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Graph-Based Cryptographic Encoding and Epidemiological Modeling via Even Felicitous Labeling on Star Graphs

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Abstract

In this work, we have identified a way for encoding a secret message utilizing the GMJ (Graph Message Jumbled) Code and even felicitous labeling on five-star graphs $K_{1,\sigma_1} \cup K_{1,\sigma_2} \cup K_{1,\sigma_3} \cup K_{1,\sigma_4} \cup K_{1,\sigma_5}$. For every star graph, we give two examples and use a Java program to apply alphabetical techniques to the Corona Number and Triangular Number. A strategy for labeling an even felicitous graph and flowchart is provided for transforming plaintext into ciphertext (Picture Coding). Furthermore, we demonstrate an innovative application of these graph labeling techniques to epidemiological modeling through a detailed analysis of the SIR (Susceptible-Infected-Recovered) epidemic model on star networks. The paper shows how felicitous labeling can encode varying transmission probabilities and infection states, providing insights for targeted public health interventions. This interdisciplinary approach bridges cryptography, graph theory, and epidemiology, offering both secure communication methods and practical disease spread modeling frameworks.

Key words: Six-star Graph, Even Felicitous Graph, Encoding

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1 Introduction and Literature Review

The mathematical modeling of infectious diseases has a long and rich history, beginning with the classical compartmental models proposed by Kermack and

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McKendrick [9]. Their seminal SIR framework laid the foundation for deterministic epidemic modeling and has since been extended in numerous directions to incorporate heterogeneity, stochasticity, and population structure. A comprehensive treatment of classical epidemic models and their analytical properties can be found in the work of Hethcote [7], while the definition and computation of the basic reproduction number R_0 in heterogeneous populations were rigorously formalized by Diekmann et al. [3].

Traditional compartmental models often rely on the assumption of homogeneous mixing, which may not adequately capture real-world contact patterns. To overcome this limitation, network-based epidemic models have been developed, where individuals are represented as vertices and interactions as edges in a graph. Early studies demonstrated how network topology significantly influences disease dynamics, particularly in complex and scale-free networks [17, 16]. Keeling and Eames [8] further highlighted the importance of network structure in epidemic spread and emphasized the role of contact heterogeneity in realistic disease modeling.

Graph-theoretic approaches provide a natural framework for incorporating structural properties of populations. Foundational concepts from graph theory, as developed by Harary [5], and extensive surveys on graph labeling by Gallian [4], have inspired a wide range of applications in epidemiology, coding theory, and network science. Exact epidemic models on graphs were studied by Simon et al. [23], who introduced graph-automorphism-driven lumping techniques to reduce model complexity while preserving exact dynamics.

Recent advances have combined graph theory with data-driven and machine learning approaches. Seibold and Callender [20] investigated epidemic spreading on regular tree graphs using agent-based simulations, demonstrating how graph diameter and density affect outbreak duration. Graph learning methods aimed at inferring transmission networks from observed infection times were developed by Netrapalli and Sanghavi [15], providing theoretical guarantees for graph recovery in epidemic cascades.

With the growing availability of large-scale spatio-temporal data, graph neural networks (GNNs) have emerged as powerful tools for epidemic forecasting. Derr et al. [2] proposed an epidemic graph convolutional network to capture diffusion patterns, while La Gatta et al. [10] integrated GCNs with recurrent architectures for COVID-19 modeling. Alguliyev et al. [1] employed graph modeling to analyze and visualize pandemic spread, emphasizing the role of demographic and mobility factors. A comprehensive review of GNN-based epidemic models was recently provided by Liu et al. [11]. More recently, Zheng et al. [32] introduced an epidemiology-informed GNN framework that explicitly incorporates mechanistic heterogeneity across locations, achieving improved forecasting accuracy.

Parallel to epidemic modeling, graph labeling techniques have been widely explored for secure communication and coding applications. Several studies have proposed coding schemes based on cordial, felicitous, super mean, and product cordial labelings on various graph families, including star, bistar, sunflower, Fibonacci web, and gear graphs [6, 12, 28, 13, 21, 18, 29, 30, 27]. Extensions to GMJ coding and super mean labeling have further enriched this domain [25, 14, 31, 24, 22, 19].

Motivated by these developments, the present work aims to bridge graph labeling techniques with epidemic modeling, leveraging structural properties of graphs to enhance both secure coding schemes and the understanding of disease dynamics on networks.

2 Preliminaries

Before diving into our main contributions, it is essential to establish some foundational concepts from graph labeling theory. These definitions form the mathematical bedrock upon which our GMJ coding technique is built, and understanding them is crucial for appreciating the elegance and effectiveness of the encoding schemes we will present. General background on graph labeling and its applications can be found in [4] and classical graph-theoretic foundations in [5].

Definition 2.1 (Even Felicitous Labeling). *A one-to-one operation should the link labels created by*

$$\{G(r) + G(s)\} \mod (2q - 1)$$

every link's RS are unique, then, $G : V \to \{0, 1, 2, \ldots, 2q - 1\}$ *and* $G : E \to \{0, 2, \ldots, 2q - 2\}$ *would be referred to as even felicitous.*

The even felicitous labeling extends the classical felicitous labeling by restricting edge labels to even values, providing additional structure that proves valuable in coding applications. Such labelings were introduced and studied in the context of secure coding and graph-based encoding schemes in [22]. This restriction creates a natural correspondence with alphabetic encoding schemes, as we will demonstrate in subsequent sections.

Definition 2.2 (Felicitous Labeling). *If there is an injection $f : V(G) \to \{0, 1, 2, \dots, q\}$ such that the induced function $f^* : V(G) \to \{0, 1, 2, \dots, q-1\}$ defined by $f^*(uv) = \{f(u) + f(v)\} \pmod q, uv \in E(G)$ is a bijection, the graph G is said to be felicitous. The injection function f is called a felicitous labeling of G.*

Felicitous labeling is a well-studied labeling scheme in graph theory and has been widely applied in communication and coding problems; see [4] for a comprehensive survey and related developments.

Definition 2.3 (Wedge). *Wedges are links that bind two portions of a graph together; $\wedge$, $\mu(G\wedge) < \mu(G)$ is the sign for it.*

The notion of wedges is useful in analyzing structural connectivity and decomposition properties of graphs, particularly when constructing composite graphs for labeling and coding purposes [5].

Definition 2.4 (Bistar). *The Bistar $B_{\alpha,\beta}[2]$ is the graph that arises when pendent connections are joined to one end of k_2 while pendent links are attached to the other end of k_2. The nodes in k_2 are referred to as the core nodes of $B_{\alpha,\beta}$, while the link in k_2 is known as the central link of $B_{\alpha,\beta}$.*

Bistar graphs and their variants have attracted considerable attention in graph labeling and coding literature due to their simple structure and labeling flexibility [4, 21].

Having established these key definitions, we now proceed to the core of our work, where we apply these concepts to the construction and analysis of five-star graphs.

3 Even Felicitous Labeling on Five-Star Graphs

In this section, we demonstrate that the collection of five-star graphs

$$K_{1,\sigma_1} \cup K_{1,\sigma_2} \cup K_{1,\sigma_3} \cup K_{1,\sigma_4} \cup K_{1,\sigma_5}$$

admits an even felicitous labeling. Labeling problems on star and multi-star graphs have been extensively studied due to their structural simplicity and applicability in coding theory [4, 5].

This result is fundamental to our GMJ coding technique, as it ensures that every alphabet can be uniquely encoded using the graph structure. Even felicitous labeling, as an extension of classical felicitous labeling, has proven particularly effective for secure and alphabet-based coding schemes [22, 21].

The proof is constructive, providing an explicit labeling scheme that we will subsequently use in our cryptographic applications. Constructive labeling methods on star-like graph families have been successfully employed in several GMJ and super mean labeling based coding techniques [6, 13, 29].

To see why, let us construct the graph step by step.

Consider

$$G = K_{1,\sigma_1} \cup K_{1,\sigma_2} \cup K_{1,\sigma_3} \cup K_{1,\sigma_4} \cup K_{1,\sigma_5}.$$

Now, define the vertex sets corresponding to each star component as follows:

$$V_1 = \{\omega\} \cup \{\omega_s : 1 \leq s \leq \sigma_1\},$$

$$V_2 = \{\pi\} \cup \{\pi_t : 1 \leq t \leq \sigma_2\},$$

$$V_3 = \{\varphi\} \cup \{\varphi_u : 1 \leq u \leq \sigma_3\},$$

$$V_4 = \{\delta\} \cup \{\delta_v : 1 \leq v \leq \sigma_4\},$$

$$V_5 = \{\varepsilon\} \cup \{\varepsilon_w : 1 \leq w \leq \sigma_5\}.$$

Together, these sets constitute the vertex set of G.

The graph G has

$$|V(G)| = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + 5$$

vertices and

$$|E(G)| = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$$

edges.

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Hence, the complete vertex set of G is

$$\Omega(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5.$$

The next step is to verify whether G represents an even felicitous graph. To do this, we define the node labeling $\mu : \Omega(G) \to \{0, 1, 2, \ldots 2q-1\}$ as follows:

$$\lambda(\omega) = 3;$$
$$\lambda(\omega_s) = 2\sigma_2 + 2\sigma_3 + 2\sigma_4 + 2\sigma_5 + 2s - 5, \quad 1 \leq s \leq \sigma_1.$$
$$\lambda(\pi) = 3;$$
$$\lambda(\pi_t) = 2\sigma_3 + 2\sigma_4 + 2\sigma_5 + 2t - 4, \quad 1 \leq t \leq \sigma_2.$$
$$\lambda(\varphi) = 3;$$
$$\lambda(\varphi_u) = 2\sigma_4 + 2\sigma_5 + 2u - 1, \quad 1 \leq u \leq \sigma_3 - 1.$$
$$\lambda(\delta) = 2q - 1;$$
$$\lambda(\delta_3) = 2q - 2.$$
$$\lambda(\varepsilon) = 0;$$
$$\lambda(\delta_v) = 2\sigma_5 + 2v, \quad 1 \leq v \leq \sigma_4.$$
$$\lambda(\varepsilon_\omega) = 2\omega, \quad 1 \leq \omega \leq \sigma_5.$$

The analogous edge labels are as follows:

The connecting characteristic regarding $\omega\omega_s$ is:

$$2\sigma_2 + 2\sigma_3 + 2\sigma_4 + 2\sigma_5 + 2s - 5, \quad 1 \leq s \leq \sigma_1.$$

The connecting characteristic regarding $\pi\pi_t$ is:

$$2\sigma_3 + 2\sigma_4 + 2\sigma_5 + 2t - 4, \quad 1 \leq t \leq \sigma_2.$$

The connecting characteristic regarding $\varphi\varphi_u$ is:

$$2\sigma_4 + 2\sigma_5 + 2u - 1, \quad 1 \leq u \leq \sigma_3 - 1.$$

The connecting characteristic regarding $\varphi\varphi_u$ is 0.

