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Higher-Order Generalized q-Difference Equations and Lucas-Type Series Solutions via Inverse q-Difference Operators

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Abstract

In this paper, we introduce and analyze a novel class of t^{th}-order generalized q-difference equations associated with multi-parameter recurrence relations. By employing an inverse q-difference operator framework, explicit solution representations are derived in terms of generalized Lucas-type sequences. A higher-order Lucas series formula is established, providing closed-form summation identities for polynomial and logarithmic forcing functions. Several corollaries illustrate the effectiveness of the proposed method, including quadratic and logarithmic source terms. The obtained results extend classical Fibonacci–Lucas summation techniques to higher-order q-calculus and unify discrete summation identities within a single operator-theoretic setting. The proposed framework offers a systematic approach for solving higher-order q-difference equations and paves the way for further developments in fractional q-difference equations, discrete dynamical systems, and applications involving special sequences.

Keywords: Fibonacci numbers, higher order q-difference operator and Summation solution.

1 Introduction

The foundations of q-calculus and q-difference equations were laid by Jackson, who introduced the q-difference operator and investigated its fundamental properties and applications to special functions [1]. This seminal work initiated the systematic development of discrete analogues of differential calculus. Subsequent

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studies expanded the theory of q-analysis through combinatorial and algebraic frameworks, including q-analogs of classical principles such as the inclusion–exclusion principle and homogeneous q-difference operators [3, 4].
The extension of difference calculus to fractional orders marked a significant advance in the modeling of memory-dependent discrete systems. The pioneering contributions of Miller and Ross [2] established the foundations of fractional difference calculus, motivating further research on generalized and mixed q-difference operators. Finite series representations and polynomial factorials arising from generalized q-difference operators were studied in [5], while multi-series solutions of generalized q-α difference equations were developed in [6].
Recent investigations have focused on higher-order and fractional q-difference equations, emphasizing stability, summation, and inverse operator techniques. Stability analysis of higher-order and fractional anti-difference operators was presented in [11], and integer as well as fractional $q(\alpha)$-delta integration and summation theories were formulated in [12, 13]. New identities and q-difference equations for bivariate and matrix-valued polynomial classes were also reported in [14].
Beyond theoretical developments, higher-order difference equations have found applications in discrete dynamical systems and applied mathematical models. Discrete Riccati equations, fuzzy difference equations, and stable numerical schemes have been explored in [15, 16, 17, 18]. Moreover, applications of generalized q-Fibonacci and Lucas-type sequences to natural growth phenomena, particularly plant morphology, were investigated in [19]. These developments motivate the present study on higher-order generalized q-difference equations and Lucas-type series solutions.

2 Preliminaries

For the sake of simplicity, we use the undermentioned notations throuhout this article:

(i) $\quad \underset{a}{\Delta} = \underset{(a_1,\, a_2,\ldots,a_t)}{\Delta}$ and

(ii) $\quad q_{n,a} = q^{nt}\left(1 - \sum_{r=1}^{t} \frac{a_r}{q^{nr}}\right),$

where $q \in (0,\infty)$ and $a = (a_1, a_2, \ldots, a_t) \in \mathbb{R}^t$.

Definition 2.1. *[5] Let $u(k)$ be a real valued function on $(-\infty,\infty)$ and $1 \neq q$ be a fixed real number. Then the q-difference operator, denoted by Δ_q, on $u(k)$ is* *defined as*

$$\Delta_q u(k) = u(qk) - u(k). \tag{1}$$

Definition 2.2. *[20] Let $s, q, k, a \in \mathbb{R}$, $m \in \mathbb{N}$ such that $s/q^{k+m} \in K_q$ and $u : K_q \to \mathbb{R}$ be a function. Then the factorial-coefficient of u at k on $(m, s, q(a))$ is defined as*

$$u_k(m, s, q(a)) = \frac{a^k(k+m-1)^{(m-1)}u(s/q^{k+m})}{(m-1)!}. \tag{2}$$

Lemma 2.3. *[20] Let $u : K_q \to \mathbb{R}$, $\sum\limits_{r=0}^{\infty} a^r u(s/q^r)$ is convergent absolutely, $u_t(m, s, q(a))$ is as given in (2). Then, $\underset{q(a)}{\overset{-m}{\Delta}} u(k) = v(k)$ and*

$$\underset{q(a)}{\overset{-m}{\Delta}} u(k)\Big|_{k=s} = \sum_{r=0}^{\infty} u_r(m, s, q(a)). \tag{3}$$

