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Existence, Stability, and Numerical Analysis of a Caputo Fractional Population Model

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Abstract

Fractional-order population models have attracted considerable attention due to their capability to describe memory-dependent biological processes more effectively than classical integer-order systems. Motivated by the limitations of traditional population growth equations in capturing hereditary effects, this paper investigates a fractional logistic population model formulated in the Caputo sense. The proposed model incorporates nonlocal memory characteristics, which play an important role in realistic biological and ecological systems. Analytical properties of the model are studied by establishing existence and uniqueness results through fixed point theory. In addition, equilibrium analysis and asymptotic stability conditions are derived for the fractional-order system. The study further demonstrates how the fractional parameter influences the growth dynamics and convergence behavior of the population. Numerical simulations and graphical illustrations are presented to validate the theoretical findings and to compare the fractional and classical models. The obtained results indicate that the Caputo fractional framework provides a more flexible and generalized approach for modeling population evolution with memory effects. The proposed methodology may also be extended to epidemic systems, predator-prey interactions, and other nonlinear biological models governed by hereditary phenomena.

Key words: Caputo derivative, fractional calculus, population dynamics, logistic model, stability analysis.
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1 Introduction

Fractional calculus has emerged as an effective mathematical framework for describing systems with memory and hereditary characteristics. Recent studies have shown that fractional-order differential equations provide more generalized and realistic representations than classical integer-order models, particularly in biological and population dynamics [1, 2].

Several researchers have investigated the application of Caputo fractional derivatives in epidemic and biological systems. Fractional epidemic models involving Caputo operators have demonstrated improved accuracy in capturing transmission dynamics and long-term memory effects [3, 4, 7, 8, 12, 13]. Stability analysis and qualitative behavior of fractional biological systems have also been extensively studied using Lyapunov techniques and fixed point approaches [3, 5].

The influence of fractional parameters on the dynamical behavior of biological populations has attracted considerable attention in recent years. Fractional population models have been shown to exhibit richer dynamical structures compared with classical logistic equations [6, 10, 11]. In particular, memory-dependent growth behavior and nonlocal interactions can be effectively represented through Caputo fractional operators.

Fractional-order models have additionally been applied in environmental and ecological systems involving human populations, forest biomass, and atmospheric interactions [9]. Comparative analyses involving Caputo, Caputo-Fabrizio, and Atangana-Baleanu operators further indicate the importance of selecting suitable fractional derivatives for modeling complex biological processes [12].

Existing studies confirm that fractional logistic and epidemic models provide improved flexibility and analytical capability for describing nonlinear biological systems. However, further investigation concerning the existence, stability, and numerical behavior of Caputo fractional population models remains an active area of research. Motivated by these observations, the present work focuses on the analytical and numerical study of a Caputo fractional logistic population model.

2 Preliminaries

In this section, some basic definitions and results related to fractional calculus are presented.

Definition 2.1 The Caputo fractional derivative of order α $(0 < \alpha < 1)$ is defined by

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \tag{1} $$

Definition 2.2 [1, 2] The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2} $$

3 Fractional Population Model

Population growth models play an important role in mathematical biology and ecology. Classical logistic equations describe the evolution of a population under limited environmental resources. However, integer-order models fail to capture memory and hereditary characteristics that naturally arise in biological systems. To overcome this limitation, a fractional-order logistic population model involving the Caputo derivative is considered.

The fractional logistic population equation is given by

$$ {}^{C}D_t^{\alpha} P(t) = rP(t)\left(1 - \frac{P(t)}{K}\right), \quad 0 < \alpha \leq 1, \tag{3} $$

subject to the initial condition

$$ P(0) = P_0 > 0, \tag{4} $$

where:

- $P(t)$ denotes the population density at time t,
- $r > 0$ represents the intrinsic growth rate,
- $K > 0$ denotes the carrying capacity,
- α is the fractional order parameter.