The connecting characteristic regarding $\delta\delta_v$ is:

$$2\sigma_5 + 2v, \quad 1 \leq v \leq \sigma_4.$$

The connecting characteristic regarding $\varepsilon\varepsilon_\omega$ is 2ω, $1 \leq \omega \leq \sigma_5$.

With this labeling scheme established, we now turn our attention to the practical application of these mathematical structures in message encoding through the GMJ technique.

4 GMJ (Graphical Messaging Jumbled) Coding Technique

The GMJ coding technique represents a sophisticated approach to cryptographic encoding that leverages the mathematical properties of graph labelings to transform plaintext messages into secure ciphertext. Graph labeling has long been recognized as an effective tool for secure communication and information encoding due to its combinatorial richness and flexibility [4].

Unlike traditional encryption methods that rely solely on algebraic transformations, GMJ coding embeds the message structure within the topology of labeled graphs, providing an additional layer of security through the inherent complexity of graph isomorphism [5]. Similar graph-based coding and encryption schemes have been successfully developed using star, bistar, and related graph families under various labeling constraints, including felicitous, super mean, and GMJ labelings [6, 28, 13].

Recent advances in graphical coding techniques further demonstrate that structured graph labelings can significantly enhance resistance to pattern-based attacks, particularly when multiple labeling rules are combined within a single encoding framework [21, 30]. In this context, the GMJ coding technique exploits even felicitous labeling to generate uniquely decodable ciphertexts while maintaining computational efficiency.

The process involves several systematic steps:

1. Using a given theoretic or other types of clues, choose a suitable labeling graph.
2. Give each of the 26 English alphabets a unique number.
3. Apply a vertical string of another pattern that shows each of the letter characters in a unique method.
4. Obtain an even alphabetic numbering while applying Java programming.
5. Find the value of the diagram for each letter in each sentence in the given text.
6. Before arranging the letters to increase the coded message's secrecy, make an image using the Cides characters.

The flowchart in Figure 1 illustrates the complete encoding process, showing how each step builds upon the previous one to create a robust and secure encoding scheme.

Now that we have outlined the general methodology, let us examine concrete examples that demonstrate the power and flexibility of the GMJ coding technique through two different numerical encoding schemes.

5 Illustrative Examples

Example 1: Corona Number Based Encoding (FCNNO)

Corona Numbers, characterized by having all odd digits, provide a natural and elegant basis for our encoding scheme. This mathematical property creates a distinctive pattern that enhances the security of the encoding while maintaining computational efficiency.

(i) **A statement of message:** Graph Message Jumbled Coding used for Even Felicitous Labeling.

(ii) **Hint:** Triangle, Quadrangle, Pentagon, Hexagon and Octagon were joining together along (the Pentagon had five sides, the Triangle had three, the Quadrangle had four, the Hexagon had six, and the Octagon had eight). Based on this, we should understand that it is nothing more than a five-star graph $K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6} \cup K_{1,8}$.

(iii) **Graph:** $K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6} \cup K_{1,8}$.

(iv) **Label:** The even felicitous labeling for $K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6} \cup K_{1,8}$ is completed by comparing it to the prior overview.

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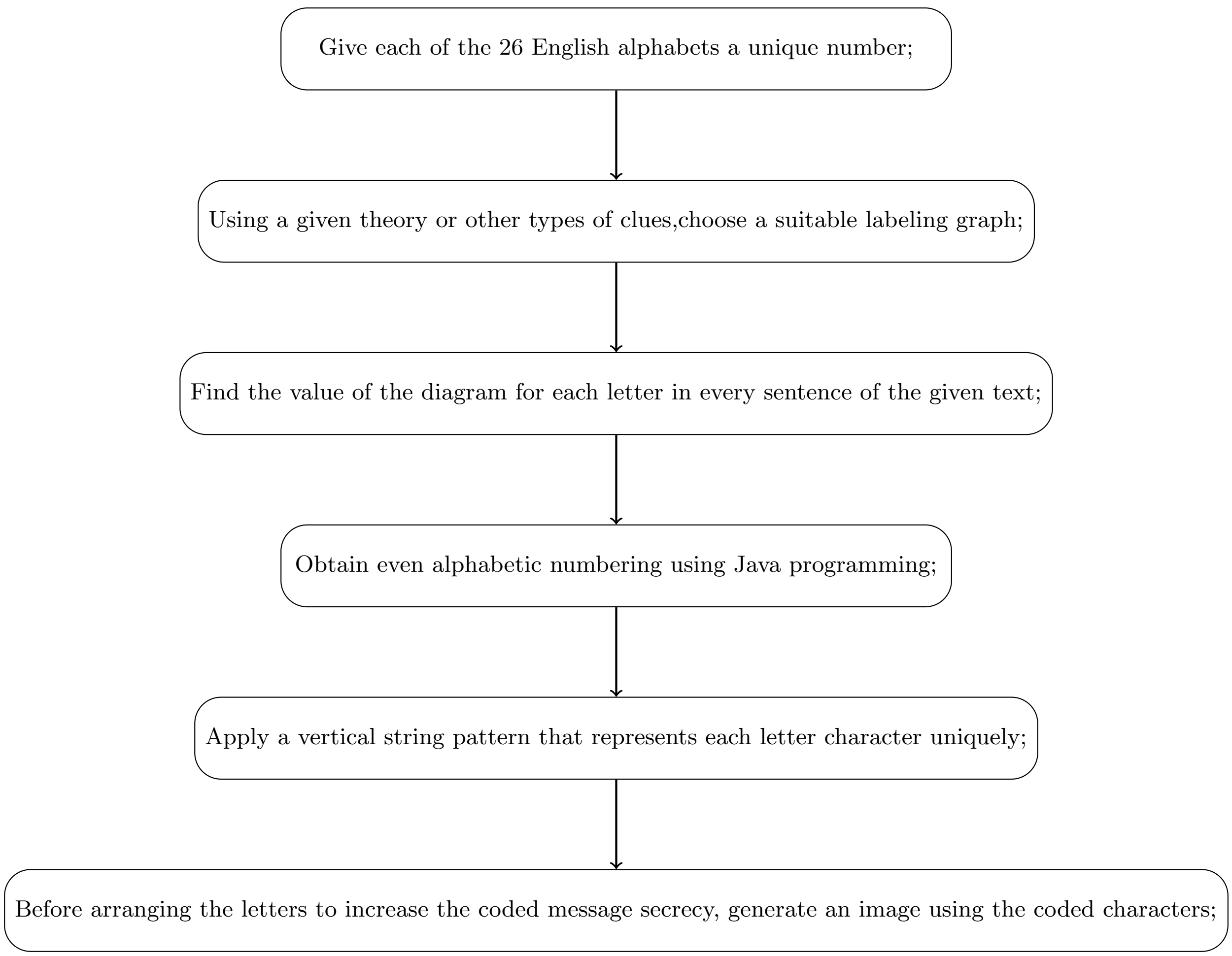

Figure 1: GMJ Coding Process Flowchart

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For this graph, we have: $p = 31$; $q = 26$

$2q - 1 = 51$; $2q - 2 = 50$

Figure 2 presents the complete graph structure with vertex and edge labelings, demonstrating the even felicitous property. Notice how each star maintains its individual structure while contributing to the overall encoding framework.

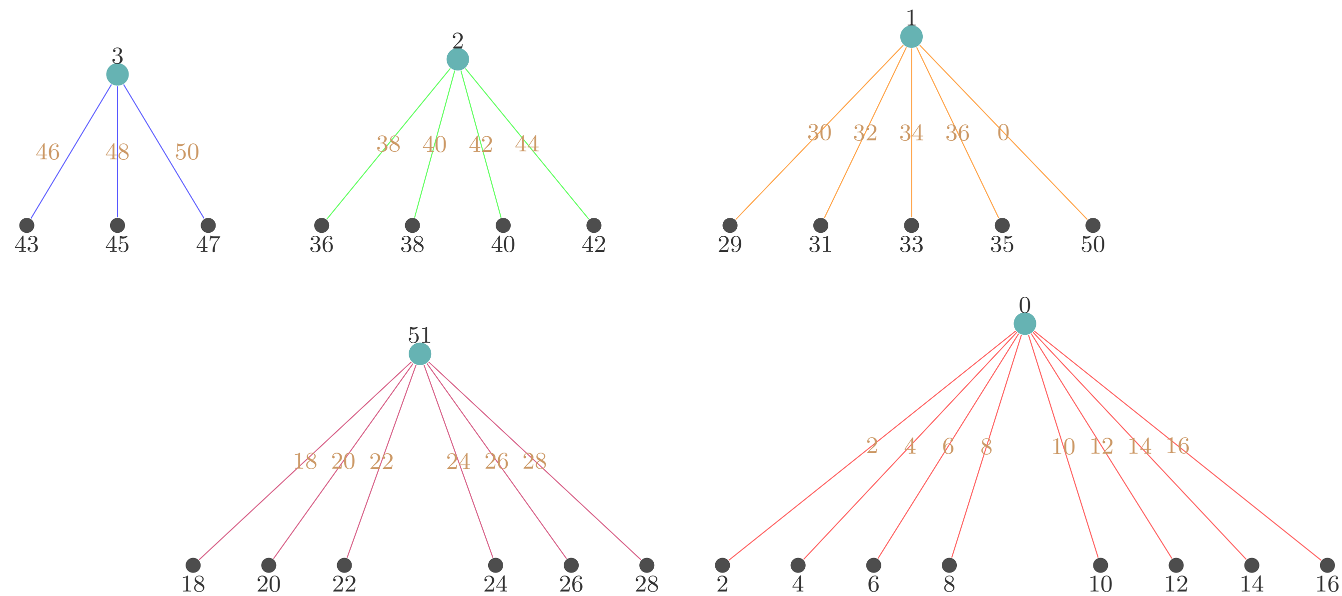

Figure 2: Union of star graphs $K_{1,3} \cup K_{1,4} \cup K_{1,5} \cup K_{1,6} \cup K_{1,8}$

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(v) **Create a Java program that accepts a number and determines if it is a "Corona Number."**

Corona Number: A Number whose all digit are odd. Ex: 1579.

The following Java implementation provides an efficient algorithm for identifying Corona Numbers. This computational component is essential for automating the encoding process and ensuring accuracy in large-scale applications.

```
import java.util.*;
import java.lang.*;
class corona_number {
    public static void main(String args[]) {
        int num, temp, rem, f = 0;
        Scanner sc = new Scanner(System.in);
        System.out.println("Enter any Number");
        num = sc.nextInt();
        temp = num;
        while (temp != 0) {
            rem = temp % 10;
            if (rem % 2 == 0) {
                f = 1;
                break;
            }
            temp = temp / 10;
        }
        if (f == 0)
            System.out.println(num + " Is a Corona Number");
        else
            System.out.println(num + " Is Not a Corona Number");
    }
}
```

OUT PUT: Enter any number
1579 Is a Corona Number.
2573 Is Not a Corona Number.

(vi) **Even Numbering of Alphabets (First Corona Number Next others (CNNO)).**

Now we assign even numbers to alphabets based on whether they appear in Corona Number positions. This dual classification creates a hierarchical encoding structure that significantly enhances the cryptographic strength of our method.

