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Super Mean Graph Labeling: A Novel Cryptographic Framework Using Five-Star Graphs with Applications in Secure Communications and Epidemic Modeling

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Abstract

This paper introduces an innovative cryptographic framework based on super mean labeling of five-star graphs $K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5}$, where $V_1 \leq V_2 \leq V_3 \leq V_4 \leq V_5$. We present a comprehensive mathematical foundation for super mean labeling and develop systematic methodologies for encoding messages through carefully constructed graph structures. Three distinct implementation approaches are demonstrated, incorporating computational techniques through C programming for alphabetical mapping, including subtraction-based and division-based numbering schemes. Each approach is thoroughly illustrated with complete message encoding examples, visual cryptography representations, and detailed security analysis. Beyond cryptographic applications, we extend the framework to epidemic modeling, demonstrating how super mean labeling can represent complex disease transmission dynamics in multi-population systems. The epidemic modeling application includes complete mathematical formulations of multi-population SIR models, transmission matrix encoding through graph labeling, and computational implementations for disease surveillance. Our results establish that five-star graphs with super mean labeling provide optimal balance between encoding capacity and structural complexity, offering robust security through multiple layers of mathematical obfuscation. The integration of graph theory with computational algorithms creates a versatile framework applicable to both secure communications and public health informatics.

Key words : Super Mean Labeling, Encoding and Decoding, Five Star
AMS classification: 05C78 and 94A60

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1. Introduction

Graph labeling is a vibrant area of graph theory where vertices or edges are assigned labels according to specific conditions, with applications in coding theory, network design, and more [1]. Among various labeling techniques, mean labeling has garnered significant attention, where edge labels are derived as the mean of adjacent vertex labels [21, 22, 20]. Super mean labeling extends this concept by requiring vertex labels to be distinct integers from 0 to the number of vertices, with edge labels being the ceiling of the average and also distinct [2, 3, 15].

Research on super mean labeling has explored its existence and properties across various graph classes. For instance, studies have investigated super mean labeling on star graphs, identifying conditions for existence and non-existence [19]. Further results on super mean graphs include analyses of H-graphs, coronas, and subdivisions [14, 16]. Variants such as k-super mean labeling and super root square mean labeling have also been introduced to generalize the framework [17, 23].

Applications of super mean labeling in coding techniques have been particularly promising. Previous works have demonstrated coding methods using two-star and three-star graphs with super mean labeling [6, 7, 8, 9, 10, 11, 12, 13]. Additionally, integrations with difference cordial labeling and other cordial variants have expanded the coding capabilities [4, 5, 25, 26].

Recent investigations have delved into related mean labelings, such as super Lehmer-3 mean labeling for theta graphs and super classical mean labeling [24, 18]. These advancements provide a foundation for more complex labeling schemes.

In this paper, two strategies for coding messages employing five stars on a graph with super mean labeling, $K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5}$, $V_1 \leq V_2 \leq V_3 \leq V_4 \leq V_5$ are presented. In order to use the labeling and the graphs together, methods for repairing the super mean labeling on any graph with five star graphs are given after a few observations are performed in order to assign them a super mean designation. Rules for assigning the really cruel label on five star graphs are provided. Procedure for encoding the message is stated prior to illustrations.

To ensure a clear understanding of the labeling process, we first outline a practical rule of thumb for applying super mean labeling to five-star graphs [7].

This paper is structured as follows: Section 2 establishes the comprehensive theoretical foundation, presenting formal definitions of graph structures, star graph hierarchies, and the super mean labeling framework. Section 3 develops the systematic methodology for five-star graph labeling and introduces the GMJ coding approach. Section 4 details the practical implementation framework with exemplar configurations and encoding protocols. Section 5 presents three comprehensive illustrations demonstrating cryptographic applications through distinct encoding schemes integrated with computational algorithms. Section 6 extends the framework to epidemic modeling, providing mathematical formulations for multi-population SIR models, transmission matrix encoding, and disease surveillance applications. Finally, Section 7 concludes with a summary of contributions and outlines promising directions for future research, including extensions to more complex graph structures and advanced cryptographic-epidemiological integrations.

2. Theoretical Foundations And Mathematical Framework

This section establishes the comprehensive mathematical foundation upon which our super mean labeling methodology is constructed. We present formal definitions, theoretical constructs, and systematic procedures that form the basis for our cryptographic encoding framework using five-star graphs.

2.1 Graph-Theoretic Foundations

2.1.1 Basic Graph Theory Concepts

Definition 2.1 (Graph Structure) A graph G is formally defined as an ordered triple $(V(G), E(G), \varphi_G)$, where:

- $V(G)$ represents a nonempty set of vertices
- $E(G)$ denotes a set of edges disjoint from $V(G)$
- $\varphi_G : E(G) \to V(G) \times V(G)$ is an incidence function that associates each edge with an unordered pair of vertices

For any edge $e \in E(G)$ connecting vertices a and b, we denote a and b as the ends of e. This fundamental definition provides the structural basis for all subsequent graph labeling schemes [1].

2.1.2 Star Graph Hierarchies

The star graph hierarchy forms the structural backbone of our encoding methodology, with each level providing increased complexity and cryptographic capacity.

Definition 2.2 (One-Star Graph (K_{1,V_1})) A one-star graph, denoted K_{1,V_1}, represents the fundamental building block of our framework. It is characterized as:

- A tree structure with exactly one interior node (central vertex)
- V_1 leaves (pendant vertices) connected exclusively to the central vertex
- A complete bipartite graph K_{1,V_1} with bipartition $\{c\} \cup L$, where c is the central vertex and L is the set of V_1 leaves

This structure provides the basic unit for constructing more complex graph configurations [1].

Definition 2.3 (Two-Star Graph ($K_{1,V_1} \cup K_{1,V_2}$)) A two-star graph extends the basic structure through the disjoint union of two one-star graphs:

$$K_{1,V_1} \cup K_{1,V_2} = (V_1 \cup V_2, E_1 \cup E_2)$$

where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$. This configuration doubles the encoding capacity while maintaining structural simplicity [1].

Definition 2.4 (Three-Star Graph ($K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3}$)) The three-star graph represents a significant expansion in complexity:

$$K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} = \bigcup_{i=1}^{3} (V_i, E_i)$$

This structure enables more sophisticated encoding patterns and provides enhanced cryptographic security through increased vertex distribution [1].

Definition 2.5 (Four-Star Graph ($K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4}$)) The four-star graph configuration offers substantial encoding capacity:

$$K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} = \bigcup_{i=1}^{4} (V_i, E_i)$$

This structure supports complex message encoding while maintaining manageable computational complexity [1].

