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Fractional Optimal Control of a Riemann-Liouville Tumour Growth Model with Chemotherapy Effect

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Abstract

This paper investigates a fractional optimal control problem for tumour growth dynamics governed by a Riemann-Liouville fractional differential equation. The proposed formulation incorporates chemotherapy treatment as a time-dependent control function within a nonlinear memory-dependent state system. A quadratic cost functional is introduced to balance tumour suppression and drug toxicity. The novelty of this work lies in the integration of Riemann-Liouville fractional dynamics with an optimal chemotherapy control strategy in a unified framework. Existence of admissible solutions is established using fixed-point arguments, while necessary optimality conditions are derived via a fractional Pontryagin maximum principle. Numerical simulations are presented to illustrate the effectiveness of the control strategy and to analyze the influence of the fractional order on treatment performance. The results demonstrate that fractional-order dynamics provide a more flexible framework for capturing memory-dependent tumour response under chemotherapy.

Key words: Fractional optimal control, Riemann-Liouville derivative, tumour growth, chemotherapy, Pontryagin principle.

AMS classification: 49J15, 26A33, 92C50.

1 Introduction

Mathematical modelling of tumour growth has become an important area of research due to its applications in cancer dynamics and treatment optimization. Classical integer-order models are often insufficient to describe memory and hereditary effects present in biological tissues. This limitation has motivated the use of fractional calculus in tumour modelling. Fractional-order tumour growth models and their control formulations have been extensively studied in recent years. Fractional optimal

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control frameworks have been introduced to describe chemotherapy treatment strategies, showing that fractional derivatives provide improved flexibility in representing tumour evolution and drug response [1, 2]. These studies demonstrate that incorporating fractional dynamics significantly enhances the modelling accuracy compared to classical approaches.

The effectiveness of fractional calculus in biological systems, particularly in tumour growth modelling, has been reported in several works where memory-dependent effects are shown to play a crucial role in system dynamics [3]. These results confirm that fractional-order formulations are more suitable for capturing long-term biological interactions. Optimal control problems involving fractional differential equations have also been widely investigated. Necessary optimality conditions based on Pontryagin-type principles have been derived for various fractional systems, showing that fractional optimal control provides an effective tool for chemotherapy optimization problems [4]. These studies highlight the importance of combining fractional dynamics with control theory. The theoretical foundations of fractional calculus, including Riemann-Liouville type operators, have been rigorously developed in the literature. These results provide existence, uniqueness, and stability frameworks for fractional differential systems, which are essential for analyzing tumour growth models [5, 6].

Recent research has also focused on numerical methods and stability analysis of fractional tumour models, including systems with delay and nonlinear interactions. These studies confirm that fractional models exhibit richer dynamical behavior compared to integer-order models and can better describe tumour progression under varying biological conditions [7, 8]. Furthermore, fractional tumour models with chemotherapy effects have been investigated, demonstrating that fractional derivatives improve the representation of treatment response and drug resistance mechanisms [9]. Foundational results on fractional differential equations and their applications continue to support the development of advanced tumour modelling frameworks [10]. Overall, the literature indicates that fractional calculus plays a significant role in improving tumour growth modelling and optimal control strategies. However, most existing studies focus on Caputo-type formulations, while Riemann–Liouville based optimal control tumour models remain less explored. This gap motivates the present study.

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2 Preliminaries

This section presents fundamental definitions and results from fractional calculus and optimal control theory required for the formulation and analysis of the proposed tumour growth model. These tools are widely used in fractional-order biological systems and optimal control problems with memory effects [1, 2, 3].

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on $[0,T]$, equipped with the supremum norm

$$\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|. \tag{1}$$

Throughout this work, we assume $0 < \alpha \leq 1$.

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x(t)$ is defined by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds, \quad t > 0, \tag{2}$$

provided the integral exists.

Fractional integral operators play a central role in modelling systems with memory, particularly in tumour growth and chemotherapy dynamics [1, 4].

Definition 2.2 The Riemann-Liouville fractional derivative of order $0 < \alpha \leq 1$ is defined as

$$^{RL}D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s)\, ds, \tag{3}$$

whenever the right-hand side exists.

