

Urban Traffic Flow and Intersection Safety Analysis through a Novel Picture Fuzzy Outerplanar Graph Approach

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Abstract

Picture fuzzy sets provide a powerful framework for representing uncertainty by incorporating degrees of membership, neutrality, and non-membership simultaneously. Motivated by the limitations of classical fuzzy and intuitionistic fuzzy planar graph models in handling complex uncertain relationships, this paper introduces the concept of picture fuzzy outerplanar graphs (PFOGs) and investigates their structural properties. The study establishes the construction of picture fuzzy outerplanar subgraphs through vertex and edge deletion operations and characterizes the notions of maximal and maximum picture fuzzy outerplanar subgraphs. Several illustrative examples are presented to demonstrate the applicability of the proposed concepts. Furthermore, a series of theorems and corollaries are developed to reveal fundamental relationships among picture fuzzy outerplanar graphs and their associated substructures. The concept of a picture fuzzy dual graph is also introduced, and its important properties and connections with picture fuzzy outerplanar graphs are examined. By combining outerplanar graph theory with picture fuzzy information, the proposed framework offers a flexible mathematical tool for modeling systems involving uncertain and hesitant interactions. Potential applications include urban traffic network analysis, risk assessment of crossing-sensitive infrastructures, transportation planning, and decision-support systems operating under uncertainty. The results presented in this work extend the scope of fuzzy graph theory and provide a foundation for future investigations on advanced picture fuzzy graph structures and their illustrative example applications.

Key Words: Fuzzy logic, Picture fuzzy graphs, Picture fuzzy planar graphs, Picture fuzzy outerplanar graphs, Picture fuzzy outerplanar subgraphs, Picture fuzzy dual graphs, Urban traffic flow, Intersection risk analysis

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1 Introduction

Graph theory is a vital area of mathematics that illuminates the intricate relationships within networks. Its development has significantly advanced our understanding of various graph structures, particularly through the study of planar and outerplanar graphs. There are several research reports on various aspects of graph theory. Primarily, graph theory, as articulated in the seminal works of Harary [41] and further developed by Bondy and Murty [17], serves as a robust mathematical framework for depicting relationships within networks. Early contributions from Chartrand and Harary [18] laid the groundwork for understanding planar permutation structures, which provided an essential structural basis for analysing the behavior of planar graphs. Subsequently, Fleisechner et al. [34] delved into the study of weak duals of outerplanar graphs, shedding light on concepts of duality and structural decomposition. Contributions from Sysło focused on critical characterizations, algorithmic strategies, and generalizations that solidified outerplanar graphs as a well-defined category within graph theory [70,71,72]. Further extending this field, Kedlaya demonstrated the capacity to partition planar graphs into multiple outerplanar components, thereby enhancing their utility in various decomposition and optimization scenarios [43].

Zadeh [75] pioneered the modeling of uncertainty through fuzzy set theory, establishing a foundation for understanding vagueness in mathematical terms. Building on this, Kaufmann and Bonaert [42] formalized fuzzy subsets, leading to structured representations of uncertainty within graph structures. Furthermore, Rosenfeld [61] expanded this concept by introducing fuzzy graphs, allowing vertices and edges to have graded memberships. Abdul-Jabbar et al. [2] developed fuzzy dual graphs, shedding light on dual relationships amid uncertainty. Similarly, Atanassov [13] introduced intuitionistic fuzzy sets, inspiring subsequent models refined by Rashmanlou et al. [60], Sahoo and Pal [63], and Alshehri and Akram [5], which enhance the representation of hesitation, incompleteness, and partial knowledge in networks. Likewise, Samanta and Pal [64,65,66] initiated fuzzy planar graph theory, while Shriram et al. [69] applied fuzzy planar subgraphs to VLSI layout design.

Deivanai et al. explored various innovative models, such as fuzzy outerplanar graphs for traffic systems [26], bipolar fuzzy outerplanar graphs for image shrinking [27], and interval-valued fuzzy outerplanar models for bypass road planning [28]. Deivanai and Sujatha [29] offered a comprehensive overview of topological indices in fuzzy graphs and multipolar fuzzy outerplanar graph approach to one-way road network construction [30]. Deivanai et al. introduced fuzzy graph theoretic approach to face shape recognition using cubic outerplanar structures [31]. Recently, Deivanai et al. [32] introduced image contraction techniques based on fuzzy soft outerplanar graph structures, demonstrating their effectiveness in handling uncertainty in graphical image representations. Advancements in m-polar and interval-valued modeling were made by Akram et al. [3], Ghorai and Pal [38], and Bera and Pal [16], enhancing the expressive capabilities of fuzzy graph systems. Mahapatra et al. [48] developed interval-valued m-polar fuzzy planar graphs, competition networks [49], and resource-controlled threshold models [50]. Similarly, Pramanik et al. [57,58] expanded planar and bipolar fuzzy frameworks to provide more refined representations of uncertainty.

Recognizing the limitations of classical fuzzy and intuitionistic models in addressing neutrality, Cuong and Kreinovich [20] introduced picture fuzzy sets, which permit positive, neutral, and negative membership degrees. Cuong and Pham [21] further developed picture fuzzy operators, while Phong et al. [59] enhanced picture fuzzy relations for practical applications. Zuo et al. [76] established picture fuzzy graphs, laying the groundwork for picture fuzzy network representation. Xiao et al. [73] proposed regular picture fuzzy graphs, Amanathulla et al. [6] developed balanced picture fuzzy models, and Das et al. [25] introduced picture fuzzy tolerance graphs for flexible relational analysis. Moreover, Amanathulla et al. [7] demonstrated picture fuzzy methods for optimizing airline route networks. Applications of picture fuzzy frameworks have expanded significantly, including Das and Ghorai's picture fuzzy road-mapping [22] and picture fuzzy medical interaction analysis. Koczy et al. [47] extended these concepts to Wi-Fi and social network environments. Other advancements comprise competition graph models [49], studies product bipolar fuzzy graphs, extending the concept of bipolar fuzzy graphs through a product-based structure [47], tolerance-based systems [55], Algorithm for colouring picture fuzzy graphs [62] and picture fuzzy line graphs by Chen et al. [19], who adapted classical line graph concepts to picture fuzzy contexts to effectively model positive, neutral, and negative uncertainty in relational structures.

Additional contributions include multisource picture fuzzy graphs by Bera et al. [15], Cayley picture fuzzy structures by Khan et al. [44], picture fuzzy chromatic polynomials for traffic control by Abbas and Mustafa et al. [1] and Khan et al. [45] investigated strong paired domination in interval-valued picture fuzzy graphs, highlighting their structural properties under uncertainty. Shi et al. [67] developed bipolar picture fuzzy graph models for applications in image segmentation and road network design, demonstrating their effectiveness in handling complex real-world systems. Khatun et al. [46] further advanced the field by introducing m-polar picture fuzzy graphs. Mohanta et al. [51] proposed Picture Dombi fuzzy graphs by integrating Dombi operators, offering a more flexible aggregation mechanism. Shoaib et al. [68] formalized key operations union, intersection, and composition on picture fuzzy graphs, enabling richer structural manipulations under uncertainty. Das et al. [24] introduced picture fuzzy threshold graphs, demonstrating their utility in medicine replenishment planning. Amanathulla et al. [8] highlighted a variety of real-world applications for discrete mathematical models, emphasizing their significance in optimization, decision-making, and network analysis. Ramya and Lavanya [59] introduced an edge-contraction technique for bipolar fuzzy graphs, simplifying graph structures while retaining essential properties, which aids in analyzing connectivity and reducing complexity for further applications, underscoring their importance in analyzing structural properties under uncertainty.

Fuzzy graph theory and topological indices have significantly expanded the applicability of graph-based mathematical models in solving real-world decision-making and optimization problems under uncertainty. Anwar et al. [9] introduced an optimal hotel chain network design model using fuzzy hyper soft topological structures integrated with the PROMETHEE decision-making technique, demonstrating the effectiveness of uncertainty-based optimization in revenue maximization problems. Furthermore, Azeem et al. [14] investigated

topological numbers of fuzzy soft graphs and highlighted their practical significance in analyzing structural properties of uncertain networks. Their work established a strong connection between fuzzy graph theoretic measures and practical decision-making frameworks. Subsequently, Anwar et al. [10] extended the concept of fuzzy soft topological numbers through vertex deletion operations and performed a comparative analysis with the TOPSIS method for car import decision-making. This study emphasized the robustness and flexibility of fuzzy topological approaches in multi-criteria decision environments. In another contribution, Anwar et al. [11] developed single-valued neutrosophic fuzzy Sombor numbers and applied them to trade flow analysis between countries connected through sea routes, illustrating the usefulness of neutrosophic graph descriptors in modeling uncertain economic and transportation systems. Moreover, graph-theoretic indices have also been widely employed in chemical graph theory. Anwar et al. [12] examined extremal values of the first reformulated Zagreb index for molecular trees and applied the results to octane isomers, thereby demonstrating the relevance of topological indices in predicting molecular characteristics and chemical behaviors. Recent developments in fuzzy planar graph theory include a comprehensive review of fuzzy planar graph applications by Deivanai et al. [33], Pythagorean fuzzy planar graphs by Akram et al. [4], vague planar graphs by Ghorai and Pal [69], inverse fuzzy mixed planar graphs by Mondal and Ghorai [52], generalized m-polar fuzzy planar graphs by Mondal et al. [53], cubic planar graphs by Muhiuddin et al. [54], and bipolar fuzzy planar graphs by Yahya Mohamed and Subashini [74]. These generalized fuzzy graph models have significantly enhanced the ability to represent uncertainty and complexity in network structures, thereby motivating further research on picture fuzzy outerplanar graph models and their applications.

Generalized graph theory, including hypergraphs, super hypergraphs, neutrosophic graphs, bipolar fuzzy graphs, and m-polar fuzzy planar graphs, have significantly improved the modeling of uncertainty and complex relationships in network systems [35,36,37]. Hypergraphs and super hypergraphs are particularly effective in representing multi-vertex interactions and higher-order relational structures, while generalized fuzzy graph models provide efficient mechanisms for handling vague, imprecise, and inconsistent information. However, many practical systems simultaneously involve uncertainty, neutrality, and planar structural constraints, which cannot be fully addressed by existing models alone. In this context, picture fuzzy outerplanar graphs offer a flexible mathematical framework capable of representing positive, neutral, and negative membership information while preserving outerplanar properties. Consequently, these graph structures possess strong potential for applications in intelligent transportation systems, communication networks, social and biological networks, and artificial intelligence-based decision-support systems. These generalized fuzzy graph models have significantly enhanced the ability to represent uncertainty and complexity in network structures, thereby motivating further research on picture fuzzy outerplanar graph models and their applications.

Classical research on outerplanar structures has demonstrated their effectiveness in representing systems characterized by restricted connectivity and hierarchical layouts [34,70,72]. Foundational studies in fuzzy and picture fuzzy sets offer valuable tools for

modelling uncertainty, hesitation, and multi-level information [48,56,75]. Further developments in picture fuzzy graph theory including planar, tolerance, line, and threshold graphs underscore the versatility of picture fuzzy frameworks in managing positive, neutral, and negative information across various applications [6,7,51,68]. The integration of these domains presents a compelling opportunity, merging the structural simplicity of outerplanar topology with the robust uncertainty-handling capabilities of picture fuzzy logic. This synergy has the potential to address complex real-world challenges in areas such as image processing, transportation networks, and intelligent decision-support systems.

This research work introduces and investigates the concept of picture fuzzy outerplanar graphs, which extend the theories of fuzzy planar graphs and intuitionistic fuzzy planar graphs by incorporating positive, neutral, and negative membership degrees. The study further examines the formation of picture fuzzy outerplanar subgraphs through vertex and edge deletion operations and analyses maximal and maximum picture fuzzy outerplanar subgraphs with supporting examples. Several theorems and corollaries are established to describe the relationships among these concepts. In addition, the notion of a picture fuzzy dual graph associated with picture fuzzy outerplanar graphs is introduced, and its important structural properties are discussed. Overall, picture fuzzy outerplanar graphs provide a robust framework for detecting high-risk intersections and representing uncertain urban traffic conditions, thereby supporting effective traffic planning, congestion minimization, and improved safety in city transportation management. The abbreviations used in this paper are given in Table 1.

Table 1: lists the abbreviations used in this paper

Abbreviation	Description
PFSS	Picture Fuzzy Sets
PFSS	Picture Fuzzy Subsets
PFGs	Picture Fuzzy Graphs
PFMS	Picture Fuzzy Multiset
PFMG	Picture Fuzzy Multigraph
PFPGs	Picture Fuzzy Planar Graphs
PDFG	Picture Fuzzy Dual Graph
PFR	Picture Fuzzy Relation
VR	Vertex Removal
ER	Edge Removal
PFOGs	Picture Fuzzy Outerplanar Graph
PFOs	Picture Fuzzy Outerplanar Subgraphs
WPFPGs	Weak Picture Fuzzy Planar Graphs
SPFPGs	Strong Picture Fuzzy Planar Graphs
VR-PFOs	Vertex Removal Picture Fuzzy Outerplanar Subgraphs
ER-PFOs	Edge Removal Picture Fuzzy Outerplanar Subgraphs
Maximum VR & ER PFOs	Maximum Vertex & Edge Removal Picture Fuzzy Outerplanar Subgraphs
Maximal VR & ER PFOs	Maximal Vertex & Edge Removal Picture Fuzzy Outerplanar Subgraphs

PFF	Picture Fuzzy Face
PFS	Picture Fuzzy Subgraph
PFCG	Picture Fuzzy Connected Graph
PFNOG	Picture Fuzzy Non-Outerplanar Graph
OPFF	Outer Picture Fuzzy Face
IPFF	Inner Picture Fuzzy Face
aFOG	Fuzzy Outerplanar Graph
VRFSs	Vertex Removal Fuzzy Subgraph
ERFSs	Edge Removal Fuzzy Subgraph
BFOG	Bipolar Fuzzy Outerplanar Graph
IvFOG	Interval-valued Fuzzy Outerplanar Graph
mPFOG	m-Polar Fuzzy Outerplanar Graph
CuFOG	Cubic Fuzzy Outerplanar Graph
VLSI	Very Large-Scale Integration

2 Motivation of the proposed research

Outerplanar graphs occupy a fundamental position in graph theory because of their simple structural properties and their effectiveness in modeling systems with restricted or hierarchical connectivity, as highlighted in classical works by Fleisechner et al. [34], Sysło [71] and later applications in network design and decomposition [14,36]. The extension of these graphs to fuzzy environments, initiated by Rosenfeld's fuzzy graph model [61], provides a means to incorporate uncertainty into network structures by assigning degrees of membership to vertices and edges. However, classical fuzzy and intuitionistic fuzzy frameworks are limited in representing scenarios where positive, neutral, and negative uncertainties coexist simultaneously a common feature in practical decision-making and real-time systems. To overcome this limitation, picture fuzzy sets and picture fuzzy graphs were introduced by Cuong and Kreinovich [20] and further developed through works such as Cuong and Pham [21], Zuo et al. [76], Xiao et al. [73], Amanathulla et al. [6] and Das and Ghorai [25]. These models provide a richer structure by accommodating three independent degrees, thereby enabling more expressive representation of vague, conflicting, or indeterminate information. Their utility has been demonstrated across diverse domains including road network analysis [23], medical diagnosis [24] and threshold-based decision systems [59]. Although extensive studies exist on picture fuzzy planar graphs, mixed graphs, tolerance graphs, and related variants, the literature contains no exploration of picture fuzzy outerplanar graphs (PFOGs). Given that outerplanar graphs provide efficient, low-complexity representations and picture fuzzy structures allow multi-faceted uncertainty modeling, their integration opens a promising research avenue. Such a model is particularly suited for real-world problems in image processing, transportation planning, and intelligent decision support, where systems naturally exhibit combinations of positive evidence, negative tendencies, and neutral or indeterminate information.