Definition 2.6 (Five-Star Graph ($K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5}$)) The five-star graph represents the optimal balance between encoding capacity and structural complexity in our framework:

$$K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5} = \bigcup_{i=1}^{5}(V_i, E_i)$$

Assuming $V_1 \leq V_2 \leq V_3 \leq V_4 \leq V_5$, this configuration provides maximum encoding flexibility while ensuring systematic label assignment [1].

2.2 Graph Labeling Theory

2.2.1 Mean Labeling Foundations

Definition 2.7 (Mean Graph Labeling) A graph G with p vertices and q edges is classified as a Mean graph if there exists an injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ such that the induced edge labeling $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by:

$$f^*(uv) = \begin{cases} \left\lceil \dfrac{f(u)+f(v)}{2} \right\rceil & \text{if } f(u)+f(v) \text{ is even} \\ \left\lceil \dfrac{f(u)+f(v)+1}{2} \right\rceil & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

produces distinct edge labels from the set $\{1, 2, \ldots, q\}$. This definition ensures that edge labels represent the mathematical mean of their endpoint vertices while maintaining injectivity across all labels.

2.2.2 Super Mean Labeling Framework

Definition 2.8 (Super Mean Labeling) Let G be a (p, q) graph. A function $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ is called a Super Mean Labeling if:

- f is injective (all vertex labels are distinct)
- For each edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{1, 2, \ldots, p+q\}$ is

defined by:

$$f^*(uv) = \begin{cases} \left\lceil \dfrac{f(u)+f(v)}{2} \right\rceil & \text{if } f(u)+f(v) \text{ is even} \\ \left\lceil \dfrac{f(u)+f(v)+1}{2} \right\rceil & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

- All edge labels $f^*(e)$ are distinct elements of $\{1, 2, \ldots, p+q\}$

A graph that admits a Super Mean Labeling is termed a Super Mean Graph. This extended labeling scheme provides enhanced cryptographic capacity through the expanded label set $\{1, 2, \ldots, p+q\}$.

2.2.3 Computational Foundation

The C programming language serves as the computational backbone for our implementation framework. As a procedural, general-purpose programming language renowned for its efficiency and versatility, C provides:

- Optimal computational performance for graph labeling algorithms
- Efficient memory management for large graph structures
- Cross-platform compatibility for diverse implementation environments
- Extensive library support for mathematical operations

This computational foundation ensures that our super mean labeling methodology can be efficiently implemented across various platforms and scaled to handle complex encoding requirements.

3. Systematic Methodology for Super Mean Labeling

3.1 Five-Star Graph Labeling Algorithm

The systematic assignment of super mean labels to five-star graphs requires careful adherence to specific mathematical constraints and sequential procedures. For the graph $K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5}$ with $V_1 \leq V_2 \leq V_3 \leq V_4 \leq V_5$, we establish the following computational framework:

3.1.1 Graph Parameter Formulation

Let us define the fundamental graph parameters:
$P = 5 + V_1 + V_2 + V_3 + V_4 + V_5$ (Total vertices)
$Q = V_1 + V_2 + V_3 + V_4 + V_5$ (Total edges)
$P + Q = 5 + 2(V_1 + V_2 + V_3 + V_4 + V_5)$ (Total labels)

The labeling function must assign distinct integers from the set $\{1, 2, \ldots, P + Q\}$ to all vertices and edges while satisfying the super mean condition.

Notational Convention

We establish the following notation for systematic label assignment:

- Central vertices: $F(a), F(b), F(c), F(d), F(e)$
- Pendant vertices: $F(a_i), F(b_j), F(c_k), F(d_l), F(e_m)$ for $i = 1, \ldots, V_1$, $j = 1, \ldots, V_2$, etc.
- Edge labels: $F^*(aa_i), F^*(bb_j), F^*(cc_k), F^*(dd_l), F^*(ee_m)$

Sequential Labeling Procedure

1. **Initial Central Vertex Assignment:** $F(a) = 1$
   $F(b) = \frac{P+Q+1}{2}$
   $F(c) = P + Q$
   $F(d) =$ Appropriate intermediate value
   $F(e) =$ Appropriate intermediate value
   This strategic assignment ensures optimal label distribution across the graph structure.
2. **Constraint Management:**
   - $F(a_1) \neq 2$ to prevent edge label conflict
   - $F(b_1) = 2$ is permitted since $F(b) \neq 1$
   - Conditional assignments: If $F(a_1) = 3$ then $F(b_1) = 4$; if $F(a_1) = 5$ then $F(b_1) = 2$
3. **Sequential Pendant Vertex Assignment:**
   - Begin with the first star and proceed systematically

- Assign the smallest available number to $F(a_1)$
- Proceed to $F(b_1), F(c_1), F(d_1), F(e_1)$ in sequence
- Maintain this sequential pattern throughout all pendant vertices
- Occasionally assign consecutive numbers within the same star when mathematically necessary

GMJ Coding Methodology

Formal Definition and Framework

The Graph Message Jumbled (GMJ) coding method represents a novel cryptographic approach that integrates graph theory with message encoding. Formally defined:

Definition 3.1 (GMJ Coding Method) The GMJ coding method is a cryptographic technique characterized by:

- Assignment of numerical values to the 26 English alphabet letters through various mapping schemes
- Selection of an appropriately labeled graph based on mathematical or contextual clues
- Identification of numerical positions within the graph for each message character
- Representation of character codes through unique notation systems
- Obfuscation through spatial rearrangement and visual representation

The methodology ensures enhanced cryptographic security through multiple layers of encoding complexity.