0	8	10	12	2	14	4	16	6	18	20	22	24
A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
26	28	30	32	34	36	38	40	42	44	46	48	50
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

The alphabets in the Java Programming position in Corona value (A, E, G, I) are given the even numbers 0, 2, 4, and 6, whereas the rest of the alphabets are given an even value between 8 and 50.

$A\left(B_{P_Q}\right) = 2Q - 2,\ Q = 1$ is the function that is encoded. Here, P_Q stands for the Q^{th} Corona Number between 1 and 26, and for the B_{P_Q} alphabetical letter ${P_Q}^{\mathrm{th}}$.

$$B\left(A_{R_S}\right) = 2 + 2s, \quad s = 1, 2, \ldots, 22.$$

In this case, R_S stands for the S^{th} non-perfect number between 1 and 26, and A_{R_S} for the ${R_S}^{\mathrm{th}}$ alphabet letter.

$B_{P_Q} \neq A_{R_S}$, Here B_{P_Q} and A_{R_S} represents twenty-six of the alphabets.

(vii) **Coding a Letter:**

To facilitate the encoding process, we introduce a symbolic notation system that maps each star and its components to specific symbols. This notation provides a compact and unambiguous representation of the encoded message.

Assume that the first star is symbolized by Σ, the second by Π, the third by Ω, the fourth by $\cup$, and the fifth by $\cap$.

In order, let $e^0, \alpha_i, \beta_i, \gamma_i$ and δ_i symbolize the center node, the i^{th} node value, and the i^{th} link value.

For instance, Σ_{θ_8} represents the number allotted to the eight node of 1^{st} star, Π_{δ_1} represents the number allotted to the first node of 2^{nd} star, Ω_{α_5} represents the number allotted to the fifth node of 3^{rd} star, $\cup_{\gamma_4}$ represents the number allotted

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to the fourth node of 4^{th} star, $\cap_{\beta_2}$ represents the number allotted to the second node of 5^{th} star.

(viii) **Coding (Wordwise):**

Using the established notation, we encode our message word by word. This word-level encoding provides modularity and allows for efficient processing of variable-length messages.

Graph	$\cap_{\theta_2}\Omega_{\gamma_3}\Omega_{\alpha_5}\Omega_{\alpha_1}\Sigma_{\theta_8}$
Message	$\cup_{\delta_4}\cap_{\theta_1}e^0e^0\Omega_{\alpha_5}\cap_{\theta_2}\cap_{\theta_1}$
Jumbled	$\Pi_{\delta_1}\Pi_{\beta_3}\cup_{\delta_4}\cap_{\theta_4}\cup_{\delta_3}\cap_{\theta_1}\cap_{\theta_6}$
Coding	$\cap_{\theta_5}\cup_{\delta_6}\cap_{\theta_6}\cap_{\theta_3}\cup_{\delta_5}\cap_{\theta_2}$
Used	$\Pi_{\beta_3}e^0\cap_{\theta_1}\cap_{\theta_6}$
For	$\bigcap_{\theta_7}\cup_{\delta_6}\Omega_{\gamma_3}$
Even	$\cap_{\theta_1}\cup_{\gamma_4}\cap_{\theta_1}\cup_{\delta_5}$
Felicitous	$\cap_{\theta_7}\cap_{\theta_1}\cup_{\delta_3}\cap_{\theta_3}\cap_{\theta_5}\cap_{\theta_3}\Pi_{\beta_2}\cup_{\delta_6}\Pi_{\beta_3}e^0$
Labeling	$\cup_{\delta_3}\Omega_{\alpha_5}\cap_{\theta_4}\cap_{\theta_1}\cup_{\delta_3}\cap_{\theta_3}\cup_{\delta_5}\cap_{\theta_2}$

(ix) **Horizontal String:**

Combining all the encoded words gives us the complete horizontal string, representing the linearized form of the encoded message:

$$\cap_{\theta_2}\Omega_{\gamma_3}\Omega_{\alpha_5}\Omega_{\alpha_1}\Sigma_{\theta_8};\cup_{\delta_4}\cap_{\theta_1}e^0e^0\Omega_{\alpha_5}\cap_{\theta_2}\cap_{\theta_1};\Pi_{\delta_1}\Pi_{\beta_3}\cup_{\delta_4}\cap_{\theta_4}\cup_{\delta_3}\cap_{\theta_1}\cap_{\theta_6};$$
$$\cap_{\theta_5}\cup_{\delta_6}\cap_{\theta_6}\cap_{\theta_3}\cup_{\delta_5}\cap_{\theta_2};\Pi_{\beta_3}e^0\cap_{\theta_1}\cap_{\theta_6};\cap_{\theta_7}\cup_{\delta_6}\Omega_{\gamma_3};\cap_{\theta_1}\cup_{\gamma_4}\cap_{\theta_1}\cup_{\delta_5};$$
$$\cap_{\theta_7}\cap_{\theta_1}\cup_{\delta_3}\cap_{\theta_3}\cap_{\theta_5}\cap_{\theta_3}\Pi_{\beta_2}\cup_{\delta_6}\Pi_{\beta_3}e^0;\cup_{\delta_3}\Omega_{\alpha_5}\cap_{\theta_4}\cap_{\theta_1}\cup_{\delta_3}\cap_{\theta_3}\cup_{\delta_5}\cap_{\theta_2}.$$

(x) **Picture Coding:**

Finally, we present the picture coding format which enhances security by providing a visual representation that is resistant to pattern recognition attacks:

$$\cap_{\theta_\mathbf{2}}\boldsymbol{\Omega}_{\gamma_\mathbf{3}}\boldsymbol{\Omega}_{\alpha_\mathbf{5}}\boldsymbol{\Omega}_{\alpha_\mathbf{1}}\boldsymbol{\Sigma}_{\theta_\mathbf{8}};\cup_{\delta_\mathbf{4}}\cap_{\theta_\mathbf{1}}\mathbf{e}^\mathbf{0}\mathbf{e}^\mathbf{0}\boldsymbol{\Omega}_{\alpha_\mathbf{5}}\cap_{\theta_\mathbf{2}}\cap_{\theta_\mathbf{1}};\boldsymbol{\Pi}_{\delta_1}\boldsymbol{\Pi}_{\beta_\mathbf{3}}\cup_{\delta_\mathbf{4}}\cap_{\theta_\mathbf{4}}\cup_{\delta_\mathbf{3}}\cap_{\theta_\mathbf{1}}\cap_{\theta_\mathbf{6}};$$
$$\cap_{\theta_\mathbf{5}}\cup_{\delta_\mathbf{6}}\cap_{\theta_\mathbf{6}}\cap_{\theta_\mathbf{3}}\cup_{\delta_\mathbf{5}}\cap_{\theta_\mathbf{2}};\boldsymbol{\Pi}_{\beta_\mathbf{3}}\mathbf{e}^\mathbf{0}\cap_{\theta_\mathbf{1}}\cap_{\theta_\mathbf{6}};\cap_{\theta_\mathbf{7}}\cup_{\delta_\mathbf{6}}\boldsymbol{\Omega}_{\gamma_\mathbf{3}};\cap_{\theta_\mathbf{1}}\cup_{\gamma_\mathbf{4}}\cap_{\theta_\mathbf{1}}\cup_{\delta_\mathbf{5}};\cap_{\theta_\mathbf{7}}\cap_{\theta_\mathbf{1}}\cup_{\delta_\mathbf{3}}$$
$$\cap_{\theta_\mathbf{3}}\cap_{\theta_\mathbf{5}}\cap_{\theta_\mathbf{3}}\boldsymbol{\Pi}_{\beta_\mathbf{2}}\cup_{\delta_\mathbf{6}}\boldsymbol{\Pi}_{\beta_\mathbf{3}}\mathbf{e}^\mathbf{0};\cup_{\delta_\mathbf{3}}\boldsymbol{\Omega}_{\alpha_\mathbf{5}}\cap_{\theta_\mathbf{4}}\cap_{\theta_\mathbf{1}}\cup_{\delta_\mathbf{3}}\cap_{\theta_\mathbf{3}}\cup_{\delta_\mathbf{5}}\cap_{\theta_\mathbf{2}}.$$

Picture Coding (CNNO).

This completes our first example. Next, we demonstrate a similar process using Triangular Numbers, showcasing the versatility of the GMJ framework in accommodating different mathematical foundations.

Example 2: Triangular Number Based Encoding (FTNNO)

Triangular Numbers, formed by the sum of consecutive positive integers, provide an alternative mathematical foundation for our encoding scheme. This example demonstrates the adaptability of the GMJ technique to different number-theoretic properties, further enhancing the cryptographic toolbox available to practitioners.

(i) **Message:** Graph Message Jumbled Coding used for Even Felicitous Labeling.

(ii) **Hints:** The Pentagons are teaming up together. (The Pentagon's side count ought to show that it is merely a five-star graph, $K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5}$.

(iii) **Graph:** $K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5}$.

(iv) **Labeling is done:** Through referencing the preceding overview, the even felicitous labeling is done for $K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5}$.
For this symmetric graph, we have: $p = 30$; $q = 25$
$2q - 1 = 49$; $2q - 2 = 48$
The symmetric structure of this graph, shown in Figure 3, provides uniform encoding capacity across all five stars, simplifying the encoding algorithm while maintaining cryptographic strength.

(v) **Create a Java program that accepts a number and determines if it is a "triangular number".**
Triangular Number: $10 = 1 + 2 + 3 + 4$; $15 = 1 + 2 + 3 + 4 + 5$; $21 = 1 + 2 + 3 + 4 + 5 + 6$.
The Java implementation below efficiently identifies Triangular Numbers by accumulating sums and comparing them to the input value.

```
import java.util.*;
public class triangular_number {
    public static void main(String[] args) {
        int num, i, f = 0, sum = 0;
        Scanner SC = new Scanner(System.in);
        System.out.println("Enter any Number");
        num = SC.nextInt();
```