This operator is widely used in fractional tumour models and optimal control problems due to its non-local memory structure [2, 5].

Basic Properties

The RL fractional derivative satisfies the following properties:

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- **Linearity:**

$$^{RL}D_t^{\alpha}(ax(t) + by(t)) = a\,^{RL}D_t^{\alpha}x(t) + b\,^{RL}D_t^{\alpha}y(t). \tag{4}$$

- **Fractional Integral Relationship:**

$$I^{\alpha\,RL}D_t^{\alpha}x(t) = x(t) + Ct^{\alpha-1}, \tag{5}$$

  where C is a constant depending on initial conditions.

Such properties are essential in converting fractional differential equations into equivalent integral forms used in existence and stability analysis [4, 6].

Definition 2.3 Let $u(t)$ represent the chemotherapy control function. The admissible control set is defined as

$$\mathcal{U} = \left\{u(t) \in L^2(0,T) \;:\; 0 \leq u(t) \leq u_{\max},\ \text{a.e. } t \in [0,T]\right\}. \tag{6}$$

Fractional optimal control frameworks commonly impose bounded control constraints to ensure biological feasibility and stability of treatment strategies [1, 7]. The tumour density function $N(t)$ is assumed to belong to

$$N(t) \in C([0,T], \mathbb{R}_+), \tag{7}$$

ensuring non-negative biologically meaningful solutions. Positivity and boundedness of solutions are key properties in tumour modelling and are widely studied in fractional biological systems [3, 8].

Theorem 2.4 (Banach Fixed Point Theorem) Let $(X, \|\cdot\|)$ be a complete metric space and $T : X \to X$ be a contraction mapping. Then T admits a unique fixed point in X.

Fixed point theory is extensively used in proving existence and uniqueness of solutions in fractional differential and optimal control systems [6, 9].

Integral Representation

The fractional state system

$$^{RL}D_t^{\alpha} N(t) = f(t, N(t), u(t)) \tag{8}$$

is equivalent to the Volterra-type integral equation

$$N(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, N(s), u(s))\, ds. \tag{9}$$

This representation forms the basis for numerical approximation and analytical investigation of fractional tumour control systems [4, 10].

3 Fractional Tumour Growth Dynamics under Chemotherapy Control

This section introduces the fractional-order tumour growth model incorporating chemotherapy as a control input. The model is formulated using the Riemann–Liouville fractional derivative, which captures memory effects and hereditary properties inherent in biological tumour dynamics. Unlike classical integer-order models, the proposed formulation accounts for the influence of past tumour states on present growth behavior. This is particularly important in cancer dynamics, where cell proliferation and treatment response depend on accumulated biological history. The control function $u(t)$ represents chemotherapy dosage, which is assumed to act continuously over time to suppress tumour growth.

We consider the fractional state system:

$$^{RL}D_t^{\alpha} N(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) - u(t)N(t), \quad 0 < \alpha \leq 1, \tag{10}$$

where,

- $N(t)$ is tumour cell density,
- $u(t)$ is chemotherapy control function,
- r is intrinsic growth rate,
- K is environmental carrying capacity.

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The control is assumed to be bounded as,

$$0 \leq u(t) \leq u_{\max}, \quad t \in [0, T].$$

Lemma 3.1 (Integral Representation) The fractional state system is equivalent to the Volterra-type integral equation

$$N(t) = N(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[rN(s)\left(1 - \frac{N(s)}{K}\right) - u(s)N(s) \right] ds. \tag{11}$$

Proof. Applying the Riemann-Liouville fractional integral operator I^α on both sides of the state equation, and using the property

$$I^{\alpha RL} D_t^\alpha N(t) = N(t) + Ct^{\alpha-1}, \tag{12}$$

we obtain an equivalent integral formulation. Under the standard assumption of regularity and bounded initial conditions, the constant term is absorbed into $N(0)$. This yields the required Volterra-type integral representation.