Motivated by this research gap, this study introduces picture fuzzy outerplanar graphs (PFOGs), a structure that has not been previously investigated. Unlike existing fuzzy, intuitionistic fuzzy, and picture fuzzy planar graph models, the proposed framework integrates outerplanar topology with picture fuzzy information, allowing positive, neutral,

and negative information to be represented simultaneously. The study further develops picture fuzzy outerplanar subgraphs, maximum and maximal structures, and vertex and edge removal operations. A conceptual application to transportation networks is also presented to illustrate the potential of the proposed framework for modelling uncertain traffic conditions and supporting future studies in intelligent transportation and complex network analysis.

3 Novelty of the article

The novelty of this work lies in introducing PFOGs as a new extension of picture fuzzy graph theory. The proposed framework incorporates positive, neutral, and negative membership degrees to model uncertainty in graph structures. The study further develops vertex-removal and edge-removal PFOGs, maximum and maximal PFOG subgraphs, and picture fuzzy dual graphs, along with their fundamental structural properties. Several theorems and corollaries are established to extend classical outerplanar graph concepts to the picture fuzzy environment. A conceptual traffic-network application is also presented to illustrate the potential usefulness of PFOGs in representing uncertain traffic and intersection conditions.

4 Objective of the study

The primary objective of this study is to introduce and analyze picture fuzzy outerplanar graphs (PFOGs) and their fundamental properties.

1. To analyze the structural characteristics of picture fuzzy outerplanar graphs and establish their theoretical foundations.
2. To explore picture fuzzy outerplanar subgraphs obtained through vertex deletion and edge removal, and to understand the preservation of outerplanarity under these operations.
3. To define and examine maximum and maximal picture fuzzy outerplanar subgraphs, highlighting their differences and behavior with respect to vertex and edge removal. The theoretical results with detailed illustrative examples, enhancing clarity and interpretability of the proposed concepts. To establish interrelationships among picture fuzzy outerplanar concepts through the formulation of relevant theorems.
4. To define the concept of the picture fuzzy dual graph and investigate its key properties in relation to picture fuzzy outerplanar graphs. Analyze the relationship between picture fuzzy outerplanar graphs and their dual structures, extending classical planar graph theory into the picture fuzzy domain.
5. To develop a picture fuzzy outerplanar graph-based framework for urban traffic systems that realistically captures multiple degrees of uncertainty in road conditions and intersection states, enable the precise identification of high-risk intersections by modeling traffic parameters using picture fuzzy information and assist traffic planning, congestion mitigation, and safety enhancement through uncertainty-aware and mathematically robust graph-based analysis. Demonstrate the potential of picture fuzzy outerplanar graphs in addressing real-world problems involving uncertainty and vagueness, including transportation networks, decision-making systems, and related domains.

5 Preliminaries

This section explains the fundamental concepts related to picture fuzzy sets (PFSs), picture fuzzy graphs (PFGs) and picture fuzzy planar graphs (PFPGs). A complete description of the symbols and terminology used in the article is provided in Table 2.

Definition 5.1 [76] Let σ be a picture fuzzy set (PFS) in a universe of discourse $\mathcal{X}$. Then σ is defined as $\sigma = \{\langle u, \mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u) \rangle | u \in \mathcal{X}\}$, where the membership functions $u, \mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u): \mathcal{X} \rightarrow [0,1]$ satisfy the condition

$$0 \leq \mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u) \leq 1$$

Here, $\mu_\sigma(u)$ denotes the positive (truth) membership degree of u in σ , $\eta_\sigma(u)$ denotes the neutral membership degree, and $\nu_\sigma(u)$ denotes the negative (falsity) membership degree. The refusal membership degree of u in σ is given by

$$\pi_\sigma(u) = 1 - \mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u).$$

Definition 5.2 [76] A picture fuzzy relation (PFR) $\mathcal{R}$ on the cartesian product $\mathcal{X} \times \mathcal{Y}$ is defined as a picture fuzzy subset (PFSS) of $\mathcal{X} \times \mathcal{Y}$ given by

$$\mathcal{R} = \{\langle (u, v), \mu_{\mathcal{R}}(u, v), \eta_{\mathcal{R}}(u, v), \nu_{\mathcal{R}}(u, v) \rangle | u \in \mathcal{X}, v \in \mathcal{Y}\},$$

where the membership functions $\mu_{\mathcal{R}}(u, v), \eta_{\mathcal{R}}(u, v), \nu_{\mathcal{R}}(u, v): \mathcal{X} \times \mathcal{Y} \rightarrow [0,1]$ satisfy the condition,

$$0 \leq \mu_{\mathcal{R}}(u, v), \eta_{\mathcal{R}}(u, v), \nu_{\mathcal{R}}(u, v) \leq 1$$

The refusal membership degree is defined as

$$\pi_{\mathcal{R}}(u, v) = 1 - \mu_{\mathcal{R}}(u, v), \eta_{\mathcal{R}}(u, v), \nu_{\mathcal{R}}(u, v).$$

Definition 5.3 [6] A picture fuzzy graph (PFG) is represented as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$, where $\mathcal{V} = \{v_1, v_2, v_3, \dots, v_n\}$ denotes the set of vertices and σ and δ respectively represent the membership functions associated with vertices and edges. Here, $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ where

- (i) $\mu_\sigma: \mathcal{V} \rightarrow [0,1], \eta_\sigma: \mathcal{V} \rightarrow [0,1]$ and $\nu_\sigma: \mathcal{V} \rightarrow [0,1]$ denote the positive, neutral and negative membership functions for the vertices, respectively. These functions satisfy the condition $0 \leq \mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u) \leq 1$ for all $u \in \mathcal{V}$.

- (ii) Similarly, $\mu_\delta: \mathcal{V} \times \mathcal{V} \rightarrow [0,1], \eta_\delta: \mathcal{V} \times \mathcal{V} \rightarrow [0,1]$ and $\nu_\delta: \mathcal{V} \times \mathcal{V} \rightarrow [0,1]$ denote the positive, neutral and negative membership functions for the edges respectively. These edge membership functions satisfy the following conditions:

$$\mu_\delta(u, v) \leq \min\{\mu_\sigma(u), \mu_\sigma(v)\}$$

$$\eta_\delta(u, v) \leq \min\{\eta_\sigma(u), \eta_\sigma(v)\}$$

$$\nu_\delta(u, v) \geq \max\{\nu_\sigma(u), \nu_\sigma(v)\}$$

Definition 5.4 [22] Let $\mathcal{V}$ be a nonempty set. A picture fuzzy multiset (PFM) is defined using three functions: $C_{\mu p}$ for the positive membership count, $C_{\eta p}$ for the neutral membership count and $C_{\nu p}$ for the negative membership count, where each of these maps elements from U to the interval $[0,1]$.

The PF multiset is expressed as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ satisfies the following relations for all edges

$$\sigma = \{\langle u: (\mu_\sigma^1(u), \dots, \mu_\sigma^n(u)), (\eta_\sigma^1(u), \dots, \eta_\sigma^n(u)), (\nu_\sigma^1(u), \dots, \nu_\sigma^n(u)) \rangle | u \in \mathcal{U}\}$$

And it satisfies the condition

$$0 \leq \max(\mu_\sigma(u)) + \max(\eta_\sigma(u)) + \max(\nu_\sigma(u)) \leq 1.$$

Now consider a nonempty set $\mathcal{V}$ with two mappings σ and δ where

$$\sigma: \mathcal{V} \rightarrow [0,1]^3, \delta: \mathcal{V} \times \mathcal{V} \rightarrow [0,1]^3.$$

Here, σ defines the membership degrees of vertices, while δ defines the membership degrees of edges. The picture fuzzy multiset $(u, v) \in \mathcal{V} \times \mathcal{V}$:

$$\mu_\delta(u, v) \leq \min\{\mu_\sigma(u), \mu_\sigma(v)\}$$

$$\eta_\delta(u, v) \leq \min\{\mu_\sigma(u), \mu_\sigma(v)\}$$

$$\nu_\delta(u, v) \geq \max\{\mu_\sigma(u), \mu_\sigma(v)\}$$

Here, n_{uv} denotes the number of edges between the vertices u and v .

Definition 5.5[22] Let $\mathcal{V}$ be a nonempty set and let $\sigma = (\mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u))$ be a picture fuzzy set (PFS) defined on $\mathcal{V}$. Also, let

$\delta = \{\omega_r = (\mu_\delta(\omega_r), \eta_\delta(\omega_r), \nu_\delta(\omega_r)): r = 1, 2, \dots, n_{uv}, uv \in \mathcal{V} \times \mathcal{V}\}$ be a picture fuzzy multiset over $\mathcal{V} \times \mathcal{V}$. Then the structure $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ is called a PFM, for all $r = 1, 2, \dots, n_{uv}$ and for all $u, v \in \mathcal{V}$. Additionally, $0 \leq (\mu_\delta(\omega_r), \eta_\delta(\omega_r), \nu_\delta(\omega_r)) \leq 1$.

If for any r , all three values are zero. (i.e., $\mu_\delta(\omega_r) = \eta_\delta(\omega_r) = \nu_\delta(\omega_r) = 0$, then no edge exists between u and v . In Contrast, if at least one of these values is nonzero, an edge exists between them. A PFM allows multiple edges between two vertices, provided those edges have different membership values.

If there is no repetition of membership values between any two vertices, the graph is said to be a simple picture fuzzy graph. Throughout the discussion, the vertices and edges of a picture fuzzy graphs are denoted as picture fuzzy vertices (PFV) and picture fuzzy edges (PFE), respectively.

Definition 5.6[15] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy multigraph (PMG). Suppose that, for a given graphical presentation of $\mathcal{G}$, there are several points where the edges intersect, denoted by

$$\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_j$$

Then, $\mathcal{G}$ is called a picture fuzzy planar graph (PFPG) if it has associated picture fuzzy planarity value (PFPV), representing as

$$\mathcal{G} = (\mathcal{G}_\mu, \mathcal{G}_\eta, \mathcal{G}_\nu)$$

Where the components of ξ are defined as:

$$\xi = \left(\frac{1}{3[1 + \sum_{i=1}^{j-1} (\mathcal{J}_\mu)_{\mathcal{R}_i}]}, \frac{1}{3[1 + \sum_{i=1}^{j-1} (\mathcal{J}_\eta)_{\mathcal{R}_i}]}, \frac{1}{3[1 + \sum_{i=1}^{j-1} (\mathcal{J}_\nu)_{\mathcal{R}_i}]} \right)$$

Each component $(\mathcal{G}_\mu, \mathcal{G}_\eta, \mathcal{G}_\nu)$ represents the planarity measure with respect to the membership, neutrality, and non-membership degrees, respectively.

The values of these components are bounded between 0 and 0.3. In other words, the picture fuzzy planarity value (PFPV) quantifies how close the graph $\mathcal{G}$ is to being planar based on the fuzzy intersection values among its edges. For certain graphical representations, if no intersections exist among the edge (that is, $\mathcal{G}$ is completely planar), then the picture fuzzy planarity value (PFPV) components approach their maximum possible values.

Definition 5.7[15] A picture fuzzy graph (PFG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ is said to be a strong picture fuzzy planar graph (PFPG) if its picture fuzzy planarity value (PFPV)

$$\xi = (\mathcal{G}_\mu, \mathcal{G}_\eta, \mathcal{G}_\nu)$$

Satisfies $\mathcal{G}_\mu \geq \frac{1}{6}$, $\mathcal{G}_\eta \geq \frac{1}{6}$, $\mathcal{G}_\nu \geq \frac{1}{6}$ (if all the three of its planarity components $(\mathcal{G}_\mu, \mathcal{G}_\eta, \mathcal{G}_\nu)$ are greater than or equal to $\frac{1}{6}$).

Otherwise, the picture fuzzy graph $\mathcal{G}$ is called a weak picture fuzzy planar graph (WPFPG) (If any of these values are less than $\frac{1}{6}$, then the graph $\mathcal{G}$ is said to be a weak picture fuzzy planar graph (WPFPG). In simple terms, a strong picture fuzzy planar graph (SPFPG) has higher planarity values, showing better or more stable planarity, while a weak picture fuzzy planar graph (WPFPG) has lower values, indicated poorer planarity.

Definition 5.8[15] Consider a single valued PFM $\mathcal{G}$, and let $\mathcal{N}_p$ represent the total number of intersection points between its edges.

The planarity value of $\mathcal{G}$ is expressed as $\mathcal{G} = (\mathcal{G}_\mu, \mathcal{G}_\eta, \mathcal{G}_\nu)$, where each component is calculated by the formula:

$$\mathcal{G}_\mu = \frac{1}{3[1 + \mathcal{N}_p]}$$

$$\mathcal{G}_\eta = \frac{1}{3[1 + \mathcal{N}_p]}$$

$$\mathcal{G}_\nu = \frac{1}{3[1 + \mathcal{N}_p]}$$

This means that as the number of intersections ($\mathcal{N}_p$) increases, the planarity value decreases, showing that a higher number of crossings leads to lower planarity. Conversely, fewer intersections result in a more planar (better) graph.

Definition 5.9[22] A picture fuzzy face (PFF) plays a significant role in constructing the picture fuzzy dual graph (PFDG) corresponding to a given picture fuzzy graph (PFG). In a PFG, a PFF is defined as the region enclosed by picture fuzzy edges. If any edge forming the boundary of this region is removed, the corresponding PFF no longer exists.

Consider a picture fuzzy planar graph (PFPG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$. Where $\delta = \{uw, (\mu_\delta(uw)_r, \eta_\delta(uw)_r, \nu_\delta(uw)_r) : r = 1, 2, \dots, n_{uv}, uv \in \mathcal{V} \times \mathcal{V}\}$ represents the picture fuzzy multiset (PFMS) associated with $\mathcal{G}$.