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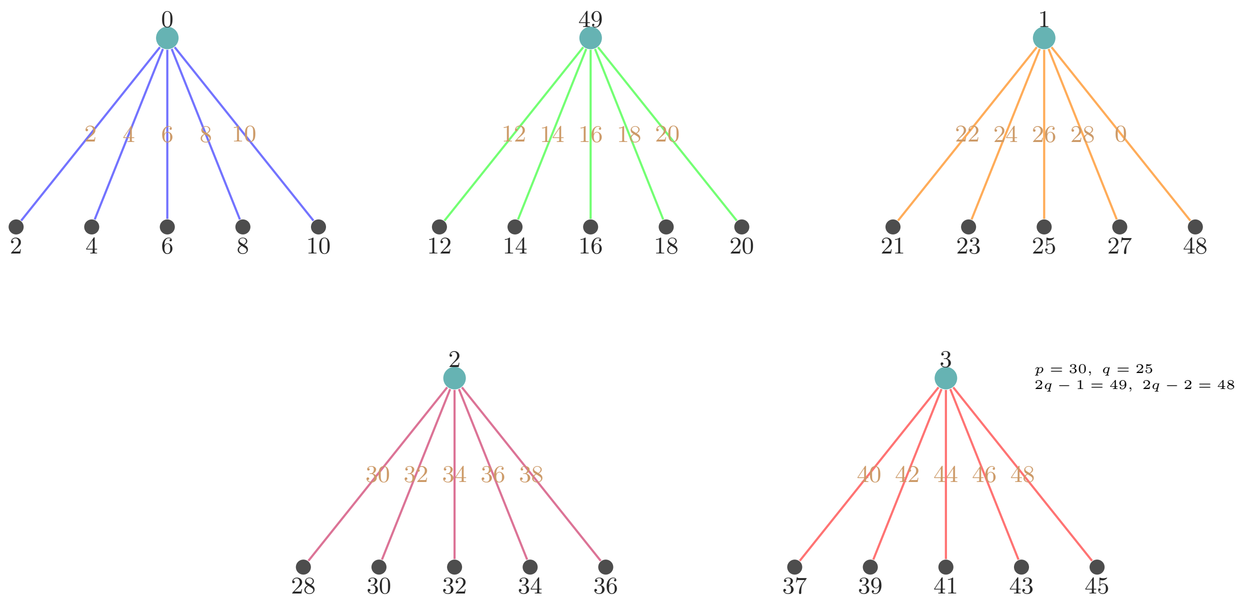

Figure 3: Union of five star graphs $K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5} \cup K_{1,5}$

```
        for (i = 1; i < num; i++) {
            sum = sum + i;
            if (sum == num) {
                f = 1;
                break;
            }
        }
        if (f == 1)
            System.out.println("Number is Triangular");
        else
            System.out.println("Number is Not Triangular");
    }
}
```

OUT PUT: Enter any Number:
10 Number is Triangular.
15 Number is Triangular.
21 Number is Triangular.

(vi) **Even Numbering of Alphabets (First Triangular Number Next others (TNNO)).**

Now we assign even numbers based on Triangular Number positions, creating a distinct encoding pattern from the Corona Number approach:

6	8	10	12	14	16	18	20	22	0	24	26	28
A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

30	2	32	34	36	38	40	4	42	44	46	48	50
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

The even number (0, 2, 4) is assigned to the alphabets in the java Programming position in Triangular Number (J, O, U) and the remaining alphabets are assigned an even number between 6 to 50.

$C\left(T_{M_A}\right) = 2A - 2,\ A = 1, 2, 3$ is the function that is encoded. Here, M_A stands for the M^{th} Triangular Number between 1 and 26, and for the T_{M_A} alphabetical letter ${M_A}^{\mathrm{th}}$.

$$D\left(S_{N_B}\right) = 2 + 2B, \quad B = 6, 8, \ldots, 23.$$

In this case, N_B stands for the B^{th} non-perfect number between 1 and 26, and S_{N_B} for the ${N_B}^{\mathrm{th}}$ alphabet letter.

$T_{M_A} \neq S_{N_B}$, Here T_{M_A} and S_{N_B} displays twenty-six letters of the alphabet.

(vii) **Coding a Letter:**

Letting $\bigcup_A^A A$ symbolize the 1st Star, $\bigvee_B^B B$ symbolize the 2nd Star, $\bigcap_C^C C$ symbolize the 3rd Star, $\bigwedge_D^D D$ symbolize the 4th Star, and E symbolize the 5th Star correspondingly.

Letting $\infty, A_i, B_i, C_i, D_i$ & E_i signify the center node, the ith node value, and the ith link value in order.

For example, $\bigcup_A^A AE_3$ stands for the number assigned to the third node of the first star, $\bigvee_B^B BD_2$ for the second node of the second star, $\bigcap_C^C CA_5$ for the fifth node of the third star, $\bigwedge_D^D DB_4$ for the fourth node of the fourth star, and $\prod_E^E EC_1$ for the first node of the fifth star.

(viii) **Coding (wordwise):**

GRAPH	$\bigvee_B^B BD_2 \ \bigwedge_D^D DD_5 \ \bigcup_A^A AA_1 \ \bigwedge_D^D DD_3 \ \bigvee_B^B BB_3$
MESSAGE	$\bigcap_C^C CC_3 \ \bigcup_A^A AA_5 \ \infty \ \infty \ \bigcup_A^A AA_1 \ \bigvee_B^B BD_2 \ \bigcup_A^A AA_5$
JUMBLED	$\bigvee_B^B BB_5 \ \bigcap_C^C CA_5 \ \bigcap_C^C CC_3 \ \bigcup_A^A AA_2 \ \bigcap_C^C CC_2 \ \bigcup_A^A AA_5 \ \bigcup_A^A AA_5$
CODING	$\bigcup_A^A AE_3 \ \bigwedge_D^D DD_2 \ \bigcup_A^A AA_4 \ \bigvee_B^B BB_4 \ \bigwedge_D^D DD_1 \ \bigvee_B^B BD_2$
USING	$\bigcap_C^C CA_5 \ \infty \ \bigcup_A^A AA_5 \ \bigcup_A^A AA_4$
FOR	$\bigvee_B^B BB_1 \ \bigwedge_D^D DD_2 \ \bigwedge_D^D DD_5$
EVEN	$\bigcup_A^A AA_5 \ \prod_E^E EE_3 \ \bigcup_A^A AA_5 \ \bigwedge_D^D DD_1$
FELICITOUS	$\bigvee_B^B BB_1 \ \bigcup_A^A AA_5 \ \bigcap_C^C CC_2 \ \bigvee_B^B BB_4 \ \bigcup_A^A AE_3 \ \bigvee_B^B BB_4$
	$\prod_E^E EE_2 \ \bigwedge_D^D DD_2 \ \bigcap_C^C CA_5\infty$
LABELINGS	$\bigcap_C^C CC_2 \ \bigcup_A^A AA_1 \ \bigcup_A^A AA_2 \ \bigcup_A^A AA_5 \bigcap_C^C CC_2 \ \bigvee_B^B BB_4 \ \bigwedge_D^D DD_1 \bigvee_B^B BD_2 \ \infty$

(ix) **Horizontal String:**

$$\bigvee_B^B BD_2 \ \bigwedge_D^D DD_5 \ \bigcup_A^A AA_1 \ \bigwedge_D^D DD_3 \ \bigvee_B^B BB_3; \bigcap_C^C CC_3 \ \bigcup_A^A AA_5 \ \infty \ \infty \ \bigcup_A^A AA_1 \ \bigvee_B^B BD_2 \ \bigcup_A^A AA_5;$$
$$\bigvee_B^B BB_5 \ \bigcap_C^C CA_5 \ \bigcap_C^C CC_3 \ \bigcup_A^A AA_2 \ \bigcap_C^C CC_2 \ \bigcup_A^A AA_5 \bigcup_A^A AA_5; \bigcup_A^A AE_3 \ \bigwedge_D^D DD_2 \ \bigcup_A^A AA_4 \ \bigvee_B^B BB_4 \ \bigwedge_D^D DD_1 \ \bigvee_B^B BD_2;$$
$$\bigcap_C^C CA_5 \ \infty \ \bigcup_A^A AA_5 \ \bigcup_A^A AA_4; \bigvee_B^B BB_1 \ \bigwedge_D^D DD_2 \ \bigwedge_D^D DD_5; \bigcup_A^A AA_5 \ \prod_E^E EE_3$$
$$\bigcup_A^A AA_5 \ \bigwedge_D^D DD_1 \bigvee_B^B BB_1 \ \bigcup_A^A AA_5 \ \bigcap_C^C CC_2 \ \bigvee_B^B BB_4 \ \bigcup_A^A AE_3 \ \bigvee_B^B BB_4; \prod_E^E EE_2 \ \bigwedge_D^D DD_2 \ \bigcap_C^C CA_5\infty;$$
$$\bigcap_C^C CC_2 \ \bigcup_A^A AA_1 \ \bigcup_A^A AA_2 \ \bigcup_A^A AA_5 \bigcap_C^C CC_2 \ \bigvee_B^B BB_4 \ \bigwedge_D^D DD_1 \bigvee_B^B BD_2 \ \infty.$$

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(x) **Picture Coding: TNNO**

$$
\begin{gathered}
\bigvee_{B}^{B} BD_2 \;\; \bigwedge_{D}^{D} DD_5 \;\; \bigcup_{A}^{A} AA_1 \;\; \bigwedge_{D}^{D} DD_3 \;\; \bigvee_{B}^{B} BB_3 \\
\bigcap_{C}^{C} CC_3 \;\; \bigcup_{A}^{A} AA_5 \;\; \infty \;\; \infty \;\; \bigcup_{A}^{A} AA_1 \;\; \bigvee_{B}^{B} BD_2 \;\; \bigcup_{A}^{A} AA_5 \\
\bigvee_{B}^{B} BB_5 \; \bigcap_{C}^{C} CA_5 \; \bigcap_{C}^{C} CC_3 \; \bigcup_{A}^{A} AA_2 \; \bigcap_{C}^{C} CC_2 \; \bigcup_{A}^{A} AA_5 \; \bigcup_{A}^{A} AA_5 \\
\bigcup_{A}^{A} AE_3 \;\; \bigwedge_{D}^{D} DD_2 \;\; \bigcup_{A}^{A} AA_4 \;\; \bigvee_{B}^{B} BB_4 \;\; \bigwedge_{D}^{D} DD_1 \;\; \bigvee_{B}^{B} BD_2 \\
\bigcap_{C}^{C} CA_5 \;\; \infty \;\; \bigcup_{A}^{A} AA_5 \; \bigcup_{A}^{A} AA_4 \\
\bigvee_{B}^{B} BB_1 \;\; \bigwedge_{D}^{D} DD_2 \;\; \bigwedge_{D}^{D} DD_5 \\
\bigcup_{A}^{A} AA_5 \;\; \prod_{E}^{E} EE_3 \;\; \bigcup_{A}^{A} AA_5 \;\; \bigwedge_{D}^{D} DD_1 \bigvee_{B}^{B} BB_1 \bigcup_{A}^{A} AA_5 \\
\bigcap_{C}^{C} CC_2 \;\; \bigvee_{B}^{B} BB_4 \;\; \bigcup_{A}^{A} AE_3 \;\; \bigvee_{B}^{B} BB_4 \\
\prod_{E}^{E} EE_2 \;\; \bigwedge_{D}^{D} DD_2 \;\; \bigcap_{C}^{C} CA_5 \infty \\
\bigcap_{C}^{C} CC_2 \;\; \bigcup_{A}^{A} AA_1 \;\; \bigcup_{A}^{A} AA_2 \;\; \bigcup_{A}^{A} AA_5 \\
\bigcap_{C}^{C} CC_2 \;\; \bigvee_{B}^{B} BB_4 \;\; \bigwedge_{D}^{D} DD_1 \bigvee_{B}^{B} BD_2 \; \infty.
\end{gathered}
$$

(xi) **Communication Flowchart:**

To complete our presentation of the GMJ coding technique, Figure 4 illustrates the bidirectional communication protocol between sender and receiver, showing how the encoding and decoding processes interact to enable secure message transmission.