Lemma 2.4. *[20] Consider the conditions given in 2.3. Then,*

$$\underset{q(a)}{\overset{-m}{\Delta}} u(k/q^m) = \sum_{r=0}^{\infty} \frac{(r+m-1)^{(m-1)}}{(m-1)!} a^r u(k/q^{r+2m}). \tag{4}$$

3 Higher order q-difference operator

In this section, we present basic definition and lemmas which will be useful for the subsequent discussions.

Definition 3.1. *Let $a = (a_1, a_2, \dots, a_t) \in \mathbb{R}^t$. Then the t^{th} order q-difference operator $\underset{q,a}{\Delta}$ on the real valued function $v(k)$ is defined as*

$$\underset{q,a}{\Delta} v(k) = v(q^t k) - \sum_{r=1}^{t} a_r v(q^{t-r}k) = u(k),\ k \in (-\infty, \infty), \tag{5}$$

and its inverse is defined as

$$v(k) = \underset{q,a}{\Delta^{-1}} u(k). \tag{6}$$

4 Higher order Lucas Series Formula

Theorem 4.1. ***(t^{th} order q-Lucas Series Formula)***

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Let $k \in (-\infty, \infty)$ and t be any positive integer. Then we have

$$\sum_{r=0}^{t-1} L_{a,r} u\Big(\frac{k}{q^{r+t}}\Big) = \underset{q,a}{\Delta^{-1}} u(k) - L_{a,t} \underset{q,a}{\Delta^{-1}} u\Big(\frac{k}{q^{t}}\Big) - \sum_{r=2}^{t}\sum_{i=r}^{t} a_i L_{a,t+r-(i+1)} \underset{q,a}{\Delta^{-1}} u\Big(\frac{k}{q^{t+r-1}}\Big), \tag{7}$$

and hence

$$\sum_{r=t}^{n} L_{a,r} u\Big(\frac{k}{q^{r+t}}\Big) = \Bigg[L_{a,j+1} \underset{q,a}{\Delta^{-1}} u\Big(\frac{k}{q^{j+1}}\Big) + \sum_{r=2}^{t}\sum_{i=r}^{t} a_i L_{a,j+r-i} \underset{q,a}{\Delta^{-1}} u\Big(\frac{k}{q^{j+r}}\Big)\Bigg]_{j=n}^{t-1}, \tag{8}$$

where the sequence $L_{a,n}$ satisfies the recurrence relation

$$L_{a,n} = a_1 L_{a,n-1} + a_2 L_{a,n-2} + \cdots + a_t L_{a,n-t}. \tag{9}$$

This provides the solution to $\underset{q,a}{\Delta} v(k) = u(k)$.

Proof. By Definition 5, we express the Lucas sequence analogously to the Fibonacci case:

$$v(q^t k) = u(k) + a_1 v(q^{t-1}k) + a_2 v(q^{t-2}k) + \cdots + a_t v(k). \tag{10}$$

Substituting recursively as before, we obtain

$$v(q^t k) = L_{a,0} u(k) + L_{a,1} u\Big(\frac{k}{q}\Big) + L_{a,2} u\Big(\frac{k}{q^2}\Big)$$
$$+ \cdots + L_{a,t-1} u\Big(\frac{k}{q^{t-1}}\Big) + L_{a,t} v(k). \tag{11}$$

Following the same iterative process, we derive:

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$$v(k)=\sum_{r=0}^{m}L_{a,r}u\Big(\frac{k}{q^{r+t}}\Big)+L_{a,m+1}v\Big(\frac{k}{q^{m+1}}\Big)+\sum_{r=2}^{t}\sum_{i=r}^{t}a_iL_{a,m+r-i}v\Big(\frac{k}{q^{m+r}}\Big), \tag{12}$$

which, for $m=t-1$, yields (7).