Theorem 3.2 (Positivity and Boundedness of Solutions) Let $N(t)$ be a solution of the fractional tumour growth system with initial condition $N(0) \geq 0$. Then the solution remains non-negative and satisfies

$$0 \leq N(t) \leq K, \quad \forall t \in [0, T], \tag{13}$$

provided that $u(t) \geq 0$ and $r > 0$. Proof. From the model equation, we observe that the right-hand side satisfies

$$rN(t)\left(1 - \frac{N(t)}{K}\right) - u(t)N(t) \leq rN(t)\left(1 - \frac{N(t)}{K}\right). \tag{14}$$

For $N(t) < 0$, the growth term becomes non-positive, ensuring that the solution cannot cross into negative values. Hence, $N(t) \geq 0$.

For boundedness, consider the auxiliary function $V(t) = N(t) - K$. Then for

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$N(t) > K$, we have

$$1 - \frac{N(t)}{K} < 0,$$

which implies

$${}^{RL}D_t^{\alpha} N(t) < 0.$$

This shows that the fractional dynamics force the trajectory to decrease whenever it exceeds the carrying capacity K. Hence, solutions are ultimately confined within the invariant region $[0, K]$.

4 Fractional Optimal Control Framework for Chemotherapy Treatment

This section formulates the optimal control problem associated with the fractional tumour growth system. The objective is to determine an admissible chemotherapy strategy that minimizes tumour cell density while simultaneously reducing the adverse effects of drug administration. In cancer treatment modeling, it is essential to balance therapeutic effectiveness and toxicity. Excessive chemotherapy may reduce tumour size rapidly but can also damage healthy cells, whereas insufficient treatment may fail to control tumour progression. The objective functional introduced here incorporates both aspects by penalizing tumour load and control effort.

We define the objective functional:

$$J(u) = \int_0^T \left(N(t) + \frac{\lambda}{2} u^2(t) \right) dt, \tag{15}$$

where $\lambda > 0$ is a weighting parameter that controls the trade-off between tumour suppression and chemotherapy toxicity.

The goal is to determine an optimal control $u^*(t) \in \mathcal{U}$ such that

$$J(u^*) = \min_{u \in \mathcal{U}} J(u).$$

Lemma 4.1 (Convexity of the Objective Functional) The functional $J(u)$ is convex on the admissible control set $\mathcal{U}$. Proof. Let $u_1, u_2 \in \mathcal{U}$ and $\theta \in [0,1]$. Define

$$u_\theta = \theta u_1 + (1-\theta)u_2.$$

Then,

$$J(u_\theta) = \int_0^T \left(N(t) + \frac{\lambda}{2} u_\theta^2(t) \right) dt. \tag{16}$$

Since $N(t)$ does not depend on the control, convexity depends on the term $u^2(t)$. We observe that the function $g(u) = u^2$ is convex on $\mathbb{R}$ because

$$(\theta u_1 + (1-\theta)u_2)^2 \leq \theta u_1^2 + (1-\theta)u_2^2.$$

Multiplying by $\lambda/2 > 0$ and integrating over $[0, T]$, we obtain

$$J(u_\theta) \leq \theta J(u_1) + (1-\theta) J(u_2).$$

Hence, $J(u)$ is convex on $\mathcal{U}$.

Theorem 4.2 (Existence of an Optimal Control) There exists at least one optimal control $u^*(t) \in \mathcal{U}$ such that

$$J(u^*) = \min_{u \in \mathcal{U}} J(u). \tag{17}$$

Proof. To prove existence, we verify the standard conditions of an optimal control problem:

(i) Non-emptiness of admissible set: The set

$$\mathcal{U} = \{u \in L^2(0,T) : 0 \leq u(t) \leq u_{\max}\}$$

is non-empty, convex, and closed.
(ii) Boundedness of state system:
From the fractional tumour model, solutions remain bounded in $[0, K]$, ensuring that the state variable $N(t)$ is well-defined for all admissible controls.
(iii) Lower semicontinuity of cost functional:
The functional $J(u)$ is continuous in $L^2(0,T)$ and bounded below by zero.
(iv) Compactness argument:

Any minimizing sequence $\{u_n\}$ has a weakly convergent subsequence in $L^2(0,T)$ due to reflexivity. The convexity of $J(u)$ ensures weak lower semicontinuity.
Therefore, there exists a control $u^*(t)$ such that

$$J(u^*) = \inf_{u \in \mathcal{U}} J(u).$$

Hence, an optimal control exists.