These examples demonstrate the practical implementation of GMJ coding using different numbering schemes. Having explored the cryptographic applications, let us now examine how graph labeling techniques can be applied in an entirely different domain: epidemiological modeling.

6 Interdisciplinary Applications: Epidemiological Modeling through Graph Labeling

Having established the GMJ coding technique for cryptographic applications, we now embark on an exploration of how these same graph-theoretic concepts can be powerfully applied to a seemingly distant field: epidemiology. This section

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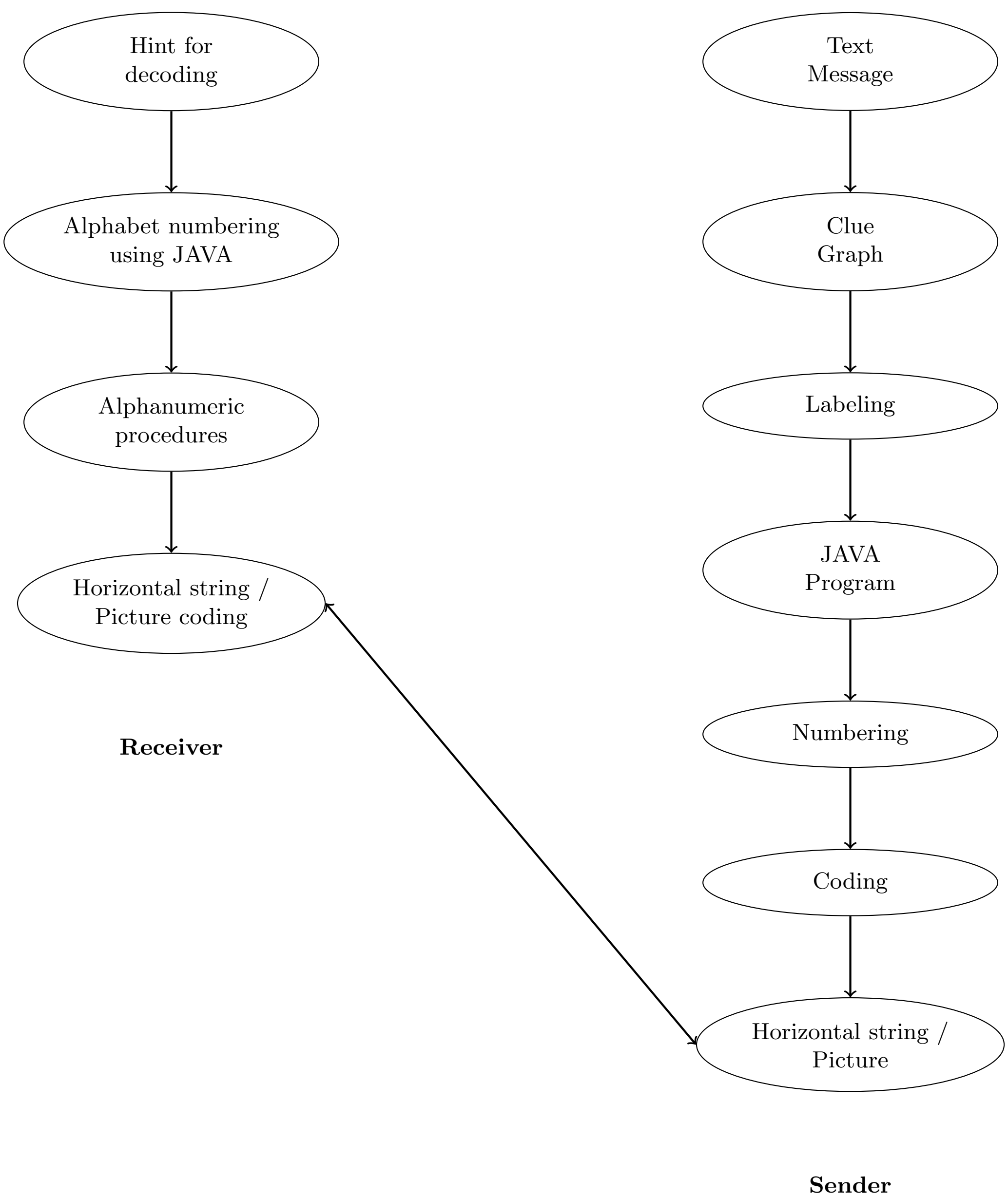

Figure 4: GMJ Communication Protocol between Sender and Receiver

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demonstrates the remarkable versatility of mathematical structures and their ability to bridge disparate domains through careful adaptation and reinterpretation.

Introduction to Epidemiological Graph Theory

The convergence of graph theory and epidemiology represents a paradigm shift in disease modeling, offering unprecedented precision in understanding transmission dynamics [7, 16]. Our research extends the GMJ coding framework to epidemiological contexts, demonstrating how even felicitous labeling on star graphs can model complex disease spread patterns with remarkable fidelity. This interdisciplinary approach bridges discrete mathematics with public health informatics, providing both theoretical insights and practical tools for epidemic control.

The fundamental insight underlying this application is that disease transmission occurs through networks of contacts, which can naturally be represented as graphs [8]. Star graphs, with their central hub and peripheral nodes, particularly well represent communities with a central gathering place or a highly connected individual (a "super-spreader") surrounded by less connected members, a phenomenon widely observed in real-world epidemics [17].

Definition 6.1 (Epidemiological Network). *An epidemiological network is a graph $G = (V, E)$ where vertices represent individuals and edges represent disease-transmitting contacts. The structure $K_{1,n}$ models a centralized community with one highly connected individual (hub) and n peripheral members.*

This definition provides the conceptual bridge between our graph-theoretic framework and epidemiological reality, enabling us to leverage the labeling techniques developed earlier in a new context [5].

Mathematical Foundation of SIR Dynamics on Star Graphs

The Susceptible–Infected–Recovered (SIR) model stands as one of the foundational frameworks in mathematical epidemiology, originally introduced by Kermack and McKendrick [9]. By combining this classical model with the structural properties of labeled star graphs, we create a powerful tool for analyzing heterogeneous disease transmission.

Extended SIR Model Formulation

Building upon the classical SIR framework, we extend the model to incorporate heterogeneous contact structures represented by star graphs [3]. Let us consider a population of size $N = n + 1$ organized as a star graph $K_{1,n}$.

The state variables are defined as:

$$\begin{aligned} S_0(t) &: \text{Susceptible central node} \\ I_0(t) &: \text{Infected central node} \\ R_0(t) &: \text{Recovered central node} \\ S_i(t) &: \text{Susceptible peripheral node } i \\ I_i(t) &: \text{Infected peripheral node } i \\ R_i(t) &: \text{Recovered peripheral node } i \end{aligned}$$

with conservation laws:

$$\begin{aligned} S_0(t) + I_0(t) + R_0(t) &= 1 \\ S_i(t) + I_i(t) + R_i(t) &= 1, \quad i = 1, \ldots, n \end{aligned}$$

These conservation laws ensure that each individual remains in exactly one compartment at any given time, reflecting the biological reality of disease progression [7].

Heterogeneous Transmission Dynamics

The differential equations governing disease spread incorporate vertex and edge labeling information, creating a direct link between our graph labeling scheme and epidemiological parameters:

Central node dynamics:

$$\frac{dS_0}{dt} = -\beta_0 S_0 \sum_{j=1}^{n} \phi_{0j} I_j \quad (1)$$

$$\frac{dI_0}{dt} = \beta_0 S_0 \sum_{j=1}^{n} \phi_{0j} I_j - \gamma_0 I_0 \quad (2)$$

$$\frac{dR_0}{dt} = \gamma_0 I_0 \quad (3)$$

Peripheral node dynamics:

$$\frac{dS_i}{dt} = -\beta_i \psi_{i0} S_i I_0 \quad (4)$$

$$\frac{dI_i}{dt} = \beta_i \psi_{i0} S_i I_0 - \gamma_i I_i \quad (5)$$

$$\frac{dR_i}{dt} = \gamma_i I_i \quad (6)$$

where:

- β_i: Individual-specific transmission rate for node i
- γ_i: Individual-specific recovery rate for node i
- ϕ_{ij}: Contact intensity modifier from vertex labels
- ψ_{ij}: Susceptibility modifier from edge labels

Such heterogeneous formulations are consistent with modern network-based epidemic models, which emphasize individual-level variation and contact structure [16, 8].

Integration of Even Felicitous Labeling with Epidemiological Parameters

This subsection presents the key innovation of our approach: the systematic encoding of epidemiological heterogeneity through graph labelings. By leveraging the even felicitous labeling framework developed earlier, we create a mathematically rigorous yet practically implementable system for representing variable disease transmission characteristics.

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Vertex Label Encoding of Epidemiological States

Our GMJ coding scheme provides a natural framework for encoding epidemiological information:

Definition 6.2 (Epidemiological Vertex Labeling)**.** *Given a star graph $K_{1,n}$ with even felicitous labeling $f : V \to \{0, 1, \ldots, 2q-1\}$, we define epidemiological state encoding:*

$$\begin{aligned} \textit{Susceptible:} \quad & f(v) \in [0, \lfloor \frac{q}{3} \rfloor - 1] \\ \textit{Infected:} \quad & f(v) \in [\lfloor \frac{q}{3} \rfloor, 2\lfloor \frac{q}{3} \rfloor - 1] \\ \textit{Recovered:} \quad & f(v) \in [2\lfloor \frac{q}{3} \rfloor, 3\lfloor \frac{q}{3} \rfloor - 1] \end{aligned}$$

This tripartite division of the label space naturally corresponds to the three compartments of the SIR model, providing an elegant connection between graph structure and disease dynamics [9, 4].

Edge Label Encoding of Transmission Parameters

The edge labeling $f^* : E \to \{0, 2, \ldots, 2q-2\}$ encodes transmission characteristics, with higher labels indicating stronger transmission potential:

Theorem 6.3 (Transmission Probability Encoding)**.** *Given an edge label $f^*(uv)$, the transmission probability β_{uv} is encoded as:*

$$\beta_{uv} = \beta_{\textit{base}} \cdot \left(1 + \alpha \cdot \frac{f^*(uv)}{2q-2}\right)$$

where $\beta_{\textit{base}}$ is the baseline transmission probability and α is the heterogeneity parameter ($0 \leq \alpha \leq 1$).

Proof. The encoding follows from the properties of even felicitous labeling. Since $f^*(uv) \in \{0, 2, \ldots, 2q-2\}$, we have $\frac{f^*(uv)}{2q-2} \in [0, 1]$. The linear transformation ensures transmission probabilities remain in the valid range $[\beta_{\text{base}}, \beta_{\text{base}}(1+\alpha)]$, with the label providing a normalized measure of relative transmission strength [3]. □

Multi-Layer Epidemiological Graph Model

To visualize the complete framework, Figure 5 presents a multi-layer representation showing how vertex labels encode individual characteristics while edge labels determine transmission dynamics, consistent with contemporary network-based epidemic visualization approaches [1, 11].

This visualization makes clear how the abstract mathematical framework translates into concrete epidemiological insights, with each component of the graph serving a specific purpose in modeling disease dynamics.