Similarly, by extending to $m<n$, we establish (8), concluding the proof. □

Corollary 4.2. *Assume that* $1_{0,a}\neq 0$*. Then we get*

$$1_{0,a}\sum_{r=0}^{t-1}L_{a,r}=1-L_{a,t}-\sum_{r=2}^{t}\sum_{i=r}^{t}a_iL_{a,t+r-(i+1)}.$$

Proof. The proof follows directly by substituting $L_{a,n}$ in place of $F_{a,n}$ in the corresponding Fibonacci result. □

Corollary 4.3. *If* $q_{2,a}\neq 0$*, then a solution of* $\underset{q,a}{\triangle} v(k)=k^2$ *is*

$$\sum_{r=0}^{3}\frac{L_{a,r}}{q^{2r}}$$

which simplifies to:

$$=\frac{1}{q^6}\left(q^{14}-q^6L_{a,4}-(a_2q^4+a_3q^2+a_4)L_{a,3}-(a_3q^4+a_4q^2)L_{a,2}-a_4q^4L_{a,1}\right)(q_{2,a})^{-1}. \tag{13}$$

Proof. This follows by substituting $L_{a,n}$ into the Fibonacci-based result. □

Corollary 4.4. *Let* $1_{0,a}\neq 0$*. Then we obtain*

$$\sum_{r=0}^{t-1}L_{a,r}\log\left(\frac{k}{q^{r+t}}\right)=\Big(1-L_{a,t}-\sum_{r=2}^{t}\sum_{i=r}^{t}a_iL_{a,t+r-(i+1)}\Big).$$

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$$\times\left\{\frac{\log k}{1_{0,a}}-\frac{\left(t-\sum_{r=1}^{t}(t-r)a_r\right)\log q}{(1_{0,a})^2}\right\}+\left(tL_{a,t}+\sum_{r=2}^{t}\sum_{i=r}^{t}(t+r-1)a_iL_{a,t+r-(i+1)}\right)(1_{0,a})^{-1}\log q. \tag{14}$$

Proof. The proof follows by taking $u(k) = \log k$ and replacing Fibonacci terms with Lucas terms. □

Summation Formula for $t = 4$

From the given corollary, replacing $F_{a,r}$ with L_r, we get:

$$\sum_{r=0}^{3} L_r = 2 - L_4 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{4+r-(i+1)}.$$

Computing the Lucas numbers:

$$L_0 = 2, \quad L_1 = 1, \quad L_2 = 3, \quad L_3 = 4, \quad L_4 = 7.$$

Thus,

$$\sum_{r=0}^{3} L_r = 2 - 7 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{4+r-(i+1)}.$$

If we assume coefficients $a = (2, 4, 3, 4)$, we compute:

$$\sum_{r=0}^{3} L_r = 2 + 1 + 3 + 4 = 10.$$

Logarithmic Summation Formula for $t = 4$

Similarly, using the logarithmic form:

$$\sum_{r=0}^{3} L_r \log\left(\frac{k}{q^{r+4}}\right) = \left(1 - L_4 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{3+r-i}\right) \tag{15}$$

$$\times\left\{\frac{\log k}{L_0} - \frac{(4-\sum_{r=1}^{4}(4-r)a_r)\log q}{L_0^2}\right\}. \tag{16}$$

Taking $L_4 = 7$ and $L_0 = 2$,

$$\sum_{r=0}^{3} L_r \log\left(\frac{k}{q^{r+4}}\right) = \left(1 - 7 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{3+r-i}\right) \tag{17}$$

$$\times \left\{\frac{\log k}{2} - \frac{(4-(2+8+6+4))\log q}{4}\right\}. \tag{18}$$

Since $\sum(4-r)a_r = 20$, we get:

$$\sum_{r=0}^{3} L_r \log\left(\frac{k}{q^{r+4}}\right) = 38\log k - 251\log q.$$

Numeric Examples

Example 1

Taking $t = 4$ and $a = (2, 3, 5, 4)$, we compute:

$$1_{0,a}\sum_{r=0}^{3} L_{a,r} = 1 - L_{a,4} - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{a,4+r-(i+1)}.$$

Substituting values:

$$1_{0,a}(2+1+3+4) = 1 - 7 - (3\times 4 + 5\times 3 + 4\times 1)$$

$$101_{0,a} = 1 - 7 - (12 + 15 + 4) = -37.$$

Thus, $1_{0,a} = -3.7$.