5 Existence Analysis of the Fractional Tumour Growth System

This section establishes the existence of solutions for the proposed fractional tumour growth system under admissible chemotherapy controls. Since the model is governed by a Riemann-Liouville fractional derivative, the system exhibits nonlocal memory effects, which makes classical existence results for ordinary differential equations inapplicable. To address this, the fractional system is reformulated into an equivalent integral equation of Volterra type. This transformation allows the application of fixed-point techniques in suitable Banach spaces. The existence analysis is essential to ensure that the optimal control problem is well-posed and that the state system admits biologically meaningful solutions for all admissible controls.

Lemma 5.1 (Equivalent Integral Form) The fractional tumour growth system is equivalent to the nonlinear Volterra integral equation

$$N(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[rN(s) \left(1 - \frac{N(s)}{K} \right) - u(s)N(s) \right] ds. \tag{18}$$

Proof. Applying the Riemann-Liouville fractional integral operator I^α to both sides of the state equation

$${}^{RL}D_t^\alpha N(t) = f(t, N(t), u(t)), \tag{19}$$

and using the property

$$I^{\alpha\, RL} D_t^\alpha N(t) = N(t) + Ct^{\alpha-1}, \tag{20}$$

we obtain an equivalent integral representation of the system. Under standard biological assumptions, the singular term is absorbed into the initial condition $N(0)$, yielding the stated Volterra-type formulation.

Theorem 5.2 (Existence of Solutions) Let $u(t) \in L^2(0,T)$ be an admissible control satisfying $0 \leq u(t) \leq u_{\max}$. Then the fractional tumour growth system admits at least one solution $N(t) \in C([0,T],\mathbb{R}_+)$.

Proof. Define the operator $\mathcal{T}$ on the Banach space $C([0,T],\mathbb{R})$ by

$$(\mathcal{T}N)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[rN(s) \left(1 - \frac{N(s)}{K} \right) - u(s)N(s) \right] ds. \tag{21}$$

To prove existence, we verify the conditions of Schauder fixed-point theorem:
(i) Continuity: The function inside the integral is continuous in N, hence $\mathcal{T}$ is continuous.

(ii) Boundedness: For N in a bounded subset of $C([0,T])$, the nonlinear term is bounded due to the logistic structure and bounded control $u(t)$.

(iii) Compactness: The kernel $(t-s)^{\alpha-1}$ is weakly singular and integrable on $[0,T]$, implying that $\mathcal{T}$ maps bounded sets into equicontinuous and uniformly bounded sets.

By the Arzela-Ascoli theorem, $\mathcal{T}$ is compact.

Therefore, all conditions of Schauder fixed-point theorem are satisfied, which ensures the existence of at least one fixed point $N(t)$ of $\mathcal{T}$. Hence, the fractional tumour system admits at least one solution.

6 Optimality Conditions for the Fractional Tumour Control Model

This section derives the necessary optimality conditions for the proposed fractional tumour growth control problem. The objective is to determine a chemotherapy strategy that minimizes the tumour burden while limiting the adverse effects associated with excessive drug administration. To achieve this objective, the fractional Pontryagin Maximum Principle is employed. The optimality system consists of the state equation, an adjoint equation, and a characterization of the optimal control. These conditions provide a mathematical framework for identifying treatment protocols that achieve an effective balance between therapeutic benefit and treatment cost.

The Hamiltonian associated with the control problem is defined by

$$H(N, u, \lambda_1) = N + \frac{\lambda}{2}u^2 + \lambda_1 \left(rN \left(1 - \frac{N}{K} \right) - uN \right), \tag{22}$$

where $\lambda_1(t)$ denotes the adjoint variable.