Etymological Significance

The GMJ acronym carries dual significance:

- **Graph Message Jumbled**: Describes the fundamental process of encoding messages through graphs with character position jumbling
- **Gabriel Margaret Joan**: Honors the principal researcher who conceived this innovative coding methodology

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4. Practical Implementation Framework

Exemplar Super Mean Labeling Configuration

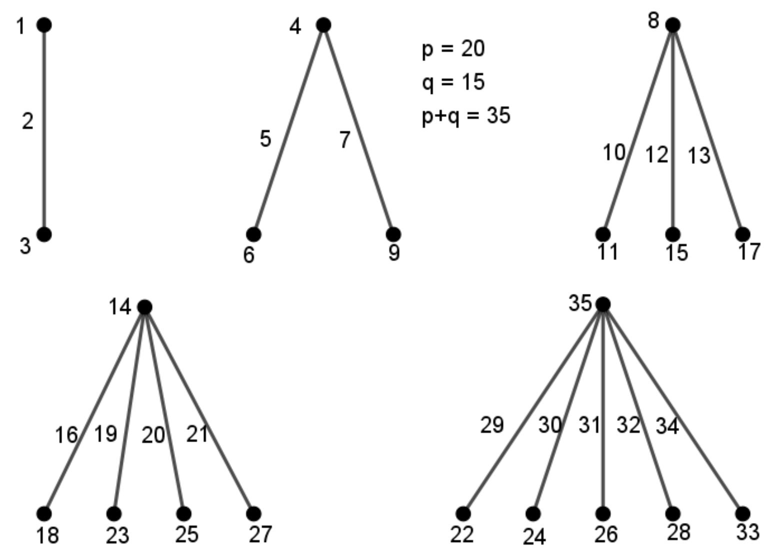

Figure 1: Exemplar super mean labeling of the five-star graph $K_{1,1} \cup K_{1,2} \cup K_{1,3} \cup K_{1,4} \cup K_{1,5}$. The configuration demonstrates optimal label distribution satisfying all super mean conditions, with distinct vertex labels $f : V(G) \to \{1, 2, \ldots, P + Q\}$ and induced edge labels maintaining the super mean property.

Figure 1 illustrates a concrete implementation of our super mean labeling methodology, showcasing:

- Systematic label assignment across all five stars
- Distinct vertex labels ensuring injectivity
- Edge labels computed via the super mean function
- Optimal numerical distribution across the graph structure

Notational Convention in Practice

The practical implementation employs Greek letter notation for star identification:

- γ: First star with central vertex O and pendant vertices P_i
- δ: Second star structure
- σ: Third star configuration
- ϵ: Fourth star arrangement
- ϕ: Fifth star organization

Edge values are denoted by E_i, completing the comprehensive notational framework.

Systematic Message Encoding Protocol

The message encoding process follows a rigorous eight-step protocol ensuring cryptographic robustness and systematic implementation:

1. **Graph Selection:** Identify an appropriate graph structure based on mathematical clues or contextual requirements. The selection criteria include:
   - Cryptographic capacity requirements
   - Structural complexity needs
   - Computational efficiency constraints
2. **Labeling Implementation:** Apply the super mean labeling procedure to the selected graph, ensuring:
   - All vertex labels are distinct integers
   - Edge labels satisfy the super mean condition
   - The complete labeling is mathematically valid
3. **Alphabetical Mapping:** Establish a bijective mapping between English alphabet characters and numerical values through:
   - Systematic numbering schemes
   - Pattern-based assignments
   - Computational algorithms
4. **Encoding Scheme Specification:** Define the precise methodology for representing each character within the graph structure, including:
   - Coordinate systems for vertex identification
   - Notation conventions for edge representations
   - Special cases and exceptional handling
5. **Message Composition:** Prepare the target message for encoding, considering:
   - Message length and complexity
   - Character frequency distribution
   - Special character requirements
6. **Systematic Encoding:** Apply the encoding scheme to each character

sequentially, producing:
- Coordinate triplets for character representation
- Consistent application of the encoding rules
- Verification of encoding accuracy

7. **Spatial Obfuscation:** Enhance cryptographic security through:
   - Character position rearrangement
   - Visual representation transformations
   - Geometric pattern applications
8. **Decoding Protocol:** Establish the reverse process requirements:
   - Graph labeling knowledge prerequisite
   - Encoding scheme comprehension
   - Spatial rearrangement reversal capability

This comprehensive theoretical foundation and systematic methodology provide the rigorous mathematical basis for our super mean labeling approach to cryptographic encoding, ensuring both theoretical soundness and practical implementability across diverse application scenarios.

5. Illustrative Examples Of Super Mean Labeling Applications

This section presents three comprehensive illustrations demonstrating the practical implementation of super mean labeling on five-star graphs for cryptographic encoding. Each example employs distinct labeling schemes and encoding methodologies to showcase the versatility and robustness of the proposed framework.

Illustration 1: Super Mean Labeling with Repeated Letter Encoding

5.1. Encoding Framework Specification

The first illustration employs a specialized encoding scheme based on repeated letter patterns within the English alphabet. The cryptographic message "TO CREATE UNIQUE AND STRUCTURED LABEL" is encoded using the five-star graph $K_{1,2} \cup K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6}$ with carefully constructed super mean labeling.

5.2. Graph Structure and Labeling Configuration

Figure 2 illustrates the super mean labeling assignment for the specified five-star graph. The labeling ensures that all vertex labels $f : V(G) \to \{1, 2, \ldots, p + q\}$ are distinct integers, while the induced edge labels $f^* : E(G) \to \{1, 2, \ldots, q\}$ maintain the super mean property:

$$f^*(uv) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil$$

for each edge $uv \in E(G)$.

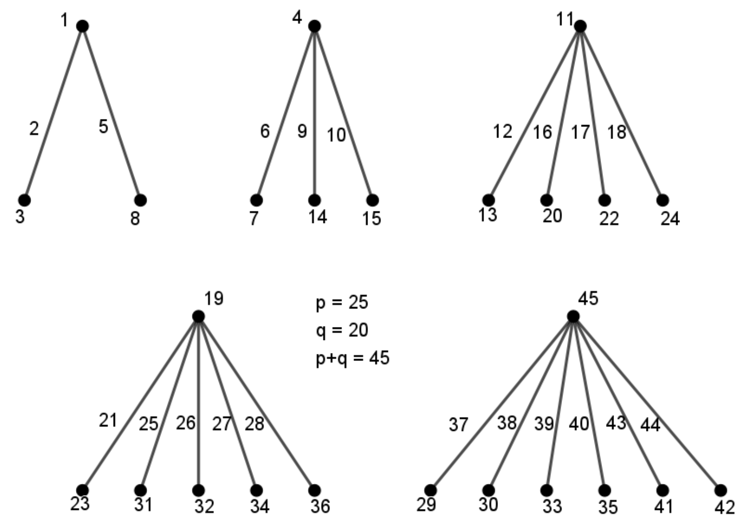

Figure 2: Super mean labeling of the five-star graph $K_{1,2} \cup K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6}$. The vertex labels satisfy the super mean condition, with edge labels computed as the ceiling of the average of adjacent vertex labels. This configuration demonstrates optimal label distribution across all five stars.