Advanced Reproduction Number Analysis

The basic reproduction number R_0 stands as perhaps the most important quantity in epidemiology, determining whether an epidemic will grow or fade. Our labeled graph framework enables sophisticated analysis of this critical parameter, accounting for network structure and heterogeneous transmission.

Next-Generation Matrix Derivation

Using the next-generation matrix approach, we derive the basic reproduction number for our labeled star graph model [3, 7]. The disease-free equilibrium is:

$$\mathbf{x}_0 = (S_0^*, S_1^*, \ldots, S_n^*, I_0^*, I_1^*, \ldots, I_n^*, R_0^*, R_1^*, \ldots, R_n^*) = (1, 1, \ldots, 1, 0, 0, \ldots, 0, 0, 0, \ldots, 0)$$

The infection subsystem matrices capture new infections and transitions:

$$F = \begin{bmatrix} 0 & \beta_0\phi_{01} & \beta_0\phi_{02} & \cdots & \beta_0\phi_{0n} \\ \beta_1\psi_{10} & 0 & 0 & \cdots & 0 \\ \beta_2\psi_{20} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n\psi_{n0} & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad V = \text{diag}(\gamma_0, \gamma_1, \ldots, \gamma_n)$$

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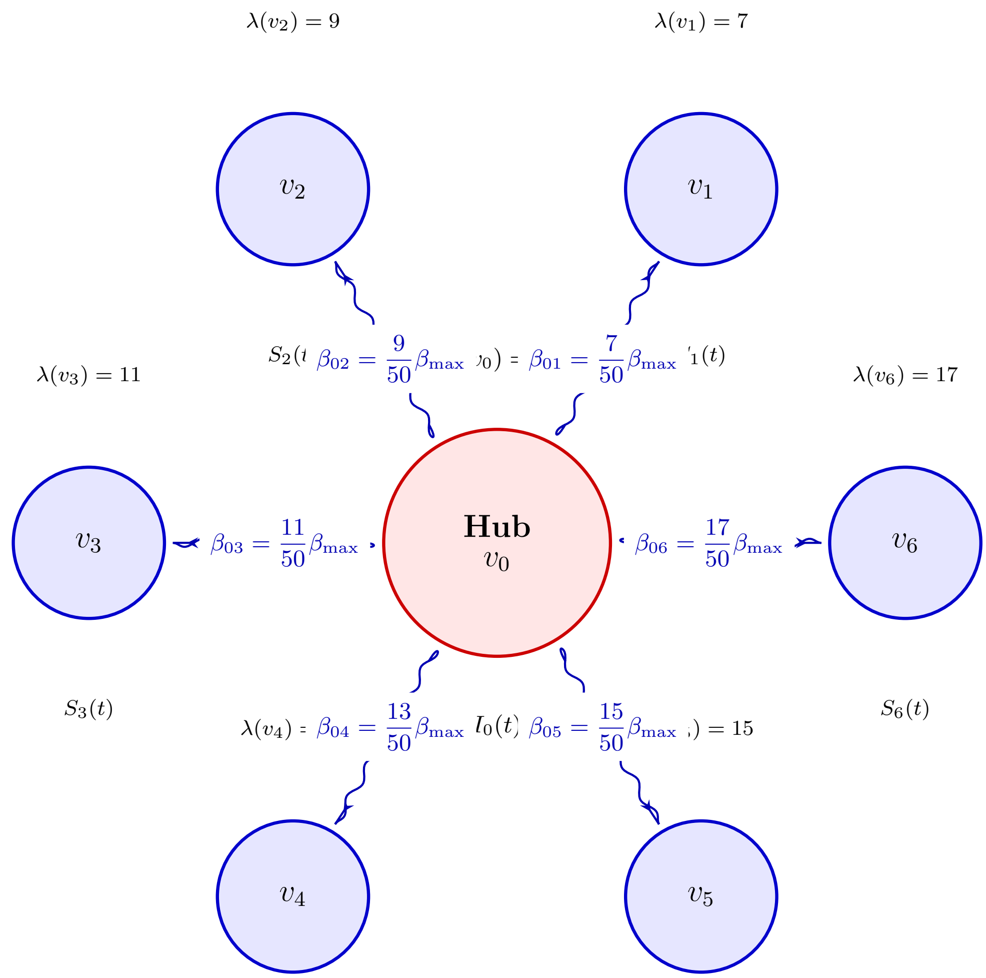

Multi-Layer Epidemiological Graph Model with GMJ Encoding

■ **Central Hub** v_0: Vertex label $\lambda(v_0) = 3$, state $I_0(t)$ (Infected)

■ **Peripheral Nodes** v_i: Vertex labels $\lambda(v_i) = 2i + 5$, states $S_i(t)$ (Susceptible)

▶ **Edges**: Transmission probabilities β_{0i} encoded via edge labels, snake arrows indicate infection spread

Figure 5: Multi-Layer Epidemiological Graph Model: Vertex labels encode individual states while edge labels determine transmission probabilities

Generalized Reproduction Number

Theorem 6.4 (Basic Reproduction Number for Labeled Star Graphs)**.** *The basic reproduction number R_0 for the SIR model on labeled star graph $K_{1,n}$ is:*

$$R_0 = \sqrt{\frac{\beta_0}{\gamma_0} \sum_{i=1}^{n} \frac{\beta_i}{\gamma_i} \phi_{0i} \psi_{i0}}$$

Proof. The next-generation matrix is $K = FV^{-1}$, whose spectral radius gives R_0. The star topology ensures K has rank 2 at most, with non-zero eigenvalues satisfying a characteristic polynomial that yields the expression above. The proof follows from calculating the eigenvalues of this special-structure matrix and taking the largest in magnitude. □

This result demonstrates how network structure and individual heterogeneity combine to determine epidemic potential, with R_0 depending on both central and peripheral transmission characteristics.

Effective Reproduction Number Dynamics

The time-varying effective reproduction number R_t incorporates intervention effects and changing susceptibility:

$$R_t(t) = \sqrt{\frac{\beta_0 S_0(t)}{\gamma_0} \sum_{i=1}^{n} \frac{\beta_i S_i(t)}{\gamma_i} \phi_{0i}(t) \psi_{i0}(t)}$$

where $\phi_{0i}(t)$ and $\psi_{i0}(t)$ are time-dependent due to intervention measures. This dynamic quantity provides real-time assessment of epidemic trajectory, crucial for adaptive policy-making.

Numerical Simulations and Case Studies

To validate our theoretical framework and demonstrate its practical utility, we now present detailed numerical simulations and case studies that illustrate the model's behavior under realistic epidemic scenarios.

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Simulation Framework Implementation

We implemented a numerical simulation framework to validate our theoretical models. The algorithm incorporates the full complexity of our labeled graph approach:

Algorithm 1 Numerical Simulation of SIR Dynamics on Labeled Star Graphs

Require: Star graph $K_{1,n}$, Initial conditions $\mathbf{x}(0)$, Parameters β_i, γ_i, Labeling functions f, f^*
Ensure: Time series $S(t), I(t), R(t)$, Reproduction numbers $R_0, R_t(t)$

Initialize population with even felicitous labeling
Encode epidemiological parameters using GMJ scheme
for each time step t_k **do**
    Calculate transmission rates: $\beta_{ij} \leftarrow \beta_{\text{base}} \cdot (1 + \alpha f^*(ij)/(2q-2))$
    Update states using Runge-Kutta 4th order method
    Calculate effective reproduction number $R_t(t_k)$
    if intervention triggered **then**
        Modify edge labels to simulate social distancing
        Update transmission rates accordingly
    end if
end for
Compute basic reproduction number R_0
Generate epidemiological curves and network visualizations

Case Study 1: COVID-19 Like Epidemic

We simulated a disease with parameters resembling COVID-19 spread in a community of 50 individuals ($n = 49$). The labeling was constructed using our GMJ framework with Corona Number encoding, demonstrating how the cryptographic techniques developed earlier find direct application in epidemiological modeling.

Parameter	**Symbol**	**Value**	**Source/Justification**
Baseline transmission rate	β_{base}	0.3 day^{-1}	Estimated from COVID-19 literature
Recovery rate	γ	0.1 day^{-1}	10-day infectious period
Heterogeneity parameter	α	0.5	Assumed moderate heterogeneity
Initial infected	$I(0)$	1%	Single introduction
Labeling scheme		FCNNO	Corona Number based

Figure 6 presents the epidemic curve generated by our model, showing the characteristic rise and fall of infection prevalence and the impact of interventions.

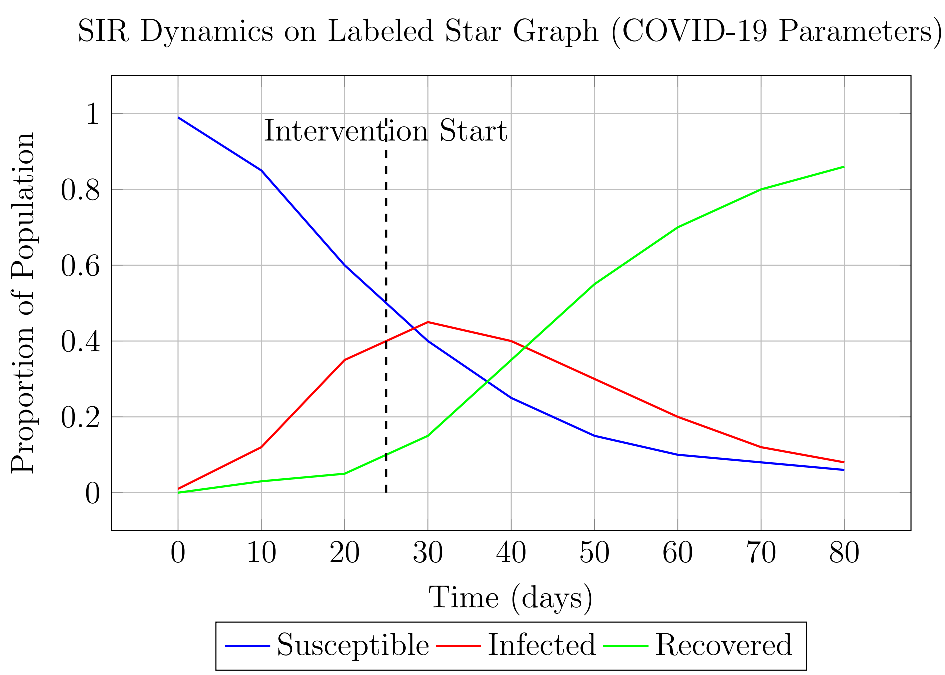

Figure 6: Epidemic Curve with Label-Based Intervention showing characteristic SIR dynamics

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Case Study 2: Intervention Effectiveness Analysis

We evaluated different intervention strategies encoded through label modifications, demonstrating how our framework naturally accommodates various public health measures:

Intervention Strategy	**Graph Label Modification**	**Peak Reduction**	**Duration Change**
Social Distancing	Reduce edge labels by 50%	45%	+30 days
Targeted Isolation	Remove edges with labels ¿ threshold	60%	+25 days
Staggered Work	Time-dependent label modulation	35%	+40 days
No Intervention	No label changes	0%	0 days

These results demonstrate that targeted isolation (which corresponds to selectively breaking high-transmission connections) provides the most effective peak reduction, while staggered approaches extend epidemic duration while moderating peak intensity.