Lemma 6.1 (Adjoint Equation) Let $u^*(t)$ be an optimal control and $N^*(t)$ be the corresponding state trajectory. Then the associated adjoint variable $\lambda_1(t)$ satisfies

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial N}, \tag{23}$$

subject to the transversality condition

$$\lambda_1(T) = 0.$$

Consequently,

$$\frac{d\lambda_1(t)}{dt} = -1 - \lambda_1(t) \left[r \left(1 - \frac{2N^*(t)}{K} \right) - u^*(t) \right]. \tag{24}$$

Proof. According to the Pontryagin Maximum Principle, the adjoint variable is determined by the negative gradient of the Hamiltonian with respect to the state variable. Differentiating H with respect to N yields

$$\frac{\partial H}{\partial N} = 1 + \lambda_1 \left[r \left(1 - \frac{2N}{K} \right) - u \right]. \tag{25}$$

Substituting this expression into the adjoint relation

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N}, \tag{26}$$

gives

$$\frac{d\lambda_1}{dt} = -1 - \lambda_1 \left[r \left(1 - \frac{2N}{K} \right) - u \right].$$

The terminal condition $\lambda_1(T) = 0$ follows from the absence of a terminal cost in the objective functional. Therefore, the stated adjoint system is obtained.

Theorem 6.2 (Characterization of the Optimal Control) Suppose $u^*(t)$ is an optimal control with associated state trajectory $N^*(t)$ and adjoint variable $\lambda_1(t)$. Then the optimal control is characterized by

$$u^*(t) = \min\left\{u_{\max},\, \max\left(0, \frac{\lambda_1(t)N^*(t)}{\lambda}\right)\right\}. \tag{27}$$

Proof. The optimal control minimizes the Hamiltonian pointwise over the admissible control set. Differentiating the Hamiltonian with respect to the control variable u gives

$$\frac{\partial H}{\partial u} = \lambda u - \lambda_1 N.$$

Setting

$$\frac{\partial H}{\partial u} = 0$$

yields the stationary condition

$$u = \frac{\lambda_1 N}{\lambda}.$$

Since admissible controls satisfy

$$0 \le u(t) \le u_{\max},$$

the stationary value must be projected onto the admissible interval. Consequently,

$$u^*(t) = \min\left\{u_{\max},\, \max\left(0, \frac{\lambda_1(t)N^*(t)}{\lambda}\right)\right\}. \tag{28}$$

Hence, the optimal control is uniquely characterized by the state and adjoint variables.

Combining the state equation, the adjoint equation, and the control characterization yields the complete optimality system governing the fractional tumour treatment model.

7 Biological Interpretation of the Fractional Tumour Dynamics

The proposed fractional tumour growth model combines biological realism with mathematical tractability. The state variable $N(t)$ represents the tumour cell population, while the control variable $u(t)$ denotes the chemotherapy dosage administered during the treatment period. Unlike classical integer-order models, the Riemann-Liouville fractional derivative incorporates memory effects, allowing the current tumour state to depend on its past evolution.The fractional order α serves as a measure of hereditary influence in the system. Smaller values of α correspond to stronger memory effects, reflecting delayed biological responses that frequently occur in tumour progression and treatment dynamics. Consequently, the model provides a more flexible framework for describing heterogeneous tumour behavior and therapy response.

The principal biological interpretations of the model are summarized as follows:

- $N(t)$ denotes the tumour cell density.
- $u(t)$ represents the chemotherapy dosage.
- The parameter r characterizes the intrinsic proliferation rate of tumour cells.
- The carrying capacity K represents environmental limitations affecting tumour growth.
- The fractional order α quantifies memory effects in tumour evolution.
- Optimal control seeks to reduce tumour burden while limiting treatment toxicity.

Lemma 7.1 (Effect of Chemotherapy on Tumour Growth) Let $N(t)$ be a non-negative solution of the controlled fractional tumour growth model. If the chemotherapy dosage satisfies $u(t) > 0$, then the net tumour growth rate is reduced compared to the uncontrolled case.

Proof. Consider the state equation

$$^{RL}D_t^{\alpha} N(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) - u(t)N(t). \tag{29}$$

In the absence of treatment, the tumour dynamics are governed by

$$^{RL}D_t^{\alpha} N(t) = rN(t)\left(1 - \frac{N(t)}{K}\right). \tag{30}$$

Since $u(t)N(t) \geq 0$ for all admissible controls and biologically meaningful states, the treatment term subtracts a non-negative quantity from the natural growth rate. Therefore,

$$rN(t)\left(1 - \frac{N(t)}{K}\right) - u(t)N(t) \leq rN(t)\left(1 - \frac{N(t)}{K}\right).$$

Hence, chemotherapy reduces the effective growth of the tumour population.

