, 136, 234, 456\}, \{126, 134, 235, 456\}, \{125, 134, 246, 356\}\}$$

$$\mathcal{F}_{\mathcal{P}_b} = \{\{136, 145, 235, 246\}, \{124, 156, 235, 346\}, \{123, 156, 246, 345\}, \{124, 136, 256, 345\}, \{123, 145, 256, 346\};$$

$$\{135, 146, 236, 245\}, \{125, 146, 234, 356\}, \{126, 135, 234, 456\}, \{126, 134, 245, 356\}, \{125, 134, 236, 456\}\}$$

$$\mathcal{F}_{\mathcal{P}_c} = \{\{134, 156, 236, 245\}, \{125, 146, 236, 345\}, \{123, 156, 246, 345\}, \{123, 146, 245, 356\}, \{125, 134, 246, 356\};$$

$$\{136, 145, 234, 256\}, \{126, 145, 235, 346\}, \{126, 135, 234, 456\}, \{124, 136, 235, 456\}, \{124, 135, 256, 346\}\}$$

$$\mathcal{F}_{\mathcal{P}_d} = \{\{136, 145, 234, 256\}, \{125, 146, 234, 356\}, \{125, 136, 246, 345\}, \{123, 146, 256, 345\}, \{123, 145, 246, 356\};$$

$$\{134, 156, 236, 245\}, \{124, 156, 235, 346\}, \{126, 135, 245, 346\}, \{126, 134, 235, 456\}, \{124, 135, 236, 456\}\}$$

$$\mathcal{F}_{\mathcal{P}_e} = \{\{134, 156, 235, 246\}, \{126, 145, 235, 346\}, \{123, 156, 245, 346\}, \{126, 134, 245, 356\}, \{123, 145, 246, 356\};$$

$$\{135, 146, 234, 256\}, \{125, 146, 236, 345\}, \{125, 136, 234, 456\}, \{124, 136, 256, 345\}, \{124, 135, 236, 456\}\}$$

$$\mathcal{F}_{\mathcal{P}_f} = \{\{134, 156, 235, 246\}, \{124, 156, 236, 345\}, \{125, 136, 246, 345\}, \{124, 136, 235, 456\}, \{125, 134, 236, 456\};$$

$$\{135, 146, 234, 256\}, \{126, 145, 234, 356\}, \{126, 135, 245, 346\}, \{123, 146, 245, 356\}, \{123, 145, 256, 346\}\}$$

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