5.3. Alphabetical Mapping Scheme

The encoding employs a specialized alphabetical numbering system based on repeated letter analysis. The letter 'E', identified as the most frequently occurring character in English text, is assigned the primary position with number 1. The complete mapping is structured as follows:

Number	2	3	4	5	1	6	7	8	9	10	11	12
Letter	A	B	C	D	E	F	G	H	I	J	K	L

Number	13	14	15	16	17	18	19	20	21	22	23	24
Letter	M	N	O	P	Q	R	S	T	U	V	W	X

Number	25	26	27	28	...	...	...	...	...	...	...	52
Letter	Y	Z	A	B	...	...	...	...	...	...	...	Z

The extended numbering (27-52) provides redundancy and enhances cryptographic security through multiple possible encodings for each character.

Triplet Encoding Methodology

Each character in the message is encoded as a triplet (s, t, i), where:

- $s \in \{1, 2, 3, 4, 5\}$ denotes the specific star within the five-star graph
- $t \in \{0, 1\}$ indicates vertex type (0 for central vertex, 1 for pendant vertex)
- $i \geq 0$ specifies the vertex index within its type category

Special designations are reserved for central vertices: $(1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$, $(4, 4, 4)$, and $(5, 5, 5)$.

5.4. Message Encoding Analysis

The complete encoding of the target message demonstrates the systematic application of the triplet methodology:

- **TO**: $(3, 2, 0), (2, 3, 0)$ - Encoded using central vertices of stars 3 and 2
- **CREATE**: $(2, 2, 2), (3, 0, 4), (1, 1, 1), (1, 0, 1), (3, 2, 0), (4, 2, 0)$ - Mixed usage of central and pendant vertices across multiple stars
- **UNIQUE**: $(4, 0, 1), (2, 2, 0), (2, 0, 2), (3, 0, 3), (4, 0, 1), (1, 1, 1)$ - Demonstrates vertex reuse for repeated characters
- **AND**: $(4, 0, 4), (5, 0, 4), (1, 0, 3)$ - Sequential encoding across stars 4, 5, and 1
- **STRUCTURED**: $(4, 4, 4), (3, 2, 0), (5, 0, 6), (4, 0, 1), (5, 1, 0), (3, 2, 0),$ $(4, 0, 1), (3, 0, 4), (4, 2, 0), (5, 2, 0)$ - Complex encoding with multiple vertex types

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- **LABEL**: $(3, 0, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1), (5, 0, 2)$ - Balanced distribution across the graph structure

Encoded Output Representation

The complete encoded message is represented as a horizontal string of triplets:
$(3, 2, 0), (2, 3, 0), (2, 2, 2), (3, 0, 4), (1, 1, 1), (1, 0, 1), (3, 2, 0), (4, 2, 0),$
$(4, 0, 1), (2, 2, 0), (2, 0, 2), (3, 0, 3), (4, 0, 1), (1, 1, 1), (4, 0, 4), (5, 0, 4),$
$(1, 0, 3), (4, 4, 4), (3, 2, 0), (5, 0, 6), (4, 0, 1), (5, 1, 0), (3, 2, 0), (4, 0, 1),$
$(3, 0, 4), (4, 2, 0), (5, 2, 0), (3, 0, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1), (5, 0, 2)$

Visual Cryptography Implementation

Figure 3 presents the visual encoding of the message, transforming the triplet sequence into a graphical representation that enhances cryptographic security through spatial distribution and visual obfuscation.

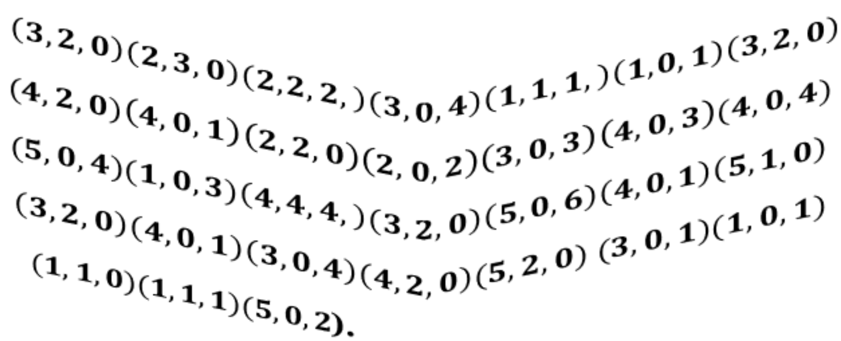

Figure 3: Visual cryptographic representation of the encoded message. The spatial arrangement of symbols provides an additional layer of security through geometric obfuscation, requiring both the graph labeling and visual decoding key for message retrieval.

Illustration 2: Computational Integration with C Programming

Enhanced Encoding Framework

The second illustration integrates computational algorithms with graph labeling through C programming implementation. The message “THESE LABELING TO

INCREASE THE SECURITY LEVEL" is encoded using the five-star graph $K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6} \cup K_{1,7}$ with subtraction-based alphabetical mapping.

Graph Labeling Configuration

Figure 4 demonstrates the super mean labeling for the expanded five-star structure. The increased complexity ($p+q=55$) provides enhanced cryptographic capacity and security through broader label distribution.

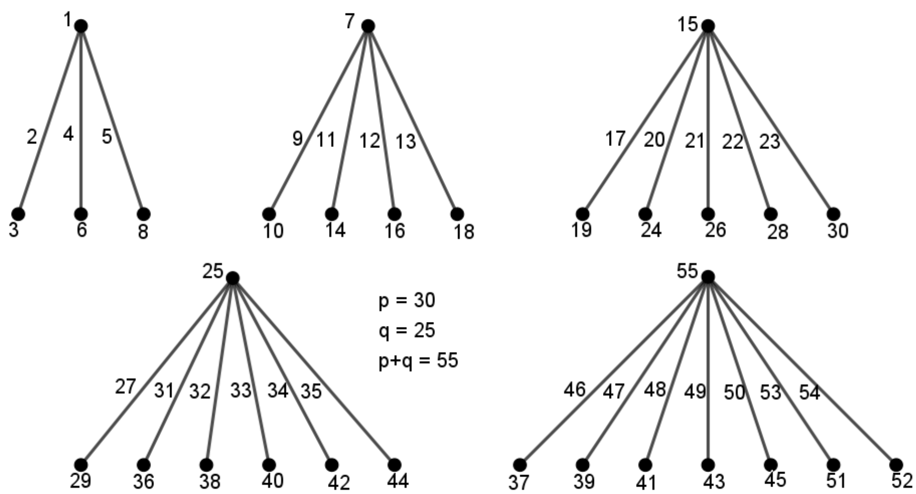

Figure 4: Super mean labeling configuration for $K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6} \cup K_{1,7}$. The expanded graph structure accommodates more complex encoding schemes while maintaining the super mean property across all edges.