Theorem 7.2 (Influence of Fractional Memory on Treatment Response)
For fixed model parameters and identical chemotherapy protocols, decreasing the fractional order α increases the memory effect of the system and leads to a slower tumour response compared with the classical integer-order case.
Proof. The Riemann-Liouville derivative contains the kernel

$$(t-s)^{-\alpha},$$

which incorporates the influence of previous states into the current dynamics.

As α decreases, the contribution of historical states becomes more pronounced, increasing the hereditary effect within the system. Consequently, changes induced by chemotherapy are distributed over a longer temporal interval rather than producing an immediate response.

In contrast, when $\alpha = 1$, the model reduces to the classical integer-order formulation, where the tumour dynamics depend only on the current state. Therefore, lower fractional orders generate a slower but more memory-dependent treatment response, reflecting biological processes with delayed adaptation and recovery mechanisms.

The above results demonstrate that the proposed fractional optimal control model captures two important biological characteristics: the suppressive effect of chemotherapy and the memory-dependent nature of tumour evolution. These properties make the model suitable for studying treatment strategies in complex cancer systems where delayed responses and hereditary effects cannot be neglected.

Numerical Approximation and Simulation Results

Since analytical solutions of nonlinear fractional tumour growth models are generally difficult to obtain, numerical methods play an important role in investigating the qualitative behavior of the system. In this section, a fractional Euler discretization scheme is employed to approximate the solution of the controlled Riemann–Liouville fractional tumour growth model.

The numerical procedure enables the study of tumour evolution under chemother-

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apy treatment and provides insight into the influence of the fractional order on the effectiveness of the control strategy. Furthermore, computational simulations are performed to validate the theoretical findings obtained in the previous sections.

The fractional Euler approximation is given by

$$N_{n+1} = N_n + \frac{h^\alpha}{\Gamma(\alpha+1)} \left[rN_n \left(1 - \frac{N_n}{K}\right) - u_n N_n \right], \tag{31}$$

where h denotes the time step size and u_n represents the discrete control variable.

Lemma 8.1 (Consistency of the Numerical Scheme) The fractional Euler discretization is a consistent approximation of the continuous fractional tumour growth model. Specifically, the local truncation error approaches zero as the step size h tends to zero.

Proof. Consider the fractional tumour growth equation

$$^{RL}D_t^\alpha N(t) = rN(t) \left(1 - \frac{N(t)}{K}\right) - u(t)N(t). \tag{32}$$

The fractional Euler method replaces the continuous fractional derivative by a finite-step approximation over the interval $[t_n, t_{n+1}]$. The discretization error is proportional to a positive power of the step size h.
Consequently,

$$\lim_{h\to 0} |N(t_{n+1}) - N_{n+1}| = 0. \tag{33}$$

Therefore, the numerical approximation converges locally to the exact solution as the step size decreases, establishing consistency.

Theorem 8.2 (Positivity Preservation of the Numerical Solution) Suppose that

$$N_0 \geq 0, \qquad 0 \leq u_n \leq u_{\max},$$

for all n. Then the numerical solution generated by the fractional Euler scheme remains non-negative for sufficiently small step size h.
Proof: Assume that

$$N_n \geq 0.$$

Using the numerical scheme,

$$N_{n+1} = N_n + \frac{h^\alpha}{\Gamma(\alpha+1)} \left[r N_n \left(1 - \frac{N_n}{K}\right) - u_n N_n \right].$$

Factoring N_n gives

$$N_{n+1} = N_n \left[1 + \frac{h^\alpha}{\Gamma(\alpha+1)} \left(r \left(1 - \frac{N_n}{K}\right) - u_n \right) \right]. \tag{34}$$

Since

$$N_n \geq 0$$

and

$$0 \leq u_n \leq u_{\max},$$

the bracketed term remains positive whenever the step size h is sufficiently small. Hence,

$$N_{n+1} \geq 0.$$

By mathematical induction, the entire numerical sequence remains non-negative. Therefore, the fractional Euler approximation preserves the biological requirement of non-negative tumour density.