The parameter α characterizes the memory effect of the system. Smaller values of α correspond to stronger memory influence on population growth dynamics.
When $\alpha = 1$, the model reduces to the classical logistic equation

$$\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right). \tag{5}$$

The fractional-order formulation generalizes the classical logistic model and provides a more realistic description of biological populations with hereditary behavior.
Applying the Riemann-Liouville fractional integral operator to the system yields the equivalent Volterra integral equation

$$P(t) = P_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} rP(\tau)\left(1 - \frac{P(\tau)}{K}\right) d\tau. \tag{6}$$

The above integral representation is useful in studying the existence, uniqueness, and stability properties of the fractional population model.

Lemma 3.1 The function

$$f(P) = rP\left(1 - \frac{P}{K}\right) \tag{7}$$

is continuous on $\mathbb{R}^+$.
Proof: Since $f(P)$ is a polynomial function in P, it is continuous for all $P \in \mathbb{R}^+$. Hence the result follows.

Lemma 3.2 The nonlinear function

$$f(P) = rP\left(1 - \frac{P}{K}\right) \tag{8}$$

satisfies the local Lipschitz condition on every bounded interval.

Proof: Let $P_1, P_2 \in [0, M]$ for some $M > 0$. Then,

$$\begin{aligned} |f(P_1) - f(P_2)| &= \left| rP_1\left(1 - \frac{P_1}{K}\right) - rP_2\left(1 - \frac{P_2}{K}\right)\right| \\ &= r\left|(P_1 - P_2) - \frac{1}{K}(P_1^2 - P_2^2)\right| \\ &= r\left|(P_1 - P_2)\left(1 - \frac{P_1 + P_2}{K}\right)\right|. \end{aligned} \tag{9}$$

Since $P_1, P_2 \in [0, M]$, there exists a positive constant L such that

$$|f(P_1) - f(P_2)| \leq L|P_1 - P_2|. \tag{10}$$

Therefore, $f(P)$ satisfies the local Lipschitz condition.

Theorem 3.3 The fractional logistic population model possesses at least one solution on the interval $[0, T]$. Proof: Consider the operator

$$(\mathcal{T}P)(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(P(\tau)) d\tau. \tag{11}$$

Since $f(P)$ is continuous, the operator $\mathcal{T}$ maps continuous functions into continuous functions. Furthermore, the operator is bounded and equicontinuous on bounded subsets of $C[0, T]$.
By the Arzela-Ascoli theorem, $\mathcal{T}$ is compact. Hence, Schauder's fixed point theorem guarantees the existence of at least one solution for the fractional population system.

Theorem 3.4 The solution of the fractional logistic population model is unique on $[0, T]$. Proof: Let $P_1(t)$ and $P_2(t)$ be two solutions of the system. Then,

$$|P_1(t) - P_2(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} |f(P_1(\tau)) - f(P_2(\tau))| d\tau. \tag{12}$$

Using the Lipschitz condition,

$$|P_1(t) - P_2(t)| \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} |P_1(\tau) - P_2(\tau)| d\tau. \tag{13}$$

Applying the fractional Gronwall inequality yields

$$|P_1(t) - P_2(t)| = 0. \tag{14}$$

Hence,

$$P_1(t) = P_2(t),$$

which proves the uniqueness of the solution.

Corollary 3.5 The solution of the fractional logistic population model continuously depends on the initial condition P_0.
Proof: The result follows directly from the uniqueness theorem and the fractional Gronwall inequality.

Theorem 3.6 All solutions of the fractional logistic population model remain positive for all $t > 0$ whenever $P_0 > 0$. Proof: Suppose there exists $t_1 > 0$ such that $P(t_1) = 0$. Since the solution is continuous and $P(0) > 0$, there exists a smallest time t_1 satisfying this property. From the integral representation,

$$P(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} r P(\tau) \left(1 - \frac{P(\tau)}{K}\right) d\tau. \tag{15}$$

Because $P(\tau) \geq 0$ for $0 \leq \tau \leq t_1$, the integral term is nonnegative. Therefore,

$$P(t_1) \geq P_0 > 0, \tag{16}$$

which contradicts the assumption that $P(t_1) = 0$. Hence, all solutions remain positive.