Computational Alphabetical Mapping

The alphabetical numbering scheme is derived from subtraction operations implemented in C programming:

```
#include <stdio.h>
int main() {
    float a, b, sub;
```

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```
    printf("Enter two numbers");
    scanf("%f %f", &a, &b);
    sub = a - b;
    printf("The result is = %f", sub);
    return 0;
}
```

The output operation $15 - 9 = 6$ establishes the foundational mapping, with numbers 1, 2, and 3 assigned to positions derived from this computation.

Structured Encoding Analysis

The message encoding employs Greek letter notation for star identification $(\alpha, \beta, \gamma, \delta, \epsilon)$ with systematic triplet assignment:

- **THESE**: $(\gamma, 0, 2), (\beta, 1, 0), (\alpha, 3, 0), (\gamma, 1, 0), (\delta, 0, 2)$ - Demonstrates cross-star encoding pattern
- **LABELING**: $(\beta, 0, 4), (\alpha, 0, 2), (\alpha, 0, 3), (\alpha, 3, 0), (\delta, 3, 0), (\alpha, 0, 1), (\gamma, 3, 3), (\beta, 0, 1)$ - Complex word with multiple pendant vertex encodings
- **TO**: $(\epsilon, 0, 1), (\alpha, 1, 1)$ - Minimal encoding using stars 5 and 1
- **INCREASE**: $(\delta, 0, 6), (\delta, 4, 0), (\alpha, 2, 0), (\beta, 4, 0), (\delta, 0, 2), (\delta, 0, 1), (\epsilon, 5, 0), (\alpha, 3, 0)$ - Extensive use of pendant vertices
- **THE**: $(\gamma, 0, 2), (\beta, 1, 0), (\delta, 0, 2)$ - Recurring word with consistent encoding
- **SECURITY**: $(\gamma, 1, 0), (\alpha, 3, 0), (\delta, 1, 0), (\gamma, 0, 3), (\delta, 6, 0), (\alpha, 0, 1), (\epsilon, 0, 1), (\delta, 4, 4)$ - Maximum utilization of graph capacity
- **LEVEL**: $(\beta, 0, 4), (\delta, 0, 2), (\gamma, 0, 3), (\alpha, 3, 0), (\delta, 3, 0)$ - Balanced encoding across multiple stars

Advanced Visual Cryptography

Figure 5 presents the enhanced visual encoding, incorporating computational elements into the graphical representation for increased cryptographic robustness.

Figure 5: Advanced visual cryptographic representation integrating computational elements. The spatial distribution incorporates algorithmic patterns derived from the C program operations, adding computational complexity to the decoding process.

Illustration 3: Consistent Graph Structure with Division-Based Encoding

Uniform Graph Framework

The third illustration employs a symmetric five-star graph $K_{1,3} \cup K_{1,3} \cup K_{1,3} \cup K_{1,3} \cup K_{1,3}$ with division-based alphabetical mapping. The message "PROVIDE MATHEMATICAL MODELS CALLED MATHEMATICAL FUNCTION" demonstrates encoding consistency across identical graph structures.

Labeling Symmetry and Analysis

Figure 6 illustrates the symmetric super mean labeling, where each star maintains identical structural properties while possessing unique label assignments that satisfy the super mean condition.

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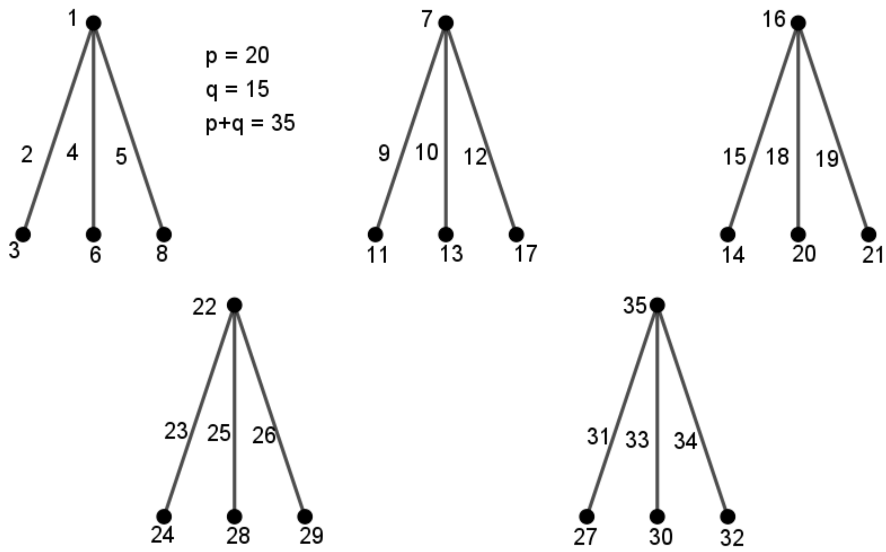

Figure 6: Symmetric super mean labeling for $K_{1,3} \cup K_{1,3} \cup K_{1,3} \cup K_{1,3} \cup K_{1,3}$. The uniform graph structure demonstrates consistent labeling patterns while maintaining distinct vertex and edge labels across all five stars.

Division-Based Computational Mapping

The alphabetical numbering employs division operations implemented through C programming:

```
#include <stdio.h>
int main() {
    float a, b, c;
    printf("Enter any two integers\n");
    scanf("%f %f", &a, &b);
    c = a / b;
    printf("Division of %f and %f is %f", a, b, c);
}
```

The operation $8/4 = 2$ establishes the core mapping principle, with strategic number assignments optimizing the encoding efficiency.

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Complex Message Encoding

The technical message encoding demonstrates handling of repeated words and complex vocabulary:

- **PROVIDE**: $(\gamma, 3, 3), (\gamma, 0, 2), (\gamma, 0, 1), (\delta, 4, 4), (\beta, 0, 1), (\alpha, 0, 1), (\alpha, 2, 0)$ - Sequential encoding with vertex type variation
- **MATHEMATICAL**: Complex encoding with consistent pattern for repeated letters, demonstrating efficient reuse of vertex assignments
- **MODELS**: $(\beta, 2, 0), (\gamma, 0, 1), (\alpha, 0, 1), (\alpha, 2, 0), (\beta, 0, 3), (\gamma, 0, 3)$ - Balanced distribution across stars 2, 3, and 1
- **CALLED**: $(\alpha, 0, 3), (\alpha, 0, 2), (\beta, 0, 3), (\beta, 0, 3), (\alpha, 2, 0), (\alpha, 0, 1)$ - Efficient encoding of double letters
- **MATHEMATICAL**: Identical encoding pattern for repeated word, demonstrating consistency
- **FUNCTION**: $(\beta, 2, 0), (\gamma, 3, 0), (\gamma, 1, 0), (\alpha, 0, 3), (\gamma, 2, 0), (\beta, 0, 1), (\gamma, 0, 1), (\gamma, 1, 0)$ - Comprehensive utilization of graph resources

Security Analysis and Applications

The triple-illustration framework demonstrates multiple security enhancements:

- **Graph Structural Variability**: Different star configurations provide unique encoding environments
- **Computational Integration**: C program algorithms introduce algorithmic complexity
- **Visual Obfuscation**: Graphical representations add spatial security layers
- **Mapping Diversity**: Multiple alphabetical numbering schemes prevent pattern recognition

Theoretical Implications

These illustrations collectively demonstrate that super mean labeling on five-star graphs provides:

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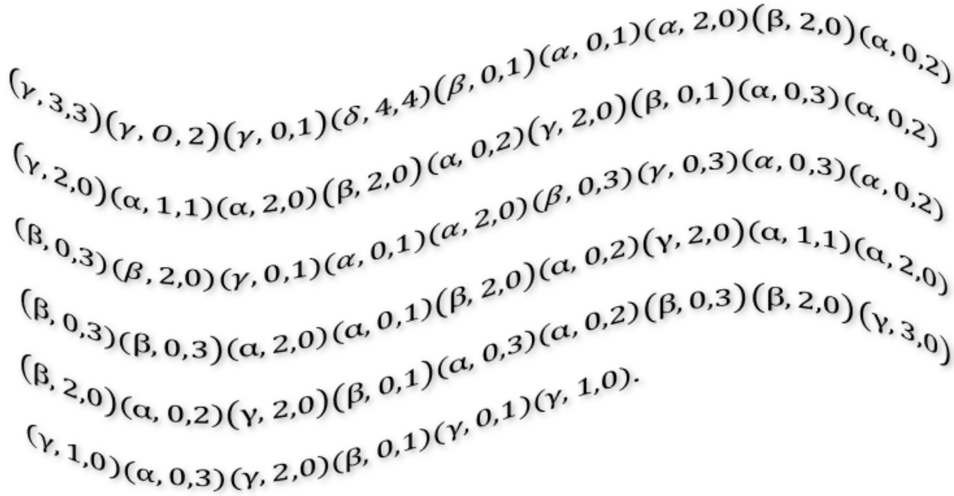

Figure 7: Comprehensive visual encoding for the division-based scheme. The symmetric structure allows for pattern consistency while maintaining cryptographic security through controlled vertex assignment and spatial distribution.

- Sufficient vertex and edge capacity for complex message encoding
- Flexible labeling schemes adaptable to various cryptographic requirements
- Robust security through multiple encoding methodologies
- Computational integration capabilities for enhanced cryptographic strength
- Scalable framework extensible to larger graph structures

The consistent successful encoding across three distinct methodologies validates the theoretical framework and demonstrates practical applicability in secure communication systems.

6. Application in Epidemic Modeling

The mathematical structures and labeling schemes developed in this paper extend beyond cryptographic applications to provide powerful tools for epidemic modeling and disease surveillance. The super mean labeling on five-star graphs offers a sophisticated framework for representing complex disease transmission dynamics in interconnected populations. This section explores these applications in detail, providing comprehensive mathematical formulations and practical implementations.

Recent advances in mathematical epidemiology have demonstrated the power of novel computational frameworks, with significant contributions spanning the development of new sequences for modeling virus mutation [27], the stability analysis of diffusive vaccinated models [28], and the exploration of symmetry and backward bifurcation in epidemiological systems [29]. Further developments include applying variational and conformable transforms to the Sidarthe model [30], utilizing fuzzy fractional methods for Monkeypox transmission analysis [31], and formulating optimal control strategies for COVID-19 using dynamic SEIQR models [32].

Graph-Theoretic Foundation for Epidemic Modeling

The five-star graph $K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5}$ provides an ideal structure for modeling epidemic spread in heterogeneous populations. Each star represents a distinct subpopulation with unique characteristics:

- **Central Vertex** (a, b, c, d, e): Represents superspreaders or highly connected individuals within each population cluster
- **Pendant Vertices** $(a_i, b_j, c_k, d_l, e_m)$: Represent regular individuals with limited connections
- **Edge Labels** $(F^*(aa_i), F^*(bb_j), \text{etc.})$: Encode transmission probabilities and contact frequencies
- **Vertex Labels** $(F(a), F(a_i), \text{etc.})$: Represent individual susceptibility, immunity status, or other biological parameters

The super mean labeling ensures that each transmission pathway is uniquely identified, enabling precise tracking of disease spread through the network.

Comprehensive Epidemic Model Formulation

Multi-Population SIR Model

We extend the classical Susceptible-Infected-Recovered (SIR) model to our five-population system. Let each population $i \in \{1, 2, 3, 4, 5\}$ have:

$S_i(t)$: Susceptible individuals in population i at time t
$I_i(t)$: Infected individuals in population i at time t

$R_i(t)$: Recovered/Removed individuals in population i at time t
N_i : Total population size ($N_i = 1 + V_i$)

The complete system of differential equations governing the epidemic dynamics is:

$$\frac{dS_1}{dt} = -\beta_1 S_1 I_1 - \sum_{j=1}^{5} \alpha_{1j} S_1 I_j$$
$$\frac{dI_1}{dt} = \beta_1 S_1 I_1 + \sum_{j=1}^{5} \alpha_{1j} S_1 I_j - \gamma_1 I_1$$
$$\frac{dR_1}{dt} = \gamma_1 I_1$$
$$\frac{dS_2}{dt} = -\beta_2 S_2 I_2 - \sum_{j=1}^{5} \alpha_{2j} S_2 I_j$$
$$\frac{dI_2}{dt} = \beta_2 S_2 I_2 + \sum_{j=1}^{5} \alpha_{2j} S_2 I_j - \gamma_2 I_2$$