A picture fuzzy face (PFF) is the area bounded by a subset of picture fuzzy edges $\mathcal{E}_1 \in \mathcal{E}$ in a planar representation of $\mathcal{G}$. The positive, neutral and negative membership values of the picture fuzzy face (PFF) are represented as

$$\mathcal{f} = (\mathcal{f}_\mu, \mathcal{f}_\eta, \mathcal{f}_\nu)$$

where

$$\mathcal{f}_\mu = \min \{(\mathcal{S}_\mu)_{uv_r}\} = \min \left\{ \frac{\mu_\delta(uv)_r}{\min\{\sigma_\mu(u), \sigma_\mu(v)\}} : r = 1, 2, \dots, n_{uv}, uv \in \mathcal{E}_1 \right\}$$

$$\mathcal{f}_\eta = \min \{(\mathcal{S}_\eta)_{uv_r}\} = \min \left\{ \frac{\eta_\delta(uv)_r}{\min\{\sigma_\eta(u), \sigma_\eta(v)\}} : r = 1, 2, \dots, n_{uv}, uv \in \mathcal{E}_1 \right\}$$

$$\mathcal{f}_\nu = \max \{(\mathcal{S}_\nu)_{uv_r}\} = \max \left\{ \frac{\nu_\delta(uv)_r}{\max\{\sigma_\nu(u), \sigma_\nu(v)\}} : r = 1, 2, \dots, n_{uv}, uv \in \mathcal{E}_1 \right\}$$

Also, a PFF is termed an effective PFF if the following conditions are satisfied:

$$\mathcal{f}_\mu \geq 0.5, \mathcal{f}_\eta \geq 0.5 \text{ and } \mathcal{f}_\nu \geq 0.5$$

Otherwise, it is referred to as a weak PFF.

A picture fuzzy face that extends to an unbounded region is known as an outer picture fuzzy face (OPFF), while all other bounded regions are called inner picture fuzzy faces (IPFF).

Definition 5.10[15] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a PFPG, where $\delta = \{uw, (\mu_\delta(uw)_r, \eta_\delta(uw)_r, \nu_\delta(uw)_r) : r = 1, 2, \dots, n_{uv}, uv \in \mathcal{V} \times \mathcal{V}\}$. Also, let $\mathcal{f}_1, \mathcal{f}_2, \dots, \mathcal{f}_s$ be the effective picture fuzzy faces (PFF) of $\mathcal{G}$. Then $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \sigma_1, \delta_1)$ is the picture fuzzy dual graph of $\mathcal{G}$. Here $\mathcal{V}' = \{p_i, i = 1, 2, \dots, s\}$, the vertex p_i of the graph $\mathcal{G}'$ is considered for the face $\mathcal{f}_i$ of $\mathcal{G}$ and $\sigma_1 = (\mu_{\sigma_1}, \eta_{\sigma_1}, \nu_{\sigma_1})$ gives the membership values of each vertex of $\mathcal{G}'$ as follows:

$$\mu_{\sigma_1}(p_i) = \max\{\mu_{\delta}(uv)_{\mathcal{r}}, \mathcal{r} = 1, 2, \dots, n_{uv}\}$$

$$\eta_{\sigma_1}(p_i) = \max\{\eta_{\delta}(uv)_{\mathcal{r}}, \mathcal{r} = 1, 2, \dots, n_{uv}\}$$

$$\nu_{\sigma_1}(p_i) = \max\{\nu_{\delta}(uv)_{\mathcal{r}}, \mathcal{r} = 1, 2, \dots, n_{uv}\}$$

Where $i = 1, 2, \dots, s$ and uv is an edge of boundary of effective PF face $\mathcal{f}_r$. The edges of $\mathcal{G}'$ exist between two vertices p_i and p_j corresponding to two faces $\mathcal{f}_i$ and $\mathcal{f}_j$ of $\mathcal{G}$. Also, there may be more than one edge between two faces $\mathcal{f}_i$ and $\mathcal{f}_j$ of $\mathcal{G}$. Also, there may be more than one edge between two vertices p_i and p_j of the picture fuzzy graph $\mathcal{G}'$. Let $\delta_1 = (p_i p_j, \mu_{\delta}(p_i p_j), \eta_{\delta}(p_i p_j), \nu_{\delta}(p_i p_j))$ be the membership function of edges of $\mathcal{G}'$. Here, the positive, neutral and negative membership function of ℓ th edge between two vertices p_i and p_j are defined as follows: $\mu_{\delta}(p_i p_j) = \mu_{\delta}(uv)_{\mathcal{r}}$, $\eta_{\delta}(p_i p_j) = \eta_{\delta}(uv)_{\mathcal{r}}$, $\nu_{\delta}(p_i p_j) = \nu_{\delta}(uv)_{\mathcal{r}}$, where $(uv)_{\mathcal{r}}$ is an edge on the boundary of two effective picture fuzzy faces $\mathcal{f}_i$ and $\mathcal{f}_j$.

In $\mathcal{G}$, if there exist any two considerable pendent edges in the picture fuzzy planar graph then there exists a self-loop in $\mathcal{G}'$ corresponding to this pendent edge and membership function is same as pendent edge. Also, here the weak edges in the planar graph are not considered for any calculation to construct a picture fuzzy dual graph.

Definition 5.11[15] The grade function corresponding to the picture fuzzy planarity value is represented by

$$\zeta = 3(\xi_{\mu} + \xi_{\eta} - \xi_{\nu})$$

where ζ denotes the grade function. The maximum value attained by the grade function is 1, whereas for a crisp graph, the grade function assumes the value 3.

Definition 5.12[68] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy graph (PFG) on $\mathcal{G}^* = (\mathcal{V}, \mathcal{E})$, where $\sigma = (\mu_{\sigma}, \eta_{\sigma}, \nu_{\sigma})$ is a picture fuzzy set on the vertex set $\mathcal{V}$. The order of $\mathcal{G}$ is defined as the triple $\mathcal{O}(\mathcal{G}) = (\mathcal{O}_{\mu}(\mathcal{G}), \mathcal{O}_{\eta}(\mathcal{G}), \mathcal{O}_{\nu}(\mathcal{G}))$, where $\mathcal{O}_{\mu}(\mathcal{G}) = \sum_{v_i \in \mathcal{V}} \mu_{\sigma}(u)$; $\mathcal{O}_{\eta}(\mathcal{G}) = \sum_{v_i \in \mathcal{V}} \eta_{\sigma}(u)$; $\mathcal{O}_{\nu}(\mathcal{G}) = \sum_{v_i \in \mathcal{V}} \nu_{\sigma}(u)$

That is, $\mathcal{O}_{\mu}(\mathcal{G})$ represents the total degree of membership of all vertices, $\mathcal{O}_{\eta}(\mathcal{G})$ represents the total degree of neutrality and $\mathcal{O}_{\nu}(\mathcal{G})$ represents the total degree of non-membership in $\mathcal{G}$.

Definition 5.13[68] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$, be a PFG on $\mathcal{G}^* = (\mathcal{V}, \mathcal{E})$, where $\sigma = (\mu_{\sigma}, \eta_{\sigma}, \nu_{\sigma})$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V}$ and $\delta = (\mu_{\delta}, \eta_{\delta}, \nu_{\delta})$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$.

The size of a picture fuzzy graph (PFG) $\mathcal{G}$ is defined as $\mathcal{S}(\mathcal{G}) = (\mathcal{S}_{\mu}(\mathcal{G}), \mathcal{S}_{\eta}(\mathcal{G}), \mathcal{S}_{\nu}(\mathcal{G}))$, where $\mathcal{S}_{\mu}(\mathcal{G}) = \sum_{\substack{v_i, v_j \in \mathcal{V} \\ v_i \neq v_j}} \mu_{\delta}(u, v)$, $\mathcal{S}_{\eta}(\mathcal{G}) = \sum_{\substack{v_i, v_j \in \mathcal{V} \\ v_i \neq v_j}} \eta_{\delta}(u, v)$ and $\mathcal{S}_{\nu}(\mathcal{G}) = \sum_{\substack{v_i, v_j \in \mathcal{V} \\ v_i \neq v_j}} \nu_{\delta}(u, v)$.

Thus, $\mathcal{S}_{\mu}(\mathcal{G})$ gives the total membership degree of all edges, $\mathcal{S}_{\eta}(\mathcal{G})$ gives the total neutrality degree of all edges and $\mathcal{S}_{\nu}(\mathcal{G})$ gives the total non-membership degree of all edges.

Table 2 Summary of notations and mathematical symbols used in the paper

Notation	Description
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$	A picture fuzzy graph consisting of a vertex set $\mathcal{V}$, edge set $\mathcal{E}$, vertex membership function σ , and edge membership function δ .
$\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$	The picture fuzzy membership function assigned to vertices.
$\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$	The picture fuzzy membership function assigned to edges.
$\mu_\sigma(u), \eta_\sigma(u), \nu_\sigma(u)$	Positive, neutral and negative membership degree of a vertex u .
$\pi_{\mathcal{R}}(u)$	Refusal membership degree of a vertex u .
$\mu_\delta(u, v), \eta_\delta(u, v), \nu_\delta(u, v)$	Positive, neutral and negative membership degree of the edge joining vertices u and v .
$\pi_{\mathcal{R}}(u, v)$	Refusal membership degree of the picture fuzzy relation $\mathcal{R}$.
w_r	The r^{th} edge between two vertices in a picture fuzzy multigraph.
n_{uv}	Number of edges between vertices u and v .
$\mathcal{R}$	A point where two or more edges intersect.
$\mathcal{I}_{\mathcal{R}} = (\mathcal{J}_\mu, \mathcal{J}_\eta, \mathcal{J}_\nu)$	Intersecting value associated with the intersection point $\mathcal{R}$.
$(\mathcal{J}_\mu)_{\mathcal{R}}, (\mathcal{J}_\eta)_{\mathcal{R}}, (\mathcal{J}_\nu)_{\mathcal{R}}$	Positive, neutral and negative membership intersecting component at the intersection point $\mathcal{R}$.
$(\mathcal{S}_\mu)_{uv}$	Stability value corresponding to the positive membership of edge uv .
$(\mathcal{S}_\eta)_{uv}$	Stability value corresponding to the neutral membership of edge uv .
$(\mathcal{S}_\nu)_{uv}$	Stability value corresponding to the negative membership of edge uv .
$\mathcal{G} = (\mathcal{G}_\mu, \mathcal{G}_\eta, \mathcal{G}_\nu)$	Picture fuzzy planarity value of a graph.
$\mathcal{G}_\mu$	Positive membership component of the planarity value.
$\mathcal{G}_\eta$	Neutral membership component of the planarity value.
$\mathcal{G}_\nu$	Negative membership component of the planarity value.
$\mathcal{N}_{\mathcal{P}}$	Total number of intersection points between edges in the graph.
$\mathcal{F} = (\mathcal{F}_\mu, \mathcal{F}_\eta, \mathcal{F}_\nu)$	Picture fuzzy face value associated with a planar region.
$\mathcal{F}_\mu$	Positive membership value of a picture fuzzy face.
$\mathcal{F}_\eta$	Neutral membership value of a picture fuzzy face.
$\mathcal{F}_\nu$	Negative membership value of a picture fuzzy face.
$\mathcal{E}_1 \subseteq \mathcal{E}$	Subset of edges forming the boundary of a picture fuzzy face.
$\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_s$	Effective picture fuzzy faces of a graph.
$\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \sigma_1, \delta_1)$	Picture fuzzy dual graph corresponding to a picture fuzzy planar graph.
$\mathcal{V}'$	Vertex set of the picture fuzzy dual graph.
$\mathcal{E}'$	Edge set of the picture fuzzy dual graph.
$\mathcal{P}_i$	Vertex in the dual graph corresponding to the face fif_ifi .
$\sigma_1 = (\mu_{\sigma_1}, \eta_{\sigma_1}, \nu_{\sigma_1})$	Vertex membership function in the picture fuzzy dual graph.

$\delta_1 = (\mu_{\delta_1}, \eta_{\delta_1}, \nu_{\delta_1})$	Edge membership function in the picture fuzzy dual graph.
$\mathcal{O}(\mathcal{G}) = (\mathcal{O}_\mu(\mathcal{G}), \mathcal{O}_\eta(\mathcal{G}), \mathcal{O}_\nu(\mathcal{G}))$	Order of a picture fuzzy graph.
$\mathcal{O}_\mu(\mathcal{G})$	Total positive membership value of all vertices in the graph.
$\mathcal{O}_\eta(\mathcal{G})$	Total neutral membership value of all vertices in the graph.
$\mathcal{O}_\nu(\mathcal{G})$	Total negative membership value of all vertices in the graph.
$\mathcal{S}(\mathcal{G}) = (\mathcal{S}_\mu(\mathcal{G}), \mathcal{S}_\eta(\mathcal{G}), \mathcal{S}_\nu(\mathcal{G}))$	Size of a picture fuzzy graph.
$\mathcal{S}_\mu(\mathcal{G})$	Total positive membership value of all edges in the graph.
$\mathcal{S}_\eta(\mathcal{G})$	Total neutral membership value of all edges in the graph.
$\mathcal{S}_\nu(\mathcal{G})$	Total negative membership value of all edges in the graph.
$\mathcal{G}/uv$	Edge contracted graph obtained after contracting the edge uv .
$\mathcal{G}^* = (\mathcal{V}, \mathcal{E})$	Underlying crisp graph corresponding to a picture fuzzy graph.

6 Picture fuzzy outerplanar graph

In this section, presents the concept of picture fuzzy outerplanar graphs, which generalize the classical idea of outerplanarity to graphs characterized by picture fuzzy values. The emphasis is placed on formulating the structural definition of such graphs and exploring their fundamental properties within the framework of picture fuzziness.

Definition 6.1 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy graph (PFG), where $\sigma = \{(v, \mu_\sigma(v), \eta_\sigma(v), \nu_\sigma(v)) | v \in \mathcal{V}\}$ is the picture fuzzy vertex set and $\delta = \{(u, v), \mu_\delta(u, v), \eta_\delta(u, v), \nu_\delta(u, v)) | (u, v) \in \mathcal{E}\}$ is the picture fuzzy edge set and the membership functions satisfy

$$0 \leq \mu_\sigma(v), \eta_\sigma(v), \nu_\sigma(v) \leq 1, 0 \leq \mu_\delta(u, v), \eta_\delta(u, v), \nu_\delta(u, v) \leq 1$$

Then $\mathcal{G}$ is called a picture fuzzy outerplanar graph (PFOG) if there exists a planar embedding of $\mathcal{G}$ such that every vertex of $\mathcal{G}$ lies on the outer boundary (exterior face) of the embedding.

Let $i(\mathcal{G})$ denote the number of vertices that do not lie on the boundary of the outer region in the embedding of $\mathcal{G}$. Then, for a picture fuzzy outerplanar graph, $i(\mathcal{G}) = 0$

The graphs given in Fig. 1 and Fig. 2 are examples of picture fuzzy outerplanar graphs.