4 Existence and Uniqueness of Solution

In this section, the existence and uniqueness of the solution for the proposed Caputo fractional logistic population model are established. The analysis is based on fixed point theory. First, a useful lemma is presented to show that the nonlinear term satisfies the Lipschitz condition. Subsequently, the Banach fixed point theorem is employed to prove the existence and uniqueness of the solution.

Lemma 4.1 Let

$$f(t,P) = rP\left(1-\frac{P}{K}\right),$$

where $r > 0$ and $K > 0$. Then f satisfies the Lipschitz condition with respect to P on every bounded interval. Proof. Let $P_1, P_2 \in [0,M]$ for some positive constant M. Then

$$\begin{aligned} |f(t,P_1) - f(t,P_2)| &= r\left|P_1\left(1-\frac{P_1}{K}\right) - P_2\left(1-\frac{P_2}{K}\right)\right| \\ &= r\left|(P_1-P_2) - \frac{1}{K}(P_1^2 - P_2^2)\right| \\ &= r|P_1-P_2|\left|1-\frac{P_1+P_2}{K}\right|. \end{aligned} \tag{17}$$

Since $P_1, P_2 \in [0,M]$, there exists a positive constant

$$L = r\left(1+\frac{2M}{K}\right)$$

such that

$$|f(t,P_1) - f(t,P_2)| \le L|P_1 - P_2|.$$

Hence, f satisfies the Lipschitz condition on every bounded interval.

Theorem 4.2 Let $f(t,P) = rP\left(1-\frac{P}{K}\right)$ be continuous and satisfy the Lipschitz condition with respect to P. Then the fractional logistic equation possesses a unique solution on $[0,T]$.

Proof. Applying the fractional integral operator to the given equation yields

$$P(t) = P_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, P(\tau))\,d\tau. \tag{18}$$

Define the operator

$$(\mathcal{T}P)(t) = P_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, P(\tau))\,d\tau. \tag{19}$$

From Lemma (4.1), the function f satisfies the Lipschitz condition. Therefore,

$$|f(t,P_1) - f(t,P_2)| \le L|P_1 - P_2|.$$

Consequently,

$$\|\mathcal{T}P_1 - \mathcal{T}P_2\| \leq \frac{LT^\alpha}{\Gamma(\alpha+1)}\|P_1 - P_2\|.$$

If

$$\frac{LT^\alpha}{\Gamma(\alpha+1)} < 1,$$

then $\mathcal{T}$ is a contraction mapping on the Banach space $C([0,T])$.
Hence, by the Banach Fixed Point Theorem, the operator $\mathcal{T}$ possesses a unique fixed point. Therefore, the fractional logistic population model admits a unique solution on $[0,T]$.

5 Stability Analysis

The investigation of equilibrium states and their stability is essential for understanding the qualitative behavior of fractional population models. Stability analysis determines whether a population returns to its equilibrium level after small perturbations and provides insight into the effects of memory on population evolution. In this section, the equilibrium points of the Caputo fractional logistic model are obtained, and the asymptotic stability of the biologically meaningful equilibrium is established. The obtained results demonstrate that the fractional-order parameter influences the rate at which the population approaches its steady-state value.

Lemma 5.1 The equilibrium points of the fractional logistic equation are

$$P_1^* = 0, \qquad P_2^* = K. \tag{20}$$

Proof. At equilibrium,

$${}^C D_t^\alpha P(t) = 0.$$

Thus,

$$rP\left(1 - \frac{P}{K}\right) = 0.$$

Therefore,

$$P = 0 \quad \text{or} \quad P = K.$$

Hence the equilibrium points are obtained.

Theorem 5.2 The equilibrium point $P^* = K$ is asymptotically stable.

Proof. Consider a perturbation

$$P(t) = K + \epsilon(t),$$

where $|\epsilon(t)|$ is small. Substituting into the model gives

$${}^{C}D_t^{\alpha}\epsilon(t) = -r\epsilon(t).$$

The solution is expressed using the Mittag-Leffler function:

$$\epsilon(t) = \epsilon_0 E_\alpha(-rt^\alpha).$$

Since

$$E_\alpha(-rt^\alpha) \to 0 \quad \text{as} \quad t \to \infty,$$

the perturbation vanishes. Therefore, $P^* = K$ is asymptotically stable.