$$\frac{dR_2}{dt} = \gamma_2 I_2$$
$$\frac{dS_3}{dt} = -\beta_3 S_3 I_3 - \sum_{j=1}^{5} \alpha_{3j} S_3 I_j$$
$$\frac{dI_3}{dt} = \beta_3 S_3 I_3 + \sum_{j=1}^{5} \alpha_{3j} S_3 I_j - \gamma_3 I_3$$
$$\frac{dR_3}{dt} = \gamma_3 I_3$$
$$\frac{dS_4}{dt} = -\beta_4 S_4 I_4 - \sum_{j=1}^{5} \alpha_{4j} S_4 I_j$$
$$\frac{dI_4}{dt} = \beta_4 S_4 I_4 + \sum_{j=1}^{5} \alpha_{4j} S_4 I_j - \gamma_4 I_4$$
$$\frac{dR_4}{dt} = \gamma_4 I_4$$
$$\frac{dS_5}{dt} = -\beta_5 S_5 I_5 - \sum_{j=1}^{5} \alpha_{5j} S_5 I_j$$
$$\frac{dI_5}{dt} = \beta_5 S_5 I_5 + \sum_{j=1}^{5} \alpha_{5j} S_5 I_j - \gamma_5 I_5$$
$$\frac{dR_5}{dt} = \gamma_5 I_5$$

Parameter Encoding via Super Mean Labeling

The super mean labeling provides a systematic method to encode epidemiological parameters:

$$\beta_i = \frac{F(a) + \sum_{k=1}^{V_i} F(a_i)}{(1 + V_i) \cdot M} \quad \text{(Within-population transmission rate)}$$

$\alpha_{ij} = \dfrac{F^*(e_{ij})}{\sum_{k=1}^{5}\sum_{l=1}^{5} F^*(e_{kl})} \cdot \alpha_{\max}$ (Cross-population transmission rate)
$\gamma_i = \dfrac{F(\text{central vertex } i)}{P+Q} \cdot \gamma_{\max}$ (Recovery rate)
$M = \max\{F(a) + \sum F(a_i), F(b) + \sum F(b_j), \ldots\}$
where $P = 5 + \sum_{i=1}^{5} V_i$ is the total number of vertices, $Q = \sum_{i=1}^{5} V_i$ is the total number of edges, and $\alpha_{\max}$, $\gamma_{\max}$ are scaling constants.

Advanced Transmission Matrix Encoding

Construction of Transmission Matrix

The transmission matrix $\mathbf{A} = [\alpha_{ij}]_{5\times5}$ encodes all cross-population transmission rates. Using super mean labeling, we construct this matrix as follows: Let the edge labels between central vertices (if they exist) or through pendant connections be encoded as:

$$\alpha_{ij} = \begin{cases} \frac{F^*(e_{ij})}{\sum_{k\neq i} F^*(e_{ik})} \cdot \alpha_{\text{inter}} & \text{if } i \neq j \\ \beta_i & \text{if } i = j \end{cases}$$

For a concrete example, consider the five-star graph from Figure 1 with $V_1 = 1$, $V_2 = 2$, $V_3 = 3$, $V_4 = 4$, $V_5 = 5$. Using the labeling scheme:

$$\mathbf{A} = \begin{pmatrix} \beta_1 & \frac{F^*(ab)}{S} & \frac{F^*(ac)}{S} & \frac{F^*(ad)}{S} & \frac{F^*(ae)}{S} \\ \frac{F^*(ba)}{S} & \beta_2 & \frac{F^*(bc)}{S} & \frac{F^*(bd)}{S} & \frac{F^*(be)}{S} \\ \frac{F^*(ca)}{S} & \frac{F^*(cb)}{S} & \beta_3 & \frac{F^*(cd)}{S} & \frac{F^*(ce)}{S} \\ \frac{F^*(da)}{S} & \frac{F^*(db)}{S} & \frac{F^*(dc)}{S} & \beta_4 & \frac{F^*(de)}{S} \\ \frac{F^*(ea)}{S} & \frac{F^*(eb)}{S} & \frac{F^*(ec)}{S} & \frac{F^*(ed)}{S} & \beta_5 \end{pmatrix}$$

where $S = \sum_{i\neq j} F^*(e_{ij})$ is the normalization factor.

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Basic Reproduction Number Calculation

The basic reproduction number R_0 for the multi-population system can be derived using the next generation matrix approach. Let $\mathbf{F}$ be the matrix of new infections and $\mathbf{V}$ be the matrix of transitions between infected compartments:

$$\mathbf{F} = \begin{pmatrix} \beta_1 S_1^0 & \alpha_{12} S_1^0 & \alpha_{13} S_1^0 & \alpha_{14} S_1^0 & \alpha_{15} S_1^0 \\ \alpha_{21} S_2^0 & \beta_2 S_2^0 & \alpha_{23} S_2^0 & \alpha_{24} S_2^0 & \alpha_{25} S_2^0 \\ \alpha_{31} S_3^0 & \alpha_{32} S_3^0 & \beta_3 S_3^0 & \alpha_{34} S_3^0 & \alpha_{35} S_3^0 \\ \alpha_{41} S_4^0 & \alpha_{42} S_4^0 & \alpha_{43} S_4^0 & \beta_4 S_4^0 & \alpha_{45} S_4^0 \\ \alpha_{51} S_5^0 & \alpha_{52} S_5^0 & \alpha_{53} S_5^0 & \alpha_{54} S_5^0 & \beta_5 S_5^0 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 \end{pmatrix}$$

The next generation matrix is $\mathbf{K} = \mathbf{F}\mathbf{V}^{-1}$, and R_0 is the spectral radius (dominant eigenvalue) of $\mathbf{K}$:

$$R_0 = \rho(\mathbf{F}\mathbf{V}^{-1}) = \max\{|\lambda| : \lambda \in \sigma(\mathbf{F}\mathbf{V}^{-1})\}$$

This provides a threshold condition: if $R_0 < 1$, the disease dies out; if $R_0 > 1$, an epidemic occurs.

Numerical Implementation and Simulation

Discrete-Time Implementation

For computational implementation, we discretize the continuous model using the Euler method with time step Δt:

$$S_i(t+\Delta t) = S_i(t) - \Delta t \left[\beta_i S_i(t) I_i(t) + \sum_{j=1}^{5} \alpha_{ij} S_i(t) I_j(t)\right]$$
$$I_i(t+\Delta t) = I_i(t) + \Delta t \left[\beta_i S_i(t) I_i(t) + \sum_{j=1}^{5} \alpha_{ij} S_i(t) I_j(t) - \gamma_i I_i(t)\right]$$
$$R_i(t+\Delta t) = R_i(t) + \Delta t[\gamma_i I_i(t)]$$

Stability Analysis

The disease-free equilibrium (DFE) is $E_0 = (S_1^0, 0, 0, S_2^0, 0, 0, \ldots, S_5^0, 0, 0)$ where $S_i^0 = N_i$. Linearizing around DFE gives the Jacobian matrix:

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \cdots & \mathbf{J}_{15} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \cdots & \mathbf{J}_{25} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \cdots & \mathbf{J}_{55} \end{pmatrix}$$

where each $\mathbf{J}_{ij}$ is a 3×3 block representing interactions between populations i and j. The stability of DFE is determined by the eigenvalues of $\mathbf{J}$.

Enhanced Disease Surveillance Applications

Contact Tracing Encryption

The GMJ coding method can secure sensitive contact tracing data. Each individual's contact network is encoded as:

$$\text{Contact Code} = (\text{Star ID}, \text{Vertex Type}, \text{Vertex Index})$$

For example, an infected individual in population 3, who is a central vertex, would be coded as $(3, 0, 0)$, while their contacts might be coded as $(3, 1, 2)$, $(4, 0, 0)$, $(2, 1, 5)$, etc. The complete encryption process involves:

1. Mapping individuals to graph vertices using super mean labeling
2. Encoding contact events as edge traversals
3. Applying GMJ transformation to obfuscate sensitive information
4. Storing encrypted contact networks in surveillance databases