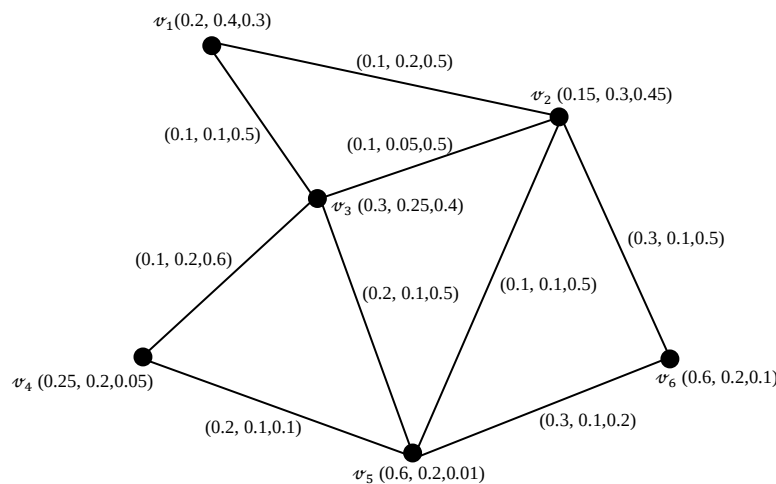


Figure 1. Picture fuzzy outerplanar graph with cycle

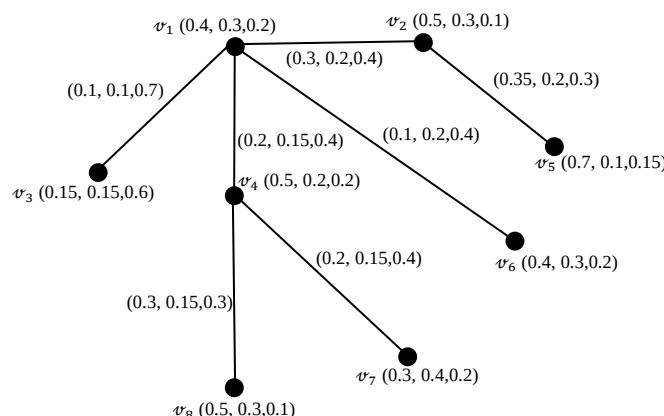


Figure 2. Picture fuzzy outer planar graph without cycle

Definition 6.2 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy graph (PFG), where $\mathcal{A} = \{ \langle v, \mu_\sigma(v), \eta_\sigma(v), \nu_\sigma(v) \rangle | v \in \mathcal{V} \}$ is the picture fuzzy vertex set and $\delta = \{ \langle (u, v), \mu_\delta(u, v), \eta_\delta(u, v), \nu_\delta(u, v) \rangle | (u, v) \in \mathcal{E} \}$ is the picture fuzzy edge set and the membership functions satisfy

$$0 \leq \mu_\sigma(v), \eta_\sigma(v), \nu_\sigma(v) \leq 1, 0 \leq \mu_\delta(u, v), \eta_\delta(u, v), \nu_\delta(u, v) \leq 1$$

Then $\mathcal{G}$ is called a picture fuzzy non-outerplanar graph (PFOG) if it cannot be embedded in the plane such that every vertex lies on the boundary of the exterior (outer) region of the embedding. Let $i(\mathcal{G})$ denote the number of vertices that do not lie on the boundary of the outer region in the embedding of $\mathcal{G}$. Hence, for a picture fuzzy non-outerplanar graph, $i(\mathcal{G}) \neq 0$.

The graphs given in Fig. 3 and Fig. 4 are examples of picture fuzzy non outerplanar graphs

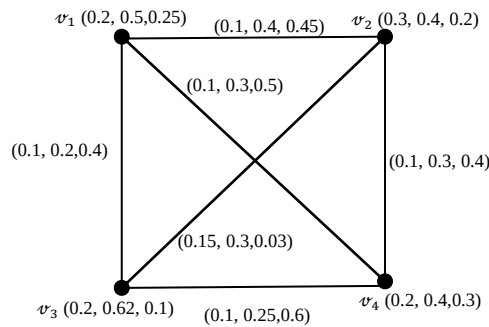


Figure 3. Picture fuzzy graph $\mathcal{K}_4$

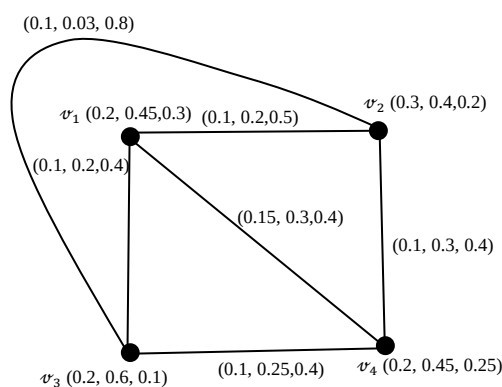


Figure 4. Picture Fuzzy planar embedding of $\mathcal{K}_4$

6.1 Characterization of Picture fuzzy outerplanar graph

In this section, we examine the fundamental theorems and illustrative conditions that define and characterize picture fuzzy outerplanar graphs.

Definition 6.1.1 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy graph, where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is the picture fuzzy vertex function and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is the picture fuzzy edge function. Let $uv \in \mathcal{E}$. A picture fuzzy subdivision of the edge uv is a picture fuzzy graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \sigma', \delta')$ obtained by replaced the edge uv by the path uw , where $w \notin \mathcal{V}$ is a newly introduced vertex. The picture fuzzy value of the new vertex w is defined componentwise by the arithmetic mean of the corresponding values of u and v namely

$$\mu_{\sigma'}(w) = \frac{\mu_\sigma(u) + \mu_\sigma(v)}{2}$$

$$\eta_{\sigma'}(w) = \frac{\eta_\sigma(u) + \eta_\sigma(v)}{2}$$

$$\nu_{\sigma'}(w) = \frac{\nu_\sigma(u) + \nu_\sigma(v)}{2}$$

Since $0 \leq \mu_\sigma(x) + \eta_\sigma(x) + \nu_\sigma(x) \leq 1$, for $x = u, v$, the newly introduced vertex satisfies $0 \leq \mu_{\sigma'}(w) + \eta_{\sigma'}(w) + \nu_{\sigma'}(w) \leq 1$. The original edge uv is deleted and the two edges uw and wv are introduced. Their picture fuzzy values are required to satisfy the picture fuzzy edge conditions

$$\mu_{\delta'}(uw) \leq \min\{\mu_{\sigma'}(u), \mu_{\sigma'}(w)\}$$

$$\eta_{\delta'}(uw) \leq \min\{\eta_{\sigma'}(u), \eta_{\sigma'}(w)\}$$

$$\nu_{\delta'}(uw) \geq \max\{\nu_{\sigma'}(u), \nu_{\sigma'}(w)\}$$

and

$$\mu_{\delta'}(uw) + \eta_{\delta'}(uw) + \nu_{\delta'}(uw) \leq 1$$

With analogous conditions for wv :

$$\mu_{\delta'}(wv) \leq \min\{\mu_{\sigma'}(w), \mu_{\sigma'}(v)\}$$

$$\eta_{\delta'}(wv) \leq \min\{\eta_{\sigma'}(w), \eta_{\sigma'}(v)\}$$

$$\nu_{\delta'}(wv) \geq \max\{\nu_{\sigma'}(w), \nu_{\sigma'}(v)\}$$

and

$$\mu_{\delta'}(wv) + \eta_{\delta'}(wv) + \nu_{\delta'}(wv) \leq 1$$

Thus, a picture fuzzy subdivision preserves the underlying subdivision structure while assigning picture fuzzy values to the newly created vertex and edges in such a way that all picture fuzzy admissibility and edge conditions remain satisfied.

Theorem 6.1.1 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy graph. Then $\mathcal{G}$ is a picture fuzzy outerplanar graph if and only if $\mathcal{G}$ contains neither a picture fuzzy subdivision of $\mathcal{K}_4$ nor a picture fuzzy subdivision of $\mathcal{K}_{2,3}$.

Proof Suppose that $\mathcal{G}$ is a picture fuzzy outerplanar graph. Then $\mathcal{G}$ admits a picture fuzzy outerplane embedding in which every picture fuzzy vertex lies on the boundary of the exterior picture fuzzy face and the picture fuzzy edges intersect only at their common picture fuzzy end vertices.

If $\mathcal{G}$ contains a picture fuzzy subdivision of $\mathcal{K}_4$, the four corresponding picture fuzzy branch vertices are joined by six pairwise internally disjoint picture fuzzy paths. Such a configuration cannot be placed on one picture fuzzy exterior face without forcing two picture fuzzy paths to intersect. Hence a picture fuzzy outerplanar embedding of $\mathcal{G}$ cannot contain a picture fuzzy subdivision of $\mathcal{K}_4$.

Similarly, if $\mathcal{G}$ contains a picture fuzzy subdivision of $\mathcal{K}_{2,3}$, its two picture fuzzy branch vertices are joined by three pairwise internally disjoint picture fuzzy paths. These three paths necessarily separate the picture fuzzy exterior region in such a way that all five picture fuzzy

vertices cannot lie on the boundary of the same picture fuzzy exterior face. Thus a picture fuzzy outerplanar embedding of $\mathcal{G}$ cannot contain a picture fuzzy subdivision of $\mathcal{K}_{2,3}$.

Hence every picture fuzzy outerplanar graph contains neither a picture fuzzy subdivision of $\mathcal{K}_4$ nor a picture fuzzy subdivision of $\mathcal{K}_{2,3}$. Conversely, suppose that $\mathcal{G}$ contains neither a picture fuzzy subdivision of $\mathcal{K}_4$ nor a picture fuzzy subdivision of $\mathcal{K}_{2,3}$. If $\mathcal{G}$ were not picture fuzzy outerplanar, then in any picture fuzzy planar representation at least one picture fuzzy vertex would lie inside the picture fuzzy exterior face. The picture fuzzy connections from this interior vertex to the boundary give either three internally disjoint picture fuzzy paths to three distinct boundary picture fuzzy vertices, forming a picture fuzzy subdivision of $\mathcal{K}_4$, or two boundary picture fuzzy vertices joined through three internally disjoint picture fuzzy paths, forming a picture fuzzy subdivision of $\mathcal{K}_{2,3}$. Both cases contradict the assumption. Therefore, $\mathcal{G}$ is a picture fuzzy outerplanar graph.

Theorem 6.1.2 If $\mathcal{G}$ is a connected picture fuzzy outerplanar graph, then its corresponding picture fuzzy dual $\mathcal{G}^*$ also exists. Moreover, there is a vertex $\nu \in \mathcal{V}(\mathcal{G}^*)$ such that the graph $\mathcal{G}^* - \nu$ contains no picture fuzzy cycle.

Proof Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a connected picture fuzzy outerplanar graph. Since the set of edges $\mathcal{E}' \subset \mathcal{E}$ contains no intersections within the graph, The graph can be divided into a finite collection of regions, where each region corresponds to a picture fuzzy face $\mathcal{f}_1, \mathcal{f}_2, \mathcal{f}_3, \dots, \mathcal{f}_k$. Accordingly, the picture fuzzy dual graph (PFDG) of $\mathcal{G}$ is represented by $\mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'', \sigma_1, \delta_1)$, where each vertex $p_i \in \mathcal{V}''$ corresponds to an effective picture fuzzy face (PFF) $\mathcal{f}_i$ of $\mathcal{G}$. Further, $\sigma_1 = (\mu_{\sigma_1}, \eta_{\sigma_1}, \nu_{\sigma_1})$ defines the picture fuzzy membership values of every vertex p_i in $\mathcal{G}''$, where $\mu_{\sigma_1}, \eta_{\sigma_1}$ and ν_{σ_1} denote the positive membership, neutral membership and negative membership values determined by⁵¹,

$$\begin{aligned}\mu_{\sigma_1}(p_i) &= \max\{\mu_{\delta}(uv)_r, r = 1, 2, \dots, n_{uv}\} \\ \eta_{\sigma_1}(p_i) &= \max\{\eta_{\delta}(uv)_r, r = 1, 2, \dots, n_{uv}\} \\ \nu_{\sigma_1}(p_i) &= \max\{\nu_{\delta}(uv)_r, r = 1, 2, \dots, n_{uv}\}\end{aligned}$$

Where $i = 1, 2, \dots, s$ and uv is an edge of boundary of effective PF face $\mathcal{f}_r$. When a picture fuzzy outerplanar graph $\mathcal{G}$ is divided into regions, each region forms a picture fuzzy face with well-defined picture fuzzy membership values. This structure allows us to construct the picture fuzzy dual graph $\mathcal{G}^* = (\mathcal{V}', \mathcal{E}', \sigma', \delta')$, where each face of $\mathcal{G}$ corresponds to a vertex of $\mathcal{G}^*$, and each edge of $\mathcal{G}$ corresponds to an edge in $\mathcal{G}^*$. Hence, the picture fuzzy dual graph $\mathcal{G}^*$ always exists for any picture fuzzy outerplanar graph.

Since $\mathcal{G}$ is outerplanar, its outer face forms a picture fuzzy cycle. In the dual graph $\mathcal{G}^*$, this outer face corresponds to a vertex ν . When this vertex ν is removed from $\mathcal{G}^*$, the resulting graph $\mathcal{G}^* - \nu$ contains no picture fuzzy cycle.

Example 6.1.1 Consider the picture fuzzy outerplanar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ illustrated in the Fig.5. If the dotted lines in Fig. 5(a) are superimposed picture fuzzy dual graph of $\mathcal{G}^*$ and separated diagram. Here $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ are four picture fuzzy faces (PFF).

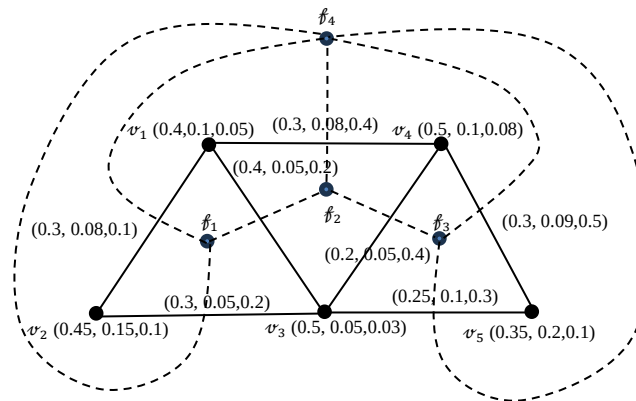


Figure.5. Connected picture fuzzy outerplanar graph $\mathcal{G}$

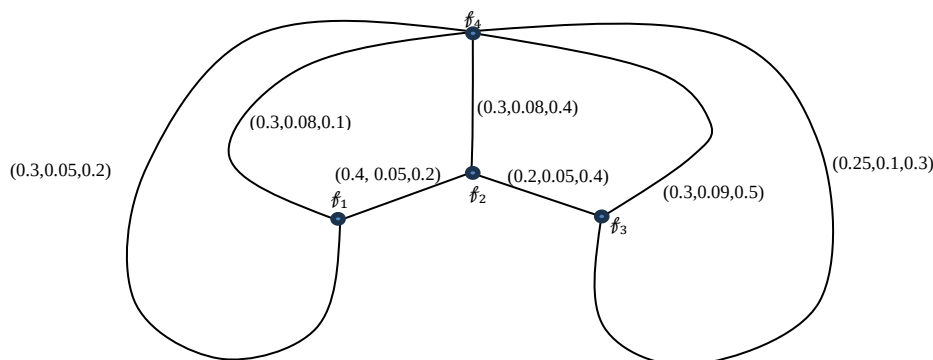


Figure.5 (a). Picture fuzzy dual graph $\mathcal{G}^*$

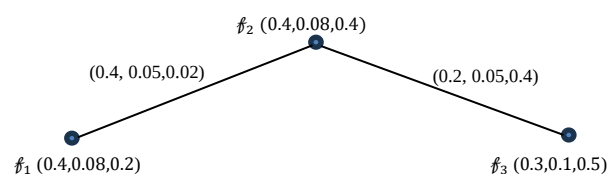


Figure.5 (b). Picture fuzzy graph but no cycle $\mathcal{G}^* - v$

Figures 5, 5(a)-(b) Example for picture fuzzy outerplanar graph $\mathcal{G}$ has picture fuzzy dual graph $\mathcal{G}^*$ then there exist ν such that $\mathcal{G}^* - \nu$ contains no picture fuzzy cycle.