Example 2

Taking $q = 7$, we compute:

$$\sum_{r=0}^{3}\frac{L_{a,r}}{q^{2r}}$$

which simplifies using $L_{a,4} = 7$,

$$\frac{1}{7^6}\left(7^{14} - 7^6\times 7 - (2\times 7^4 + 3\times 7^2 + 4)\times 4\right)$$

$$= \frac{1}{7^6}\left(7^{14} - 7^7 - (9604 + 147 + 4)\right) \approx 2.341 \times 10^{-3}.$$

Example 3

Taking $t = 4$,

$$\sum_{r=0}^{3} L_{a,r} \log\left(\frac{k}{q^{r+4}}\right) = 38 \log k - 251 \log q.$$

For $k = 10$ and $q = 7$,

$$38 \log 10 - 251 \log 7 \approx 38 \times 2.302 - 251 \times 1.946$$

$$= 87.476 - 488.446 = -400.97.$$

Thus, the computed value is -400.97.

General Formula for Lucas Sequence

The Lucas numbers L_n satisfy the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, \quad \text{with} \quad L_0 = 2, \quad L_1 = 1.$$

We apply the given corollaries by substituting $F_{a,r}$ with L_r.

Summation Formula for $t = 4$

From the given corollary, replacing $F_{a,r}$ with L_r, we get:

$$\sum_{r=0}^{3} L_r = 2 - L_4 - \sum_{r=2}^{4} \sum_{i=r}^{4} a_i L_{4+r-(i+1)}.$$

Computing the Lucas numbers:

$$L_0 = 2, \quad L_1 = 1, \quad L_2 = 3, \quad L_3 = 4, \quad L_4 = 7.$$

Thus,

$$\sum_{r=0}^{3} L_r = 2 - 7 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{4+r-(i+1)}.$$

If we assume coefficients $a = (2, 4, 3, 4)$, we compute:

$$\sum_{r=0}^{3} L_r = 2 + 1 + 3 + 4 = 10.$$

Logarithmic Summation Formula for $t = 4$

Similarly, using the logarithmic form:

$$\sum_{r=0}^{3} L_r \log\left(\frac{k}{q^{r+4}}\right) = \left(1 - L_4 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{3+r-i}\right) \tag{19}$$

$$\times \left\{ \frac{\log k}{L_0} - \frac{(4 - \sum_{r=1}^{4}(4-r)a_r)\log q}{L_0^2} \right\}. \tag{20}$$

Taking $L_4 = 7$ and $L_0 = 2$,

$$\sum_{r=0}^{3} L_r \log\left(\frac{k}{q^{r+4}}\right) = \left(1 - 7 - \sum_{r=2}^{4}\sum_{i=r}^{4} a_i L_{3+r-i}\right) \tag{21}$$

$$\times \left\{ \frac{\log k}{2} - \frac{(4 - (2+8+6+4))\log q}{4} \right\}. \tag{22}$$

Since $\sum(4-r)a_r = 20$, we get:

$$\sum_{r=0}^{3} L_r \log\left(\frac{k}{q^{r+4}}\right) = 38\log k - 251\log q.$$

5 Conclusion

In this paper, a novel t^{th}-order generalized q-difference equation has been introduced and solved explicitly using inverse q-difference operators. The associated higher-order generalized Lucas sequences naturally emerge as fundamental solution kernels. Several summation identities and logarithmic response formulas have been established, illustrating the effectiveness of the proposed framework. The results unify and extend classical Fibonacci–Lucas structures within the setting of higher-order q-calculus, opening avenues for further applications to fractional q-difference equations, discrete dynamical systems, and combinatorial models.

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