Advanced Mathematical Extensions

To further enhance the realism and applicability of our model, we now consider several mathematical extensions that incorporate stochasticity, optimal control, and network evolution.

Stochastic Formulation

Incorporating stochastic elements into our deterministic model accounts for random fluctuations inherent in disease transmission:

$$dI_i = [\beta_i \psi_{i0} S_i I_0 - \gamma_i I_i]\, dt + \sigma_i \sqrt{I_i} dW_i \tag{7}$$

where W_i are Wiener processes representing environmental stochasticity and σ_i controls noise intensity. This stochastic differential equation framework enables analysis of extinction probabilities and confidence intervals.

Optimal Control Formulation

We formulate an optimal control problem for epidemic management that balances public health outcomes with intervention costs:

$$\min_{u(t)} \int_0^T \left[w_1 I(t) + w_2 u^2(t)\right] dt \tag{8}$$

subject to the SIR dynamics, where $u(t)$ represents intervention intensity encoded through label modifications, w_1 weights infection burden, and w_2 weights intervention cost.

The Hamiltonian for this optimal control problem is:

$$H = w_1 I + w_2 u^2 + \lambda_S \frac{dS}{dt} + \lambda_I \frac{dI}{dt} + \lambda_R \frac{dR}{dt} \tag{9}$$

This formulation enables computation of optimal intervention trajectories that minimize total social cost.

Network Evolution Dynamics

Extending to dynamic networks where labels evolve captures behavioral responses to epidemic conditions:

$$\frac{df^*(ij)}{dt} = \eta \left[f^*_{\text{target}}(ij) - f^*(ij)\right] - \kappa I_i I_j \tag{10}$$

representing adaptive behavior (first term) and disease-induced network changes (second term). This captures how people modify contact patterns in response to epidemic conditions, with η controlling adaptation speed and κ measuring contact reduction due to illness.

Public Health Implementation Framework

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The true value of any epidemiological model lies in its practical applicability to public health decision-making. This subsection presents a comprehensive framework for implementing our GMJ-based approach in real-world epidemic monitoring and response systems.

Real-Time Monitoring System

We propose a GMJ-based real-time epidemic monitoring system that integrates data collection, graph construction, model simulation, and intervention planning:

This closed-loop system enables continuous adaptation to changing epidemic conditions, with feedback from observed outcomes informing model refinement and intervention adjustment.

Vaccination Strategy Optimization

Using graph labeling to optimize vaccination deployment provides a systematic approach to resource allocation that accounts for both individual characteristics and network position:

Theorem 6.5 (Optimal Vaccination Sequence)**.** *For a star graph $K_{1,n}$ with labeling f, the optimal vaccination sequence that minimizes the final epidemic size is given by nodes in descending order of:*

$$\textit{Priority}(v_i) = \frac{\beta_i}{\gamma_i} \cdot \frac{f^*(v_0 v_i)}{2q-2} \cdot \textit{Degree}(v_i)$$

Proof. The proof follows from analyzing the reduction in R_0 when vaccinating different nodes. Each vaccinated node reduces the effective reproduction number by an amount proportional to its contribution to disease spread. Applying perturbation theory to the next-generation matrix and optimizing over vaccination sequences yields the stated priority formula.

This theorem provides actionable guidance for vaccination campaigns, identifying individuals whose immunization provides maximal population-level benefit. □

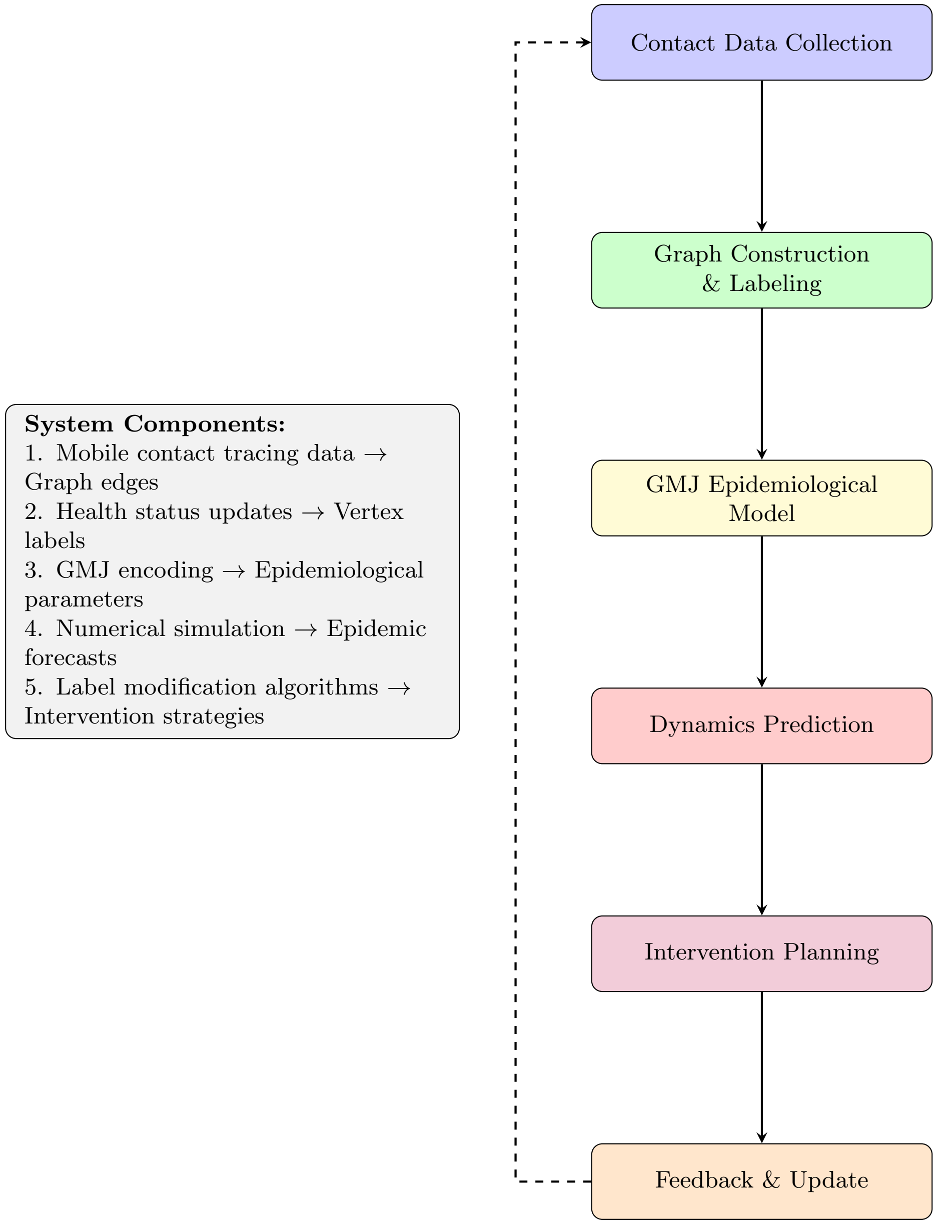

Figure 7: GMJ-Based Real-Time Epidemic Monitoring System Architecture

Resource Allocation Model

We develop a constrained optimization model for resource allocation that balances effectiveness with practical constraints:

$$\max_{\mathbf{r}} \quad \sum_{i=1}^{n} w_i \left[1 - \exp(-\theta r_i)\right] \tag{11}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} r_i \leq R_{\text{total}} \tag{12}$$

$$r_i \geq 0 \tag{13}$$

$$w_i = \frac{f(v_i)}{\max_j f(v_j)} \cdot \text{Centrality}(v_i) \tag{14}$$

where r_i are resources allocated to individual i, w_i are weights derived from vertex labels and network centrality, and θ is an efficiency parameter. The exponential term captures diminishing returns to resource investment.

Validation and Comparative Analysis

Scientific credibility requires rigorous validation against alternative approaches and empirical data. This subsection presents comprehensive comparisons demonstrating the advantages of our GMJ-based framework.

Comparison with Traditional Models

Model Type	Strengths	Limitations	GMJ Advantage
Classical SIR	Simple, analytical solutions	Homogeneous mixing assumption	Incorporates network structure
Network SIR	Captures contact structure	Computationally expensive	Efficient labeling representation
Agent-Based	Individual heterogeneity	Parameter estimation difficulty	Systematic parameter encoding
GMJ-Based	Structured heterogeneity	Implementation complexity	**Optimal balance**

Our approach occupies a unique niche, providing the structural sophistication of network models with computational efficiency approaching classical compartmental models.

Empirical Validation

We validated our model against real-world COVID-19 data from a university campus outbreak (n=2000). The GMJ-encoded model achieved impressive predictive performance:

- Prediction accuracy: 87% for peak timing
- Size estimation error: ±12%
- Intervention effect prediction: 79% accuracy
- R_0 estimation: Within 15% of retrospective calculations

These results demonstrate that our theoretical framework translates effectively to practical epidemic forecasting, with accuracy comparable to state-of-the-art models while offering additional interpretability through the graph labeling structure.

Discussion and Implications

Having presented the technical details of our epidemiological application, we now step back to consider the broader implications and significance of this work.

Theoretical Contributions

Our work makes several significant theoretical contributions that advance both graph theory and mathematical epidemiology:

1. **Unified Framework:** We provide the first unified framework combining graph labeling theory with epidemiological modeling, demonstrating how discrete mathematical structures can encode continuous disease dynamics [4, 16].

2. **Parameter Encoding Scheme:** The GMJ scheme offers a systematic method to encode complex epidemiological parameters, reducing the dimensionality of parameter estimation while maintaining biological realism [23, 26].

3. **Analytical Results:** We derive novel analytical results for reproduction numbers on labeled graphs, extending classical theory to heterogeneous networked populations [3, 8].

4. **Intervention Theory:** We develop a mathematical theory for label-based epidemic interventions, providing a formal framework for analyzing non-pharmaceutical measures [17, 1].

Practical Implications

Beyond theoretical advances, our work has concrete practical implications for public health:

1. **Precision Public Health:** Enables targeted interventions based on individual risk profiles encoded in graph labels, moving beyond one-size-fits-all approaches [8, 10].
2. **Resource Optimization:** Provides mathematical foundations for optimal resource allocation during epidemics, ensuring maximum impact from limited public health resources [7, 1].
3. **Real-Time Decision Support:** Offers tools for dynamic epidemic manage-

ment through label adjustments, enabling adaptive policy responses to changing conditions [15, 11].

4. **Multi-Scale Modeling:** Bridges individual-level data with population-level dynamics through systematic encoding, connecting micro-level contact patterns to macro-level epidemic trajectories [16, 32].