Picture fuzzy face $\mathcal{f}_1$ is bounded by the edges $((\nu_1, \nu_2), (0.3, 0.08, 0.1))$, $((\nu_2, \nu_3), (0.3, 0.05, 0.2))$ and $((\nu_1, \nu_3), (0.4, 0.05, 0.2))$;

$$\begin{aligned}\mu_{\sigma_1}(\mathcal{f}_1) &= \max\{0.3, 0.3, 0.4\} = 0.4 \\ \eta_{\sigma_1}(\mathcal{f}_1) &= \max\{0.08, 0.05, 0.05\} = 0.08 \\ \nu_{\sigma_1}(\mathcal{f}_1) &= \max\{0.1, 0.2, 0.2\} = 0.2 \\ \mathcal{f}_1 &= (0.4, 0.08, 0.2)\end{aligned}$$

Picture fuzzy face $\mathcal{f}_2$ is bounded by the edges $((\nu_1, \nu_4), (0.3, 0.08, 0.4))$, $((\nu_1, \nu_3), (0.4, 0.05, 0.2))$ and $((\nu_3, \nu_4), (0.2, 0.05, 0.4))$;

$$\begin{aligned}\mu_{\sigma_1}(\mathcal{f}_2) &= \max\{0.3, 0.4, 0.2\} = 0.4 \\ \eta_{\sigma_1}(\mathcal{f}_2) &= \max\{0.08, 0.05, 0.05\} = 0.08 \\ \nu_{\sigma_1}(\mathcal{f}_2) &= \max\{0.4, 0.2, 0.4\} = 0.4 \\ \mathcal{f}_2 &= (0.4, 0.08, 0.4)\end{aligned}$$

Picture fuzzy face $\mathcal{f}_3$ is bounded by the edges $((\nu_4, \nu_3), (0.2, 0.05, 0.4))$, $((\nu_3, \nu_5), (0.25, 0.1, 0.3))$ and $((\nu_4, \nu_5), (0.3, 0.09, 0.5))$;

$$\begin{aligned}\mu_{\sigma_1}(\mathcal{f}_3) &= \max\{0.2, 0.25, 0.3\} = 0.3 \\ \eta_{\sigma_1}(\mathcal{f}_3) &= \max\{0.05, 0.1, 0.09\} = 0.1 \\ \nu_{\sigma_1}(\mathcal{f}_3) &= \max\{0.4, 0.3, 0.5\} = 0.5 \\ \mathcal{f}_3 &= (0.3, 0.1, 0.5)\end{aligned}$$

Picture fuzzy face $\mathcal{f}_4$ is the outer face $((\nu_1, \nu_4), (0.3, 0.08, 0.4))$, $((\nu_1, \nu_2), (0.3, 0.08, 0.1))$, $((\nu_2, \nu_3), (0.3, 0.05, 0.2))$, $((\nu_3, \nu_5), (0.25, 0.1, 0.3))$ and $((\nu_4, \nu_5), (0.3, 0.09, 0.5))$.

$$\begin{aligned}\mu_{\sigma_1}(\mathcal{f}_4) &= \max\{0.3, 0.3, 0.3, 0.25, 0.3\} = 0.3 \\ \eta_{\sigma_1}(\mathcal{f}_4) &= \max\{0.08, 0.08, 0.05, 0.1, 0.09\} = 0.1 \\ \nu_{\sigma_1}(\mathcal{f}_4) &= \max\{0.4, 0.1, 0.2, 0.3, 0.5\} = 0.5 \\ \mathcal{f}_4 &= (0.3, 0.1, 0.5)\end{aligned}$$

For the picture fuzzy dual graph shown in Fig. 8(a), the picture fuzzy faces $\mathcal{f}_1 = (0.4, 0.08, 0.2)$, $\mathcal{f}_2 = (0.4, 0.08, 0.4)$, $\mathcal{f}_3 = (0.3, 0.1, 0.5)$, $\mathcal{f}_4 = (0.3, 0.1, 0.5)$ shown in the Table 3. According to the effectiveness condition $\mathcal{f}_\mu \geq 0.5$, $\mathcal{f}_\eta \geq 0.5$ and $\mathcal{f}_\nu \geq 0.5$, none of these faces satisfy all the required threshold conditions simultaneously. Hence, all the considered picture fuzzy faces are categorized as weak picture fuzzy faces (PFFs).

Among these picture fuzzy faces, f_4 represents the outer face of the embedding. Let $v \in \mathcal{V}'$ be the dual vertex corresponding to the outer face f_4 . Then the picture fuzzy graph $\mathcal{G}^* - v$ represents the weak dual of the connected picture fuzzy outerplanar graph $\mathcal{G}$. Hence, $\mathcal{G}^* - v$ contains no picture fuzzy cycle. Therefore, there exists a vertex v in the dual graph such that the removal of v produces an acyclic picture fuzzy dual subgraph. Furthermore, if $\mathcal{G}$ is a tree, then no bounded faces exist and consequently the weak dual is empty. Similarly, bridges in $\mathcal{G}$ correspond to cut edges in the dual and do not generate cycles in the weak dual structure. In this study, only effective picture fuzzy faces satisfying the threshold condition are included in the dual representation.

Table 3 Calculation of picture fuzzy face

S. No	Picture fuzzy faces	Positive membership value (μ)	Neutral membership value (η)	Negative membership value (ν)
1.	f_1	0.4	0.08	0.2
2.	f_2	0.4	0.08	0.4
3.	f_3	0.3	0.1	0.5
4.	f_4	0.3	0.1	0.5

Definition 6.3 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy outerplanar graph. The graph $\mathcal{G}$ is called a maximal picture fuzzy outerplanar graph if no additional edge can be added to $\mathcal{G}$ without destroying its outerplanarity. That is, for every edge $e \notin \mathcal{E}$, the graph $\mathcal{G} + e$ is not picture fuzzy outerplanar.

The graphs shown in Fig. 6 and Fig. 7 illustrate examples of maximal picture fuzzy outerplanar graphs.

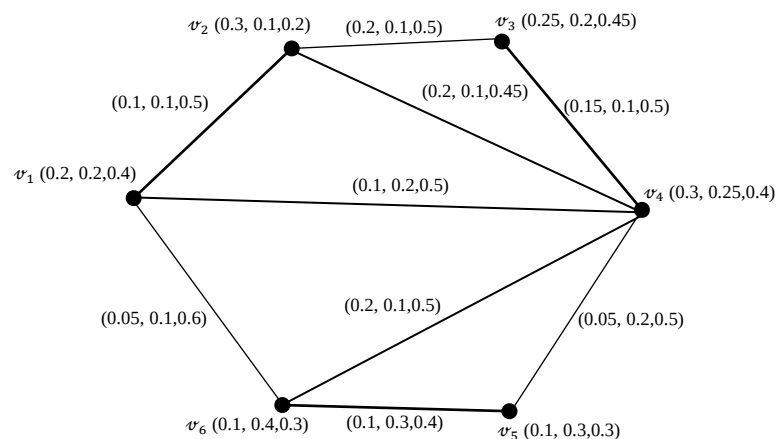


Figure 6. Maximal picture fuzzy outerplanar graph

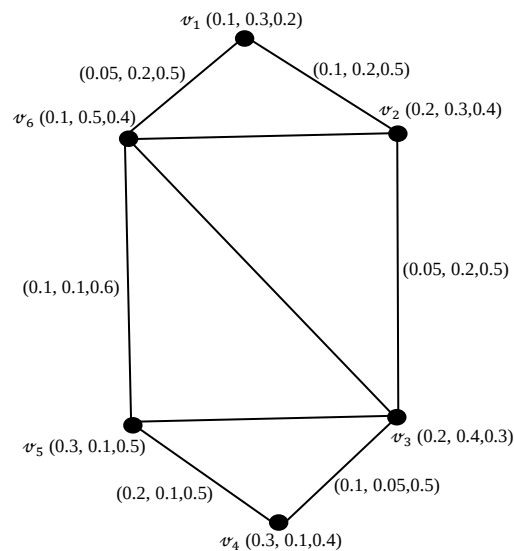


Figure 7. Maximal picture fuzzy outerplanar graph

Definition 6.4 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy planar graph. The graph $\mathcal{G}$ is said to be minimally picture fuzzy non-outerplanar if $i(\mathcal{G}) \neq 0$ and there exists at most one vertex $v \in \mathcal{G}$ such that its picture fuzzy membership value satisfies $\sigma(v) > 0$ in the interior region of the embedding.

The graphs presented in Fig. 8 and Fig. 9 are examples of minimally picture fuzzy non-outerplanar graphs.

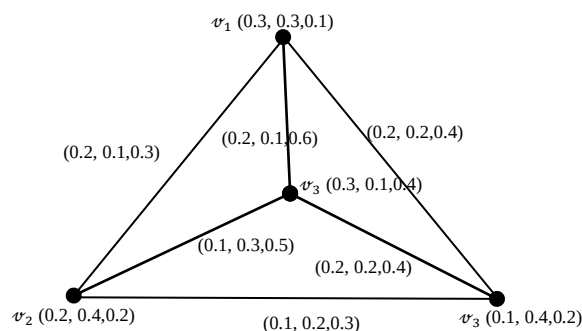


Figure 8. Picture fuzzy graph $\mathcal{K}_4$

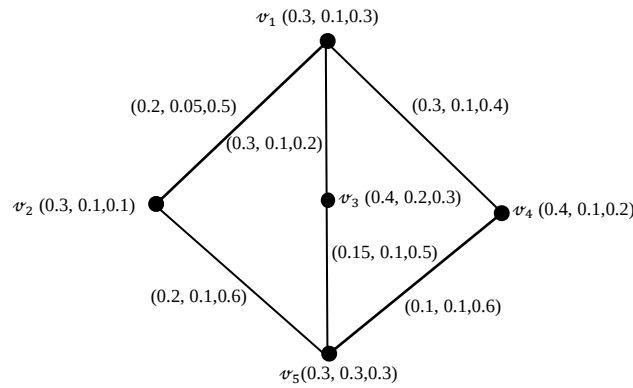


Figure 9. Picture fuzzy graph $\mathcal{K}_{2,3}$

7 Vertex removal Picture fuzzy outerplanar subgraph

In this section, a subgraph is obtained by removing selected vertices from the picture fuzzy graph, and the concept of picture fuzzy outerplanar graphs is introduced along with suitable illustrative examples. Furthermore, the notion of picture fuzzy edge contraction is defined

Definition 7.1 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy planar graph. If $\mathcal{G}'$ is a picture fuzzy subgraph obtained by removing a vertex subset from $\mathcal{G}$, and $\mathcal{G}'$ satisfies the outerplanarity property, then $\mathcal{G}'$ is called a vertex removal picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Example 7.1 Let us consider the picture fuzzy graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ shown in Figure 10. Where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$: $\sigma(v_1) = (0.6, 0.20, 0.10)$, $\sigma(v_2) = (0.6, 0.22, 0.12)$, $\sigma(v_3) = (0.56, 0.2, 0.2)$, $\sigma(v_4) = (0.5, 0.28, 0.18)$, $\sigma(v_5) = (0.5, 0.28, 0.18)$ and $\sigma(v_6) = (0.48, 0.30, 0.2)$, and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$: $\delta(v_1, v_2) = (0.5, 0.10, 0.2)$, $\delta(v_2, v_3) = (0.48, 0.12, 0.3)$, $\delta(v_3, v_1) = (0.5, 0.11, 0.3)$, $\delta(v_2, v_5) = (0.4, 0.13, 0.3)$, $\delta(v_3, v_6) = (0.42, 0.14, 0.3)$, $\delta(v_5, v_6) = (0.4, 0.15, 0.25)$, $\delta(v_6, v_5) = (0.38, 0.14, 0.3)$, $\delta(v_2, v_4) = (0.41, 0.13, 0.2)$, $\delta(v_3, v_4) = (0.41, 0.14, 0.4)$, $\delta(v_4, v_6) = (0.4, 0.15, 0.3)$, $\delta(v_4, v_5) = (0.5, 0.15, 0.3)$.