Corollary 5.3 For $0 < \alpha < 1$, the convergence toward equilibrium is slower than the classical integer-order case. Proof. The Mittag-Leffler function decays slower than the exponential function associated with classical differential equations. Therefore, the memory effect introduced by the fractional derivative slows the convergence rate.

6 Numerical Investigation of the Fractional Population Model

This section presents numerical simulations of the proposed Caputo fractional logistic population model to examine the influence of the fractional-order parameter on population growth dynamics. The simulations are performed for different values of α while maintaining fixed biological parameters. The initial population and model parameters are selected as

$$r = 0.5, \qquad K = 100, \qquad P(0) = 10.$$

The fractional-order parameter is varied to investigate the impact of memory effects on the evolution of the population. The obtained numerical results are illustrated in Fig. 1.

Figure 1 illustrates the temporal evolution of the population for several values

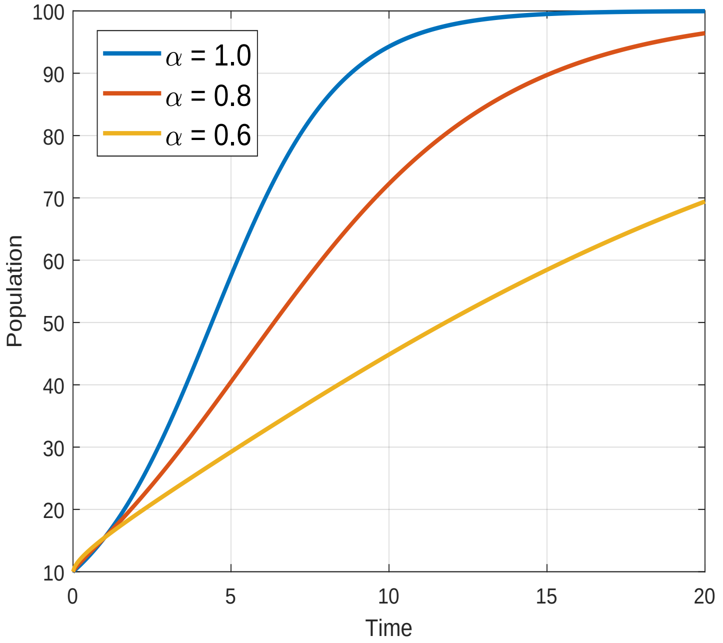

Figure 1: Population trajectories of the Caputo fractional logistic model for different fractional orders α.

of the fractional-order parameter. It can be observed that all solution trajectories remain positive and eventually approach the carrying capacity $K = 100$, which is consistent with the theoretical stability results established in the previous section.

Furthermore, the fractional-order parameter has a significant influence on the growth rate of the population. For values of α closer to unity, the population approaches the carrying capacity more rapidly, exhibiting behavior similar to the classical logistic model. In contrast, smaller values of α produce slower convergence due to the stronger memory effects introduced by the fractional derivative.

The results indicate that the Caputo fractional model provides additional flexibility in describing population growth processes. The incorporation of memory effects enables the model to capture gradual evolutionary behavior that cannot be represented adequately by classical integer-order differential equations. Consequently, fractional-order models offer a more realistic mathematical framework for studying biological populations characterized by hereditary and nonlocal interactions.

7 Conclusion

In this paper, a Caputo fractional logistic population model was investigated. Existence and uniqueness of solutions were established using Banach fixed point theory. Stability analysis showed that the carrying-capacity equilibrium is asymptotically stable. Numerical simulations demonstrated the influence of the fractional-order parameter on population growth.

The study confirms that fractional calculus provides a useful mathematical framework for biological systems exhibiting memory and hereditary properties. Future work may include predator-prey systems, epidemic models, and stochastic fractional population dynamics.

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