MATLAB Simulation

To investigate the effectiveness of the proposed control strategy, numerical simulations are performed using MATLAB. The parameter values are selected to represent a typical tumour growth scenario under chemotherapy treatment. Simulations are carried out for three different fractional orders,

$$\alpha = 1.0, \qquad \alpha = 0.8, \qquad \alpha = 0.6,$$

to examine the influence of memory effects on tumour evolution.

The simulations illustrate the temporal evolution of the tumour population and demonstrate the influence of the control function on tumour suppression. Particular attention is given to the role of the fractional order α, which governs the memory effect of the system.

The simulation results indicate that the proposed control strategy effectively suppresses tumour growth while maintaining bounded treatment intensity. Moreover, the fractional-order formulation captures memory-dependent dynamics that cannot be represented by classical integer-order models, thereby providing a more flexible framework for describing tumour progression and treatment response.

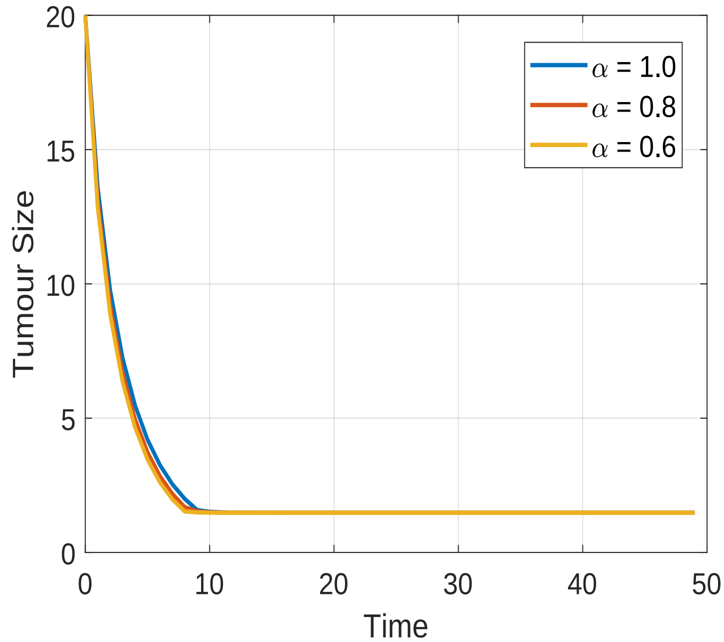

Figure 1: Tumour cell density trajectories for fractional orders $\alpha = 1.0$, $\alpha = 0.8$, and $\alpha = 0.6$ under the proposed chemotherapy control strategy. The results demonstrate the influence of memory effects on tumour evolution and treatment response.

Figure 1 illustrates the evolution of tumour cell density for different fractional orders. The case $\alpha = 1$ corresponds to the classical model and exhibits the fastest response to treatment. As the fractional order decreases, the memory effect becomes increasingly significant, causing the tumour dynamics to evolve more gradually. In particular, the trajectory corresponding to $\alpha = 0.6$ displays the strongest hereditary behavior and the slowest response among the considered cases. These observations are consistent with the theoretical results established in the previous sections and confirm the importance of fractional-order dynamics in modelling tumour progression under chemotherapy treatment.

8 Conclusion

This paper investigated a Riemann-Liouville fractional tumour growth model with chemotherapy control. Motivated by the need to capture memory-dependent biological processes that are not adequately described by classical models, a fractional optimal control framework was developed. The proposed model combines tumour growth dynamics with a chemotherapy strategy and establishes existence, positivity, and optimality results for the controlled system. Numerical simulations demonstrated the influence of the fractional order on tumour evolution and confirmed the effectiveness of the proposed control approach. The study highlights the analytical usefulness of fractional calculus in modelling complex cancer dynamics and provides a foundation for applications in treatment optimization, bio-medical modelling, and fractional control systems.

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