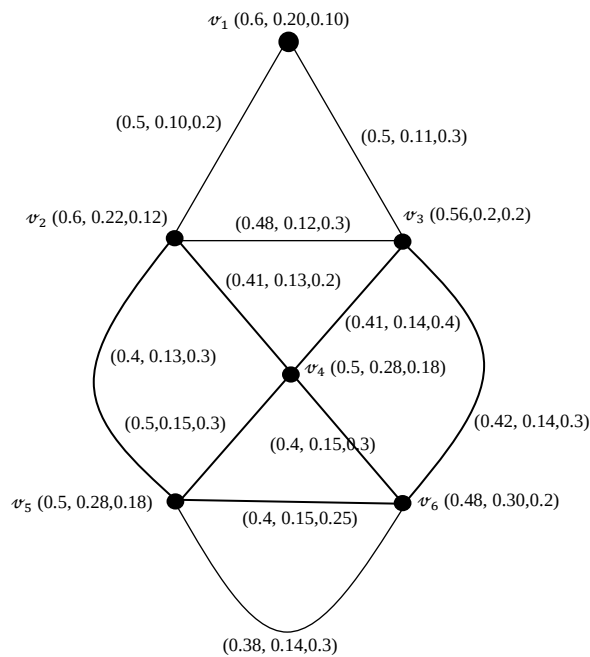


Figure 10. Picture fuzzy graph $\mathcal{G}$

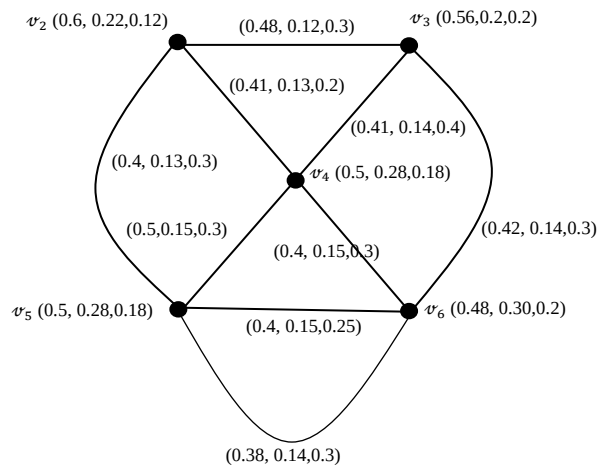


Figure 10 (a). Vertex removal picture fuzzy subgraph

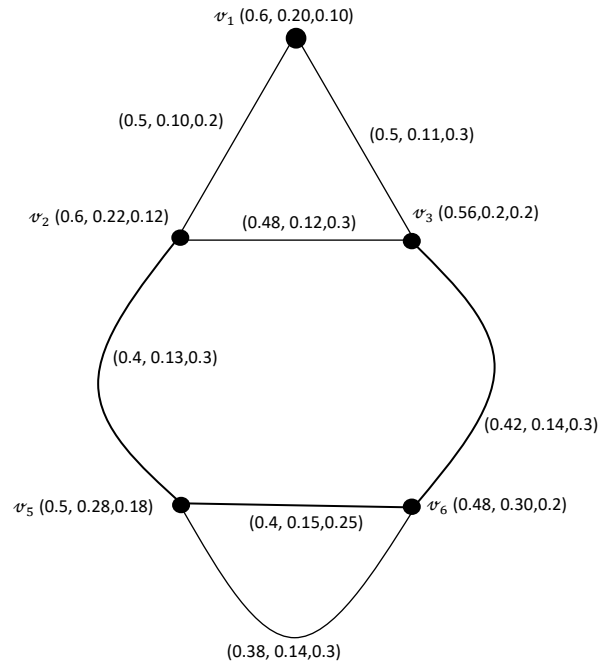


Figure 10 (b). Vertex removal picture fuzzy outerplanar subgraph

The edges intersect between the sets $\sigma(v_2, v_6)$ and $\sigma(v_3, v_5)$ within picture fuzzy graph $\mathcal{G}$. Let us define subsets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{V}$, where $\mathcal{X} = \{(v_1(0.6, 0.20, 0.10))\}$ and $\mathcal{Y} = \{(v_4(0.5, 0.28, 0.18))\}$ in the picture fuzzy graph $\mathcal{G}$.

The picture fuzzy subgraphs $\mathcal{G} - \mathcal{X}$, represented as $\mathcal{G}_1$ can be seen in Fig. 10(a). Similarly, the picture fuzzy subgraph $\mathcal{G} - \mathcal{Y}$ represented as $\mathcal{G}_2$ is shown in Fig. 10(b). It can be noted that, picture fuzzy subgraph $\mathcal{G}_1$ is categorized as a vertex removal picture fuzzy subgraph, while $\mathcal{G}_2$ is classified as a vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Note 7.1 It is not necessary for the vertex removal picture fuzzy subgraph of $\mathcal{G}$ to be the vertex removal picture fuzzy outerplanar subgraph of $\mathcal{G}$. The picture fuzzy graphs in Figs 10(a) and 10(b) allow for the observation of this.

Theorem 7.1 A picture fuzzy outerplanar graph $\mathcal{G}$ always has a vertex removed picture fuzzy outerplanar subgraph that is also a vertex removed picture fuzzy subgraph of $\mathcal{G}$.

Proof Let the picture fuzzy outerplanar graph be $\mathcal{G}$ and $\mathcal{H}$ be the vertex removed picture fuzzy subgraph of $\mathcal{G}$. The vertices of the picture fuzzy graph $\mathcal{G}$ will all be in the outer region since it is outerplanar. As a result, the picture subgraph that was created by removing vertices still has outerplanarity. Any vertex removed picture subgraph in $\mathcal{G}$ is therefore a vertex removal picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Theorem 7.2 Let $\mathcal{G}$ be a picture fuzzy outerplanar graph which is connected and let $\mathcal{W}$ be a subset of its vertices, such that $\mathcal{W} \subseteq \mathcal{V}$. The picture fuzzy outerplanar subgraph of $\mathcal{G}$ with a vertex removed is $\mathcal{G}' = \mathcal{G} \setminus \mathcal{W}$. Its picture fuzzy dual graph is $\mathcal{G}''$.

Proof Let $\mathcal{G}$ be a connected picture fuzzy outerplanar graph and let $\mathcal{W} \subseteq \mathcal{V}$. By removing the vertex set $\mathcal{W}$ from $\mathcal{G}$, we obtain the vertex removed picture fuzzy outerplanar subgraph, denoted by $\mathcal{G}'$. Since no picture fuzzy edges intersect in the resulting subgraph $\mathcal{G}'$, The graph can be divided into a finite collection of regions, where each region corresponds to a picture fuzzy face $\mathcal{f}_1, \mathcal{f}_2, \mathcal{f}_3, \dots, \mathcal{f}_k$. Accordingly, the picture fuzzy dual graph (PFDG) of $\mathcal{G}$ is represented by $\mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'', \sigma_1, \delta_1)$, where each vertex $p_i \in \mathcal{V}''$ corresponds to an effective picture fuzzy face (PFF) $\mathcal{f}_i$ of $\mathcal{G}$. Further, $\sigma_1 = (\mu_{\sigma_1}, \eta_{\sigma_1}, \nu_{\sigma_1})$ defines the picture fuzzy membership values of every vertex p_i in $\mathcal{G}''$, where $\mu_{\sigma_1}, \eta_{\sigma_1}$ and ν_{σ_1} denote the positive membership, neutral membership and negative membership values determined by⁵¹,

$$\mu_{\sigma_1}(p_i) = \max\{\mu_{\delta}(uw)_r, r = 1, 2, \dots, n_{uw}\}$$

$$\eta_{\sigma_1}(p_i) = \max\{\eta_{\delta}(uw)_r, r = 1, 2, \dots, n_{uw}\}$$

$$\nu_{\sigma_1}(p_i) = \max\{\nu_{\delta}(uw)_r, r = 1, 2, \dots, n_{uw}\}$$

Where $i = 1, 2, \dots, s$ and uw is an edge of boundary of effective PF face $\mathcal{f}_r$. Using the fact that each picture fuzzy graph with planarity value as 1 has picture fuzzy dual graph, $\mathcal{G} - \mathcal{W}$ has a picture fuzzy dual graph $\mathcal{G}''$. Consequently, the picture fuzzy dual graph exists for the vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Example 7.2 Consider the connected picture fuzzy outerplanar graph $\mathcal{G}$ illustrated in the Fig.11, it is evident that the picture subgraph $\mathcal{G}''$ is the vertex removal picture fuzzy outerplanar subgraph of $\mathcal{G}$ in Figure 11. The dotted lines represent the superimposed picture fuzzy dual graph of $\mathcal{G}''$ in Figure 11(a). Here $\mathcal{f}_1, \mathcal{f}_2, \mathcal{f}_3$ and $\mathcal{f}_4$ are four picture fuzzy faces.

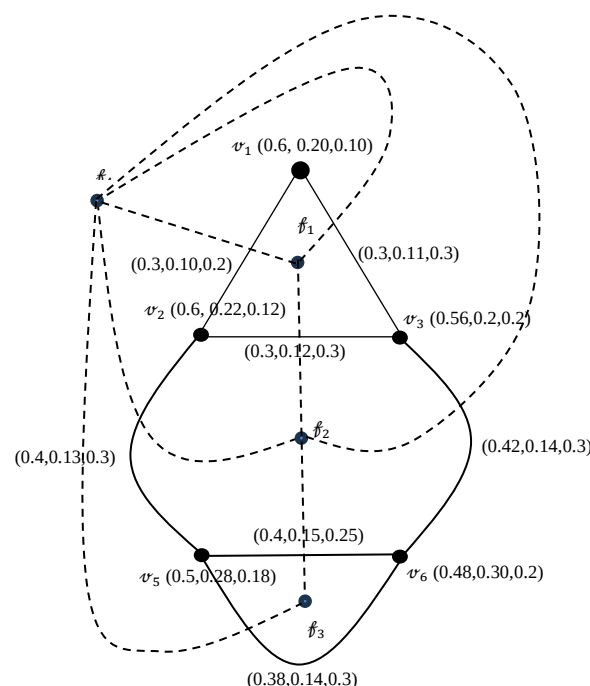


Figure 11. Connected picture fuzzy outer planar graph $\mathcal{G}$

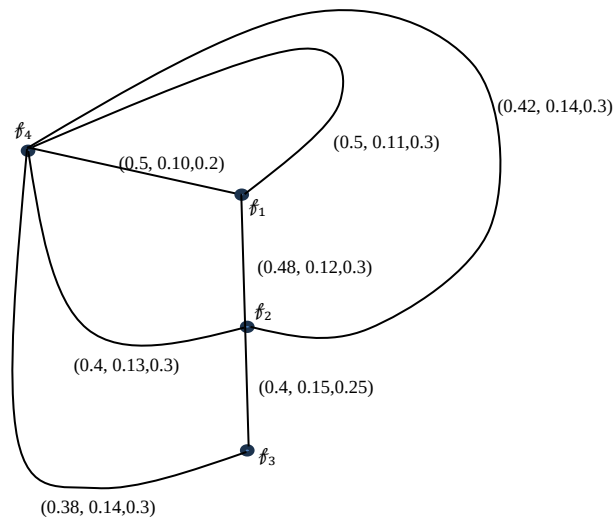


Figure 11(a). Picture fuzzy dual graph $\mathcal{G}'$

Figures 11,11(a). Example for vertex removal picture fuzzy outerplanar subgraph and its picture fuzzy dual graph $\mathcal{G}'$. Picture fuzzy face f_1 is bounded by the edges $((v_1, v_2), (0.3, 0.10, 0.2))$, $((v_2, v_3), (0.3, 0.12, 0.3))$ and $((v_1, v_3), (0.3, 0.11, 0.3))$;

$$\begin{aligned}\mu_{\sigma_1}(f_1) &= \max\{0.3, 0.3, 0.3\} = 0.3 \\ \eta_{\sigma_1}(f_1) &= \max\{0.10, 0.12, 0.3\} = 0.3 \\ \nu_{\sigma_1}(f_1) &= \max\{0.08, 0.10, 0.3\} = 0.3 \\ f_1 &= (0.3, 0.3, 0.3)\end{aligned}$$

Picture fuzzy face f_2 is bounded by the edges $\delta(v_2, v_3) = (0.3, 0.12, 0.3)$, $((v_3, v_6), (0.42, 0.14, 0.3))$, $((v_5, v_6), (0.4, 0.15, 0.25))$ and $((v_2, v_5), (0.4, 0.13, 0.3))$;

$$\begin{aligned}\mu_{\sigma_1}(f_2) &= \max\{0.3, 0.42, 0.4, 0.4\} = 0.42 \\ \eta_{\sigma_1}(f_2) &= \max\{0.12, 0.14, 0.15, 0.13\} = 0.15 \\ \nu_{\sigma_1}(f_2) &= \max\{0.3, 0.3, 0.25, 0.3\} = 0.3 \\ f_2 &= (0.48, 0.15, 0.3)\end{aligned}$$

Picture fuzzy face f_3 is bounded by the edges $((v_5, v_6), (0.4, 0.15, 0.25))$, $((v_6, v_5), (0.38, 0.14, 0.3))$;

$$\begin{aligned}\mu_{\sigma_1}(f_3) &= \max\{0.4, 0.38\} = 0.4 \\ \eta_{\sigma_1}(f_3) &= \max\{0.15, 0.14\} = 0.15 \\ \nu_{\sigma_1}(f_3) &= \max\{0.25, 0.3\} = 0.3 \\ f_3 &= (0.4, 0.15, 0.3)\end{aligned}$$

Similarly, picture fuzzy face f_4 is the outer face $((v_1, v_3), (0.3, 0.11, 0.3)), ((v_1, v_2), (0.3, 0.10, 0.2)), ((v_2, v_5), (0.4, 0.13, 0.3)), ((v_6, v_5), (0.38, 0.14, 0.3))$ and $((v_3, v_6), (0.42, 0.14, 0.3))$.

$$\begin{aligned}\mu_{\sigma_1}(f_4) &= \max\{0.3, 0.3, 0.4, 0.38, 0.42\} = 0.42 \\ \eta_{\sigma_1}(f_4) &= \max\{0.11, 0.10, 0.13, 0.14, 0.14\} = 0.14 \\ \nu_{\sigma_1}(f_4) &= \max\{0.3, 0.2, 0.3, 0.3, 0.3\} = 0.3 \\ f_4 &= (0.42, 0.14, 0.3)\end{aligned}$$

The picture fuzzy dual graph is associated with picture fuzzy face values, with all picture fuzzy faces having membership values of $f_1 = (0.3, 0.3, 0.3)$, $f_2 = (0.48, 0.15, 0.3)$, $f_3 = (0.4, 0.15, 0.3)$ and $f_4 = (0.42, 0.14, 0.3)$ shown in the Table.4. According to the effectiveness condition $f_\mu \geq 0.5$, $f_\eta \geq 0.5$ and $f_\nu \geq 0.5$, none of these faces satisfy all the required threshold conditions simultaneously. Hence, all the picture fuzzy faces are weak PF faces. Therefore, the vertex removal picture fuzzy outerplanar subgraph (VR-PFOS) $\mathcal{G}''$ indeed possesses a picture fuzzy dual graph (PFDG).

Table 4 Calculation of picture fuzzy face (PFF)

S. No	Picture fuzzy faces	Positive membership value(μ)	Neutral membership value(η)	Negative membership value (ν)
1.	f_1	0.3	0.3	0.3
2.	f_2	0.48	0.15	0.3
3.	f_3	0.4	0.15	0.3
4.	f_4	0.42	0.14	0.3

Definition 7.2 Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ be a picture fuzzy graph (PFG), where each vertex $x \in \mathcal{V}$ has picture fuzzy membership $(\mu_\sigma(x), \eta_\sigma(x), \nu_\sigma(x))$. Let $uv \in \mathcal{E}$ be an edge of $\mathcal{G}$. The picture fuzzy edge contracted graph of $\mathcal{G}$ with respect to the edge uv is denoted by $\mathcal{G}/uv$. The graph $\mathcal{G}/uv$ is defined as follows. $\mathcal{V}' = (\mathcal{V} \setminus \{u, v\} \cup \{w\})$, where the edge uv is contracted into a single new vertex w .

$$\begin{aligned}\mu_\sigma(w) &= \min\{\mu_\sigma(u), \mu_\sigma(v)\} \\ \eta_\sigma(w) &= \min\{\eta_\sigma(u), \eta_\sigma(v)\} \\ \nu_\sigma(w) &= \max\{\nu_\sigma(u), \nu_\sigma(v)\}\end{aligned}$$

And two vertices x and y are adjacent in $\mathcal{G}/uv$ if any one of the following conditions hold.