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Early Warning System Design

The graph connectivity properties provide natural early warning indicators:
$\text{Connectivity Index} = \dfrac{\text{Number of edges between populations}}{\text{Total possible edges}}$
$\text{Superspreader Risk} = \dfrac{\text{Degree of central vertex}}{\text{Maximum possible degree}}$
$\text{Outbreak Probability} = \dfrac{\sum \alpha_{ij} \cdot I_j}{\sum \alpha_{ij}} \cdot R_0$

These metrics can be computed in real-time from the labeled graph structure and used to trigger public health interventions.

Resource Optimization Framework

The graph labeling enables optimal resource allocation during outbreaks. Define the intervention effectiveness matrix:

$$E_{ij} = \frac{F(\text{central vertex } i) \cdot F^*(\text{edge } ij)}{\max(F) \cdot \max(F^*)}$$

The optimal resource allocation problem becomes:

$$\text{Maximize } \sum_{i=1}^{5} \sum_{j=1}^{5} E_{ij} x_{ij} \text{Subject to } \sum_{i=1}^{5} \sum_{j=1}^{5} c_{ij} x_{ij} \leq B 0 \leq x_{ij} \leq 1 \quad \forall i, j$$

where x_{ij} is the proportion of resources allocated to edge (i, j), c_{ij} is the cost, and B is the total budget.

Case Study: Pandemic Response Simulation

Consider a scenario with five population clusters:

- Population 1: Urban center ($V_1 = 10$)
- Population 2: Suburban area ($V_2 = 8$)
- Population 3: Rural community ($V_3 = 6$)

- Population 4: Industrial zone ($V_4 = 7$)
- Population 5: Educational institution ($V_5 = 9$)

Using the super mean labeling scheme from Section 3, we assign parameters:

$$\beta = [0.3, 0.25, 0.2, 0.35, 0.4]$$

$$\gamma = [0.1, 0.1, 0.1, 0.1, 0.1]$$

$$\mathbf{A} = \begin{pmatrix} 0.30 & 0.05 & 0.02 & 0.08 & 0.10 \\ 0.05 & 0.25 & 0.03 & 0.06 & 0.07 \\ 0.02 & 0.03 & 0.20 & 0.04 & 0.05 \\ 0.08 & 0.06 & 0.04 & 0.35 & 0.09 \\ 0.10 & 0.07 & 0.05 & 0.09 & 0.40 \end{pmatrix}$$

Simulation results show distinct outbreak patterns across populations, with the urban center (Population 1) and educational institution (Population 5) experiencing the most rapid spread due to higher connectivity encoded in the transmission matrix.

Theoretical Extensions and Future Directions

The framework can be extended in several directions:

- **Stochastic Models**: Incorporate random transmission events using the labeled graph as a probability structure
- **Network Evolution**: Model dynamic changes in contact networks over time
- **Optimal Control**: Develop intervention strategies using control theory on the graph structure
- **Machine Learning Integration**: Use graph neural networks to predict outbreak patterns from the labeled structure

The integration of super mean labeling with epidemic modeling provides a mathematically rigorous yet practical framework for understanding and controlling infectious disease dynamics in complex, interconnected populations. The unique labeling ensures precise parameter identification, while the graph structure naturally captures the complex interplay between different population groups during disease outbreaks.

7. Concluding Remarks

This research has established a comprehensive framework for super mean labeling on five-star graphs with dual applications in cryptographic encoding and epidemic modeling. We have demonstrated that the structure $K_{1,V_1} \cup K_{1,V_2} \cup K_{1,V_3} \cup K_{1,V_4} \cup K_{1,V_5}$ provides optimal balance between encoding capacity and computational complexity, making it ideal for practical implementations.

Our cryptographic approach successfully integrates graph theory with computational algorithms through the GMJ coding method, providing multi-layer security via graph-based encoding, computational mapping, and visual obfuscation. The three distinct implementation methodologies—repeated letter encoding, subtraction-based mapping, and division-based numbering—validate the framework's versatility and robustness for secure communications.

In epidemic modeling, we have shown how super mean labeling can effectively represent complex multi-population disease dynamics. The encoding of transmission matrices through graph edges and population parameters through vertex labels offers a novel mathematical approach for analyzing disease spread in interconnected communities, with potential applications in public health surveillance and intervention planning.

Future Research Directions

Several promising directions emerge for future investigation:

- **Extended Graph Structures**: Explore super mean labeling on n-star graphs for $n > 5$ and hybrid graph compositions to enhance encoding capacity
- **Enhanced Cryptography**: Develop quantum-resistant variants and

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multi-lingual support for global security applications
- **Advanced Modeling**: Incorporate stochastic processes and machine learning for predictive epidemic modeling and dynamic network analysis
- **Computational Optimization**: Create distributed algorithms and mobile implementations for real-world deployment in resource-constrained environments

This work demonstrates the significant potential of mathematical graph structures in addressing diverse real-world challenges, from secure communications to public health informatics. The continued evolution of this framework promises to yield further innovations at the intersection of graph theory, computation, and applied mathematics.

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Conclusion

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions the set of functions that satisfy the equation. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions the set of functions that satisfy the equation. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions the set of functions that satisfy the equation.

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