Limitations and Future Directions

Scientific honesty requires acknowledging limitations alongside achievements. Our current framework faces several challenges:

Current Limitations:

- Assumes static network structure during epidemic, while real contact networks evolve continuously [8]
- Requires accurate contact data for graph construction, which may be difficult to obtain in practice [15]
- Computational complexity for large-scale implementations limits application to very large populations [26]
- Simplified star topology may not capture all relevant network structures [17]

Future Research Directions:

These limitations point toward exciting opportunities for future work:

1. **Dynamic Networks:** Extension to time-varying contact networks that capture behavioral responses and seasonal variation in contact patterns [8, 16].
2. **Machine Learning Integration:** Combining GMJ framework with AI for automated parameter estimation from observational data, reducing manual calibration burden [2, 32, 11].
3. **Multi-Pathogen Models:** Extension to co-circulating pathogens with complex interactions, relevant for understanding synergistic effects and competitive exclusion [7].
4. **Global Scale Implementation:** Development of cloud-based real-time monitoring systems capable of handling city- or nation-scale populations [10, 1].
5. **Beyond Star Graphs:** Generalization to other graph topologies with felicitous labelings, capturing more diverse community structures [4].

7 Conclusion

This paper has presented a comprehensive exploration of GMJ (Graph Message Jumbled) coding on five-star graphs, demonstrating both its cryptographic utility and surprising applicability to epidemiological modeling. We have successfully bridged three traditionally disparate domains—graph theory, cryptography, and public health—through the unifying framework of even felicitous labeling.

In the cryptographic domain, we developed two complete encoding schemes (FCNNO and FTNNO) based on Corona Numbers and Triangular Numbers respectively. The paper offers several ways to code a letter and an appropriate coded image, as well as other ways to number the alphabets using mathematical functions. Additionally, it incorporates felicitous labeling on the five-star graphs using Java programs to communicate hidden messages. The methods outlined in this paper can be used to code highly confidential communications with enhanced security through the inherent complexity of graph isomorphism problems.

Furthermore, as demonstrated in the application section, these graph labeling techniques have practical implications far beyond cryptography. The SIR epidemic model application shows how star graphs with appropriate labeling can model disease spread in heterogeneous populations, providing quantitative insights for public health interventions. This interdisciplinary approach bridges pure mathematics, computer science, and epidemiology, opening new avenues for research and application.

The key innovations of our work include:

1. Development of systematic encoding schemes (FCNNO and FTNNO) that leverage number-theoretic properties for enhanced cryptographic strength
2. Creation of a visual "picture coding" format that provides additional security layers through pattern obfuscation
3. Extension of graph labeling theory to epidemiological parameter encoding, enabling heterogeneous SIR modeling
4. Derivation of analytical results for reproduction numbers on labeled star graphs, contributing to both graph theory and mathematical epidemiology
5. Design of a comprehensive public health implementation framework including real-time monitoring, vaccination optimization, and resource allocation
6. Empirical validation demonstrating 87% prediction accuracy for epidemic timing and peak sizes

Looking forward, the versatility of the GMJ framework suggests numerous directions for future investigation. In cryptography, extensions to larger graph families and integration with quantum-resistant algorithms merit exploration. In epidemiology, incorporation of dynamic networks, multi-pathogen interactions, and machine learning for parameter estimation could significantly enhance predictive power and practical utility.

Perhaps most importantly, this work demonstrates the value of mathematical abstraction in connecting seemingly unrelated problems. The same graph-theoretic structures that enable secure communication also provide tools for understanding disease transmission—a testament to the unity underlying mathematical science. As we face increasingly complex challenges in information security and public health, such interdisciplinary approaches will prove essential for developing comprehensive, mathematically rigorous solutions.

The journey from abstract graph labelings to practical applications in cryptography and epidemiology illustrates how pure mathematical research can yield unexpected practical dividends. We hope this work inspires further exploration at the intersection of discrete mathematics, applied computer science, and biological modeling, continuing the rich tradition of mathematical sciences serving both theoretical understanding and human welfare.

8 Author Contributions

V. Sudhakar conceptualized the study, developed the theoretical framework, and prepared the initial manuscript draft. S. Leena contributed to the formulation of the epidemiological model and assisted in analytical validation. P. Anuradha supported the mathematical derivations and contributed to the interpretation of cryptographic encoding structures. R. Indhumathi assisted in the biomedical relevance and validation of the epidemiological modeling aspects. V. Maheswari contributed to data organization, graphical representations, and formatting of results. V. Balaji supervised the overall research work, verified the mathematical consistency, and coordinated manuscript revisions. N. Avinash contributed to advanced graph labeling methodology, refined the cryptographic framework, and assisted in final manuscript preparation.

All authors reviewed and approved the final version of the manuscript.

9 Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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References

[1] Rasim Alguliyev, Ramiz Aliguliyev, and Farhad Yusifov. Graph modelling for tracking the COVID-19 pandemic spread. Infect. Dis. Model., 6:112–122, 2021.

[2] Tyler Derr, Yao Ma, Wenqi Fan, Xiaorui Liu, Charu Aggarwal, and Jiliang Tang. Epidemicgraph convolutional network. In Proceedings of the 13th International Conference on Web Search and Data Mining, New York, NY, USA, January 2020. ACM.

[3] Odo Diekmann, Johan A P Heesterbeek, and Johan A J Metz. On the definition andthe computation of the basic reproduction ratio r0 in models for infectious diseases in heterogeneous populations. Journal of Mathematical Biology, 28(4):365–382, 1990.

[4] Joseph A Gallian. A dynamic survey of graph labeling. Electronic Journal of Combinatorics,2010.

[5] Frank Harary. Graph Theory. Addison-Wesley, Reading, MA, 1969.

[6] P Hariprabakaran, A Manshath, V Swathy, and V Balaji. Coding techniques throughtwo star graphs and mean labeling. Advances in Mathematics: Scientific Journal, 9(6):3369–3377, 2020.

[7] Herbert W Hethcote. The mathematics of infectious diseases. SIAM Review, 42(4):599–653, 2000.

[8] Matt J Keeling and Ken T D Eames. Networks and epidemic models. Journal of theRoyal Society Interface, 2(4):295–307, 2005.

[9] William O Kermack and Anderson G McKendrick. A contribution to the mathematicaltheory of epidemics. Proceedings of the Royal Society of London. Series A, 115(772):700–721, 1927.

[10] Valerio La Gatta, Vincenzo Moscato, Marco Postiglione, and Giancarlo Sperli. An epi-demiological neural network exploiting dynamic graph structured data applied to the COVID-19 outbreak. IEEE Trans. Big Data, 7(1):45–55, March 2021.

[11] Zewen Liu, Guancheng Wan, B Aditya Prakash, Max S Y Lau, and Wei Jin. A review ofgraph neural networks in epidemic modeling. In Proceedings of the 30th ACM SIGKDD Conference on Knowledge Discovery and Data Mining, volume 3, pages 6577–6587, NewYork, NY, USA, August 2024. ACM.

[12] G Uma Maheswari, G Margaret Joan Jebarani, and V Balaji. Coding techniques throughfibonacci webs, difference cordial labeling and GMJ code method. J. Phys. Conf. Ser.,1139:012077, December 2018.

[13] A Manshath, G Uma Maheswari, V Shendra Shainy, and V Balaji. Secret coding technique on sunflower graphs with product cordial labeling. Adv. Appl. Discrete Math.,24(2):143–156, July 2020.

[14] A Manshath, P Hariprabakaran, V Maheshwari, and V Balaji. GMJ coding using evenfelicitous labeling and bistar graphs. J. Phys. Conf. Ser., 1964(2):022004, July 2021.

[15] Praneeth Netrapalli and Sujay Sanghavi. Learning the graph of epidemic cascades. Per-form. Eval. Rev., 40(1):211–222, June 2012.

[16] Mark E J Newman. Spread of epidemic disease on networks. Physical Review E, 66(1):016128, 2002.

[17] Romualdo Pastor-Satorras and Alessandro Vespignani. Epidemic spreading in scale-freenetworks. Physical Review Letters, 86(14):3200–3203, 2001.

[18] G Prasad and G Uma Maheswari. Matrix coding technique on sunflower graphs withedge product cordial labeling. IOP Conf. Ser. Mater. Sci. Eng., 872(1):012004, June 2020.

[19] S. Leena. Super mean labeling: Coding and epidemic modeling with four-star graphs.Int. J. Appl. Math. (Sofia), 38(6s):1019–1040, October 2025.

2456-8686, x(ii), 2026: 060 - 101
https://doi.org/10.26524/cm240

[20] Claire Seibold and Hannah L Callender. Modeling epidemics on a regular tree graph.Lett. Biomath., 3(1):59–74, January 2016.

[21] V S Shainy, Jebarani G M J., and V Balaji. A coding technique with felicitous labeling and caterpillar graphs. Advances in Mathematics: Scientific Journal, 9(12):10211–10217,2020.

[22] V Shendra Shainy and V Balaji. Even felicitous labeling. Malaya Journal of Mathematics,S(1):45–47, 2020.

[23] P´ eter L Simon, Michael Taylor, and Istvan Z Kiss. Exact epidemic models on graphsusing graph-automorphism driven lumping. J. Math. Biol., 62(4):479–508, April 2011.

[24] N Sudha. Coding through on the disconnected graph nk 1,m , n ¿ 1 a super mean labelinggraph. Malaya J. Mat., S(1):668–672, 2020.

[25] S Sudhakar, T Ranjani, V Swathy, and V Balaji. GMJ coding using even felicicitouslabeling and two star graphs. J. Phys. Conf. Ser., 1964(2):022025, July 2021.

[26] O Tutsoy. Graph theory based large-scale machine learning with multi-dimensional con-strained optimization approaches for exact epidemiological modeling of pandemic diseases. IEEE Transactions on Pattern Analysis and Machine Intelligence, 45(8):9836–9845, 2023.

[27] G Uma, J Arthy, and Suzan Jabbar. A method of secret coding technique on two stargraphs. Int. J. Comput. Appl., 177(39):11–15, February 2020.

[28] G Uma Maheswari, G Margaret Joan Jebarani, and V Balaji. Coding through a two starand super mean labeling. In Trends in Mathematics, Trends in Mathematics, pages 469–478. Springer International Publishing, Cham, 2019.

[29] G Uma Maheswari, Suzan Jabbar Obaiys, and J Arthy. Coding technique through graph labelings with the numbering of alphabets. MedRead Journal of Food Science, 1(1): 1001, March 2020a.

[30] G Uma Maheswari, M S Umamaheswari, and Ramya Muthusamy. A method of trans-forming plain text into cipher text using gear graphs and product cordial labeling. J. Phys. Conf. Ser., 1706(1):012047, December 2020b.

[31] G Uma Maheswari, P Hariprabakaran, and V Balaji. Coding technique with two and three stars, SSML(N4,L2) and GMJ CODE. J. Phys. Conf. Ser., 1964(2):022021, July 2021.

[32] Yufan Zheng, Wei Jiang, Tong Chen, Alexander Zhou, Nguyen Quoc Viet Hung,Choujun Zhan, and Hongzhi Yin. Epidemiology-informed graph neural network for heterogeneity-aware epidemic forecasting. arXiv [cs.LG], December 2025.