- (i) If $x, y \in \mathcal{V}$ & x, y are adjacent in $\mathcal{G}$, then $(\mu_\delta, \eta_\delta, \nu_\delta)_{\mathcal{G}/uv}(xy) = (\mu_\delta, \eta_\delta, \nu_\delta)_{\mathcal{G}}(xy)$.
- (ii) If $x \in \mathcal{V}$ & $y = w$, then x and y are adjacent in $\mathcal{G}/uv$ if either x is adjacent to u or v and
$$\begin{aligned}\mu_{\mathcal{G}/uv}(xy) &= \min\{\mu_\sigma(x), \mu_\sigma(w)\} \\ \eta_{\mathcal{G}/uv}(xy) &= \min\{\eta_\sigma(x), \eta_\sigma(w)\} \\ \nu_{\mathcal{G}/uv}(xy) &= \max\{\nu_\sigma(x), \nu_\sigma(w)\}\end{aligned}$$
- (iii) If $x \in \mathcal{V}$ & $y = w$, then x and y are adjacent in $\mathcal{G}/uv$ if x is adjacent to both u or v in $\mathcal{G}$ and

$$\begin{aligned}\mu_{\mathcal{G}/uv}(xy) &= \min\{\mu_{\delta}(xu), \mu_{\delta}(xv)\} \\ \eta_{\mathcal{G}/uv}(xy) &= \min\{\eta_{\delta}(xu), \eta_{\delta}(xv)\} \\ \nu_{\mathcal{G}/uv}(xy) &= \max\{\nu_{\delta}(xu), \nu_{\delta}(xv)\}\end{aligned}$$

Note 7.2 The picture fuzzy edge contraction operation preserves the picture fuzzy graph structure. In particular, the resulting graph $\mathcal{G}/uv$ satisfies the picture fuzzy graph conditions because the contracted vertex and the newly induced edge values are defined using the minimum membership and neutrality values together with the maximum non-membership values. Hence, for every edge xy in $\mathcal{G}/uv$, the corresponding picture fuzzy edge values remain consistent with the associated vertex values and satisfy the standard picture fuzzy graph constraints.

Example 7.3 Let us consider the picture fuzzy graph (PFG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ shown in Figure 12. Where $\sigma = (\mu_{\sigma}, \eta_{\sigma}, \nu_{\sigma})$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$: $\sigma(v_1) = (0.2, 0.3, 0.4)$, $\sigma(v_2) = (0.3, 0.2, 0.3)$, $\sigma(v_3) = (0.3, 0.2, 0.5)$, $\sigma(v_4) = (0.35, 0.45, 0.05)$, $\sigma(v_5) = (0.2, 0.5, 0.1)$, $\sigma(v_6) = (0.3, 0.4, 0.3)$, $\sigma(v_7) = (0.4, 0.1, 0.4)$, $\sigma(v_8) = (0.3, 0.3, 0.1)$, $\sigma(v_9) = (0.2, 0.2, 0.2)$ and $\delta = (\mu_{\delta}, \eta_{\delta}, \nu_{\delta})$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$: $\delta(v_1, v_2) = (0.1, 0.2, 0.4)$, $\delta(v_2, v_3) = (0.2, 0.05, 0.3)$, $\delta(v_3, v_4) = (0.1, 0.2, 0.05)$, $\delta(v_4, v_5) = (0.2, 0.1, 0.1)$, $\delta(v_5, v_6) = (0.05, 0.3, 0.3)$, $\delta(v_6, v_7) = (0.2, 0.1, 0.4)$, $\delta(v_7, v_8) = (0.2, 0.1, 0.4)$, $\delta(v_8, v_9) = (0.05, 0.2, 0.04)$, $\delta(v_9, v_{10}) = (0.1, 0.2, 0.04)$, $\delta(v_{10}, v_1) = (0.1, 0.2, 0.05)$, $\delta(v_1, v_9) = (0.1, 0.2, 0.04)$, $\delta(v_2, v_9) = (0.2, 0.1, 0.3)$, $\delta(v_2, v_{10}) = (0.2, 0.1, 0.3)$, $\delta(v_3, v_{10}) = (0.2, 0.05, 0.3)$, $\delta(v_4, v_{10}) = (0.1, 0.1, 0.3)$, $\delta(v_5, v_{10}) = (0.1, 0.1, 0.3)$, $\delta(v_6, v_{10}) = (0.2, 0.2, 0.3)$, $\delta(v_6, v_9) = (0.1, 0.2, 0.3)$, $\delta(v_7, v_9) = (0.05, 0.1, 0.4)$, $\delta(v_8, v_9) = (0.1, 0.1, 0.3)$, $\delta(v_9, v_{10}) = (0.1, 0.1, 0.3)$.

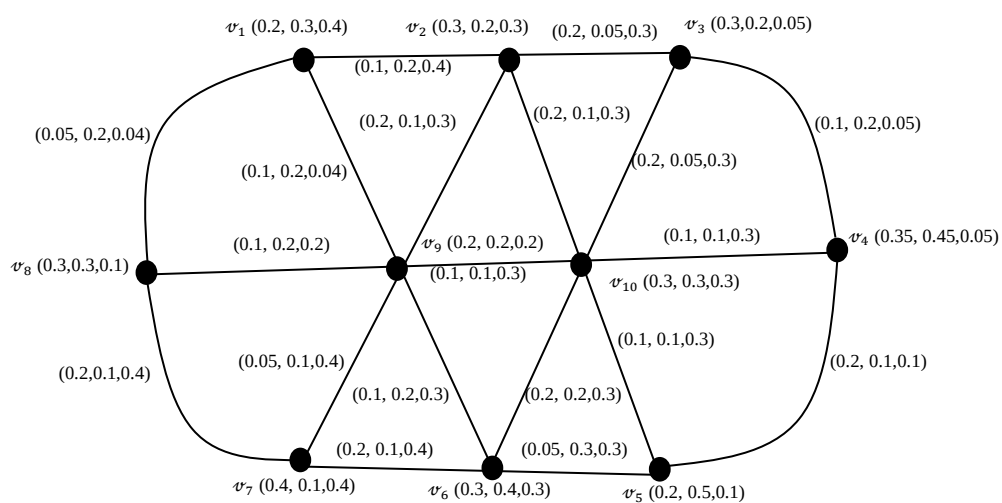


Figure. 12 Picture fuzzy graph $\mathcal{G}$

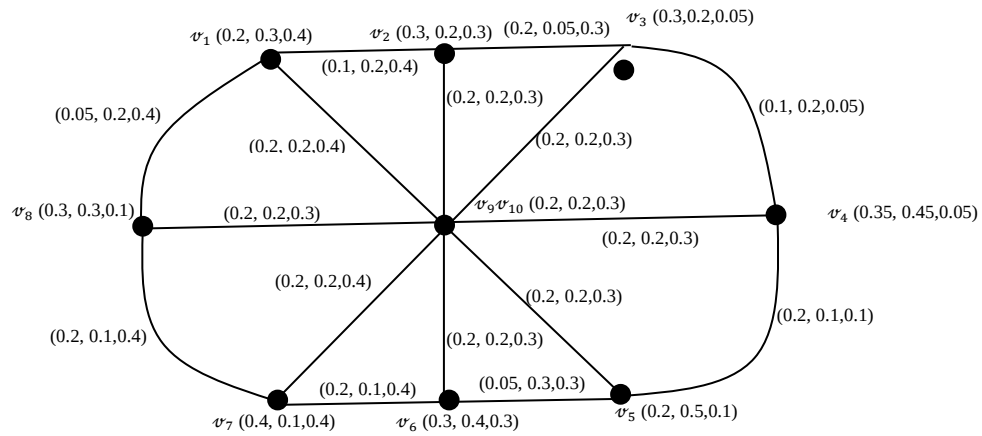


Figure. 12 (a) Picture fuzzy edge contraction $\mathcal{G}_1$

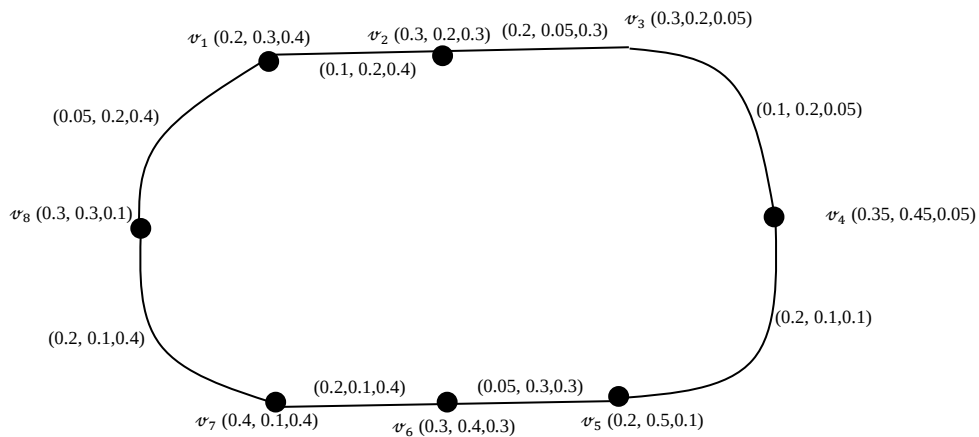


Figure.12 (b) Picture vertex removed fuzzy outerplanar graph $\mathcal{G}_2$

Let $\mathcal{G}$ be the picture fuzzy graph in the Figure.12, where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is the picture fuzzy set defined on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is the picture fuzzy set defined on the edge set $\mathcal{E}$.

The edge contracted picture fuzzy graph, denoted by $\mathcal{G} \setminus v_9 v_{10}$ is obtained from $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ by contracting the edge $v_9 v_{10}$.

After contraction, the new vertex set becomes $\mathcal{V}' = [\mathcal{V} \setminus \{v_9, v_{10}\} \cup \{v_9 v_{10}\}]$ as shown in the Figure.12(b). For every vertex $x \in \mathcal{V} \setminus \{v_9, v_{10}\}$, the picture fuzzy membership values $(\mu_\sigma(x), \eta_\sigma(x), \nu_\sigma(x))$ remain unchanged in the contracted graph. For the newly created vertex $w = v_9 v_{10}$.

The adjacency relation in the contracted graph is determined according to the conditions given in Definition (7.2)

The resulting edge contracted picture fuzzy graph $\mathcal{G} \setminus v_9 v_{10}$, represented as $\mathcal{G}_1$ can be seen in Figure 12 (a). Similarly, the vertex $\mathcal{V} = v_9 v_{10}$ removed, the picture fuzzy graph $\mathcal{G}_1 - \mathcal{V}$ represented as $\mathcal{G}_2$ is shown in Figure 12 (b). It is observed that $\mathcal{G}_2$ an edge removal picture fuzzy outerplanar graph.

8 Maximum and maximal vertex removal picture fuzzy outerplanar subgraph

In this section, the concepts of maximum and maximal vertex removal picture fuzzy outerplanar subgraphs are defined, and the relationship between these two notions is analyzed with suitable illustrative examples.

Definition 8.1 Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \sigma', \delta')$ be a vertex removed picture fuzzy outerplanar subgraph of a picture fuzzy graph $\mathcal{G}$. If no other vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ has an order $\mathcal{O}(\mathcal{G}) = (\mathcal{O}_\mu(\mathcal{G}), \mathcal{O}_\eta(\mathcal{G}), \mathcal{O}_\nu(\mathcal{G}))$ or size $\mathcal{S}(\mathcal{G}) = (\mathcal{S}_\mu(\mathcal{G}), \mathcal{S}_\eta(\mathcal{G}), \mathcal{S}_\nu(\mathcal{G}))$ respectively. From these values, the corresponding picture fuzzy planarity values $(\xi_\mu, \xi_\eta, \xi_\nu)$ are obtained and the associated grade function is computed as $\zeta = 3(\xi_\mu + \xi_\eta - \xi_\nu)$. If no other vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ has a grade value greater than that of $\mathcal{G}'$. Then $\mathcal{G}'$ is called the maximum vertex-removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Note 8.1 Let $\mathcal{G}'$ and $\mathcal{G}''$ be two vertices removed picture fuzzy outerplanar subgraphs of $\mathcal{G}$. Their orders and sizes are given by the triples $\mathcal{O}(\mathcal{G}') = (\mathcal{O}_\mu', \mathcal{O}_\eta', \mathcal{O}_\nu')$, $\mathcal{S}(\mathcal{G}') = (\mathcal{S}_\mu', \mathcal{S}_\eta', \mathcal{S}_\nu')$ and $\mathcal{O}(\mathcal{G}'') = (\mathcal{O}_\mu'', \mathcal{O}_\eta'', \mathcal{O}_\nu'')$, $\mathcal{S}(\mathcal{G}'') = (\mathcal{S}_\mu'', \mathcal{S}_\eta'', \mathcal{S}_\nu'')$ respectively. Using these order and size values, the corresponding picture fuzzy planarity values are determined and the grade values $\zeta(\mathcal{G}')$ and $\zeta(\mathcal{G}'')$ are computed. $\zeta(\mathcal{G}') > \zeta(\mathcal{G}'')$, then $\mathcal{G}'$ is called the maximum vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$. $\zeta(\mathcal{G}') = \zeta(\mathcal{G}'')$, then both $\mathcal{G}'$ and $\mathcal{G}''$ are regarded as maximum vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Example 8.1 Let us consider the picture fuzzy graph (PFG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ shown in Figure. 13. Where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$: $\sigma(v_1) = (0.45, 0.18, 0.15)$, $\sigma(v_2) = (0.43, 0.17, 0.14)$, $\sigma(v_3) = (0.42, 0.19, 0.13)$, $\sigma(v_4) = (0.44, 0.16, 0.14)$, $\sigma(v_5) = (0.15, 0.10, 0.05)$, $\sigma(v_6) = (0.40, 0.18, 0.12)$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$: $\delta(v_1, v_6) = (0.040, 0.18, 0.20)$, $\delta(v_2, v_6) = (0.39, 0.17, 0.21)$, $\delta(v_4, v_6) = (0.39, 0.12, 0.20)$, $\delta(v_5, v_6) = (0.07, 0.06, 0.12)$, $\delta(v_1, v_2) = (0.35, 0.15, 0.20)$, $\delta(v_2, v_3) = (0.42, 0.15, 0.24)$, $\delta(v_3, v_4) = (0.36, 0.16, 0.23)$, $\delta(v_4, v_5) = (0.14, 0.09, 0.14)$, $\delta(v_5, v_1) = (0.15, 0.06, 0.15)$.

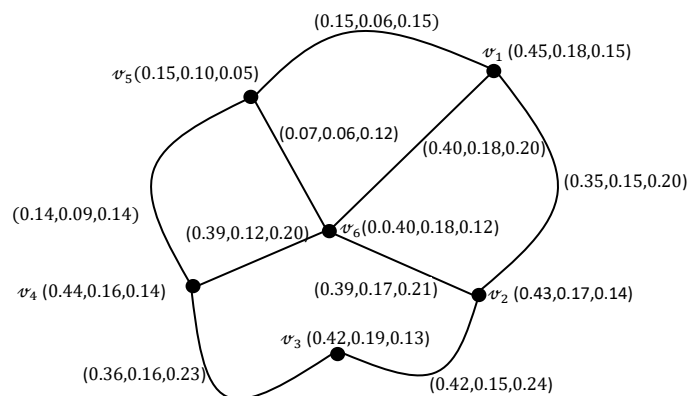


Figure. 13 Picture fuzzy graph $\mathcal{G}$

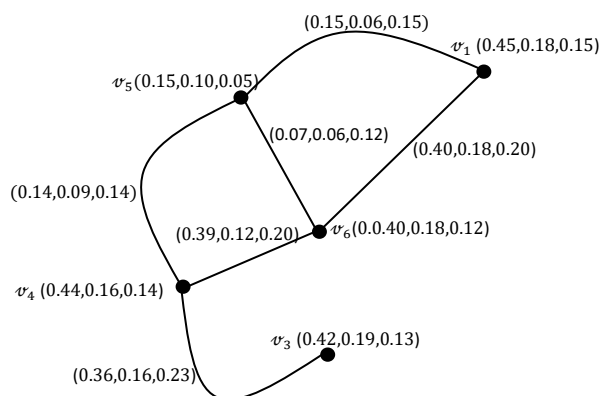


Figure. 13(a) $\mathcal{G} - v_2$

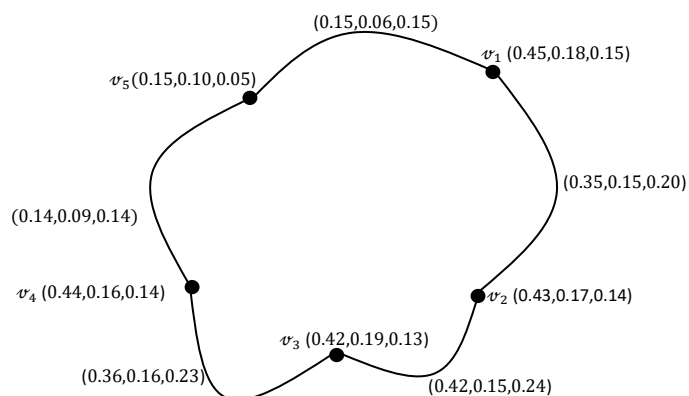


Figure. 13(b) $\mathcal{G} - v_6$

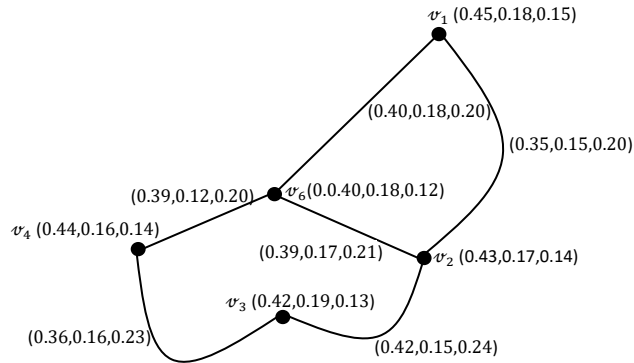


Figure. 13(c) $\mathcal{G} - v_5$

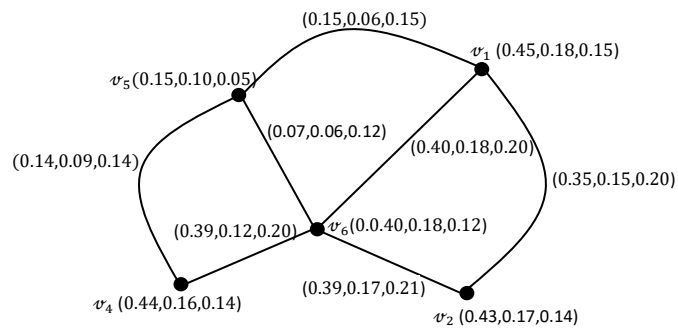


Figure. 13(d) $\mathcal{G} - v_3$

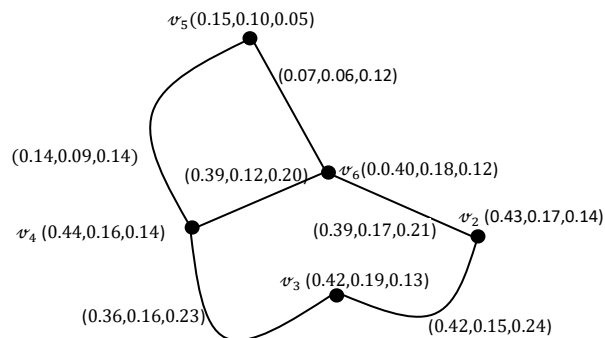


Figure. 13(e) $\mathcal{G} - v_1$

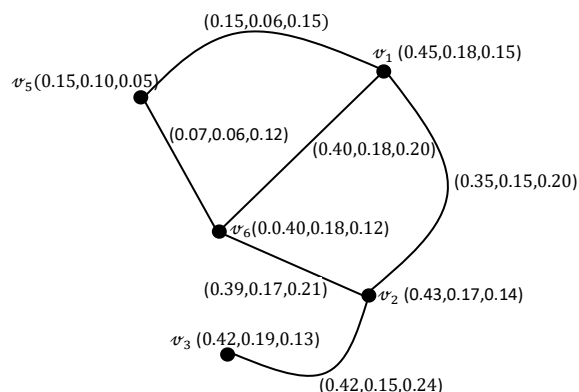


Figure. 13(f) $\mathcal{G} - v_4$

Table 5 Calculation of order and size of these vertex removal picture fuzzy outerplanar subgraphs

Subset	Vertex removed picture fuzzy outerplanar subgraph	$\mathcal{O}_\mu(\mathcal{G})$	$\mathcal{O}_\eta(\mathcal{G})$	$\mathcal{O}_\nu(\mathcal{G})$	Grade function $\zeta = 3(\xi_\mu + \xi_\eta - \xi_\nu)$	$\mathcal{S}_\mu(\mathcal{G})$	$\mathcal{S}_\eta(\mathcal{G})$	$\mathcal{S}_\nu(\mathcal{G})$	Grade function $\zeta = 3(\xi_\mu + \xi_\eta - \xi_\nu)$
$\mathcal{W}_1 = v_1(0.45, 0.18, 0.15)$	$\mathcal{G} - \mathcal{W}_1$	1.84	0.80	0.58	6.18	1.77	0.75	1.14	4.14
$\mathcal{W}_2 = v_2(0.43, 0.17, 0.14)$	$\mathcal{G} - \mathcal{W}_2$	1.86	0.81	0.59	6.24	1.51	0.67	1.04	3.42
$\mathcal{W}_3 = v_3(0.42, 0.19, 0.13)$	$\mathcal{G} - \mathcal{W}_3$	1.87	0.79	0.60	6.18	1.89	0.83	1.22	4.50
$\mathcal{W}_4 = v_4(0.44, 0.16, 0.14)$	$\mathcal{G} - \mathcal{W}_4$	1.85	0.82	0.59	6.24	1.78	0.77	1.12	4.29
$\mathcal{W}_5 = v_5(0.15, 0.10, 0.05)$	$\mathcal{G} - \mathcal{W}_5$	2.14	0.88	0.68	7.02	2.31	0.93	1.28	5.88
$\mathcal{W}_6 = v_6(0.40, 0.18, 0.12)$	$\mathcal{G} - \mathcal{W}_6$	1.89	0.80	0.61	6.24	1.42	0.61	0.96	3.21

The vertex removed picture fuzzy outerplanar subgraphs of $\mathcal{G}$ are illustrated in Figures 13(a)-13(f). From Table 5, it can be observed that the subset $\mathcal{W}_5$ corresponds to the maximum vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Theorem 8.1 The maximum vertex removed picture fuzzy outerplanar subgraph of a graph possesses a picture fuzzy dual graph, but the converse is not necessarily true.

Proof It follows directly from Theorem 3 that a maximum vertex removed picture fuzzy outerplanar subgraph has a picture fuzzy dual. However, the existence of a picture fuzzy dual does not guarantee that the corresponding subgraph is a maximum vertex removed picture fuzzy outerplanar subgraph.

Definition 8.2 Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \sigma', \delta')$ be a vertex-removed picture fuzzy outerplanar subgraph of a picture fuzzy outerplanar graph $\mathcal{G}$. Then $\mathcal{G}'$ is said to be the maximal vertex-removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ if the following condition holds: For any vertex $v \in \mathcal{V} \setminus \mathcal{V}'$, the picture fuzzy subgraph of $\mathcal{G}$ induced by the enlarged vertex set $\mathcal{V}'' = \mathcal{V}' \cup \{v\}$ fails to remain outerplanar.

In other words, once $\mathcal{G}'$ is formed by removing vertices from $\mathcal{G}$, adding back any removed vertex destroys the outerplanarity property. Hence, $\mathcal{G}'$ is maximal with respect to being a vertex removed picture fuzzy outerplanar subgraph.

Theorem 8.2 The picture fuzzy outerplanar subgraph obtained from each maximum vertex removal is a maximal vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Proof Let $\mathcal{G}' \subseteq \mathcal{G}$ denote a maximum vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$. Consider a nonempty subset $\mathcal{W} \subseteq \mathcal{V}$ representing the vertices removed from $\mathcal{G}$ to form $\mathcal{G}'$. Take any vertex $u \in \mathcal{W}$ and defined by $\mathcal{G}'' = \mathcal{G}' \cup \{u\}$ where $\mathcal{G}'' = \{x \in \mathcal{V}' \cup \frac{\{u\}}{\delta(x,y)} \neq \emptyset, x, y \in \mathcal{V} \setminus \mathcal{W} \cup \{u\}\}$.

Since $\mathcal{G}'$ is a maximum induced picture fuzzy outerplanar subgraph, adding any vertex u from $\mathcal{W}$ would violate the outerplanarity property (i.e., $\mathcal{f}_{\mathcal{G}} \neq 1$). Therefore, $\mathcal{G}'$ is also a maximal induced vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$. Hence, every maximum vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ is also maximal.

Consequently, these picture fuzzy subgraphs are considered picture fuzzy outerplanar subgraphs with maximal vertex removal of $\mathcal{G}$. Hence, every maximum vertex removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ is necessarily maximal.

Theorem 8.3 The statement of Theorem 8.2 does not necessarily hold in the reverse direction.

Proof Let $\mathcal{G}$ be a picture fuzzy graph with $\mathcal{f}_{\mathcal{G}} \neq 1$ and let $\mathcal{G}'$ and $\mathcal{G}''$ be vertex removed picture fuzzy outerplanar subgraphs of $\mathcal{G}$. Let $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{V}$ denote the sets of vertices removed from $\mathcal{G}$ to obtain $\mathcal{G}'$ and $\mathcal{G}''$ respectively.

Suppose that $\mathcal{G}'$ and $\mathcal{G}''$ are maximal picture fuzzy outerplanar subgraphs; that is, for every $u \in \mathcal{W}_1$ and $v \in \mathcal{W}_2$, the graphs $\mathcal{G}' \cup \{u\}$ and $\mathcal{G}'' \cup \{v\}$ are picture fuzzy non outerplanar. Hence, neither $\mathcal{G}'$ and $\mathcal{G}''$ can be enlarged by adding any removed vertex while preserving

picture fuzzy outerplanarity. Let $\mathcal{G}'$ and $\mathcal{G}''$ be two vertices removed picture fuzzy outerplanar subgraphs of $\mathcal{G}$. Their orders and sizes are given by the triples $\mathcal{O}(\mathcal{G}') = (\mathcal{O}_{\mu}', \mathcal{O}_{\eta}', \mathcal{O}_{\nu}')$, $\mathcal{S}(\mathcal{G}') = (\mathcal{S}_{\mu}', \mathcal{S}_{\eta}', \mathcal{S}_{\nu}')$ and $\mathcal{O}(\mathcal{G}'') = (\mathcal{O}_{\mu}'', \mathcal{O}_{\eta}'', \mathcal{O}_{\nu}'')$, $\mathcal{S}(\mathcal{G}'') = (\mathcal{S}_{\mu}'', \mathcal{S}_{\eta}'', \mathcal{S}_{\nu}'')$ respectively. Using these order and size values, the corresponding picture fuzzy planarity values are computed, and the grade values $\zeta(\mathcal{G}')$ and $\zeta(\mathcal{G}'')$ are obtained. A maximal picture fuzzy outerplanar subgraph is maximum only if its grade value is not less than that of any other vertex removed picture fuzzy outerplanar subgraph. Since maximal subgraphs may have different order and size and consequently different grade values, it is possible that $\zeta(\mathcal{G}') < \zeta(\mathcal{G}'')$. Therefore, although both $\mathcal{G}'$ and $\mathcal{G}''$ are maximal, only the subgraph with the largest grade value is maximum. Consequently, a maximal vertex removed picture fuzzy outerplanar subgraph need not be a maximum vertex removed picture fuzzy outerplanar subgraph. Hence, the converse of Theorem 8.2 does not hold.

Note 8.2 Only when both picture fuzzy subgraphs are maximum vertex removal picture fuzzy outerplanar subgraphs as defined by Note 8.3 are two vertex removal picture fuzzy outerplanar subgraphs maximal.

Example 8.2 Let us consider the picture fuzzy graph (PFG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ shown in Fig. 14. Where $\sigma = (\mu_{\sigma}, \eta_{\sigma}, \nu_{\sigma})$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$: $\sigma(v_1) = (0.43, 0.20, 0.22)$, $\sigma(v_2) = (0.44, 0.19, 0.21)$, $\sigma(v_3) = (0.45, 0.18, 0.20)$, $\sigma(v_4) = (0.46, 0.17, 0.20)$, $\sigma(v_5) = (0.42, 0.16, 0.19)$ and $\delta = (\mu_{\delta}, \eta_{\delta}, \nu_{\delta})$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$: $\delta(v_1, v_2) = \delta(v_2, v_1) = (0.43, 0.15, 0.32)$, $\delta(v_2, v_3) = \delta(v_3, v_2) = (0.43, 0.15, 0.32)$, $\delta(v_3, v_4) = \delta(v_4, v_3) = (0.43, 0.15, 0.32)$, $\delta(v_1, v_4) = \delta(v_4, v_1) = (0.43, 0.15, 0.32)$, $\delta(v_1, v_5) = (0.28, 0.05, 0.22)$, $\delta(v_3, v_5) = (0.28, 0.05, 0.20)$, $\delta(v_2, v_5) = (0.28, 0.05, 0.20)$, $\delta(v_4, v_5) = (0.28, 0.05, 0.20)$.

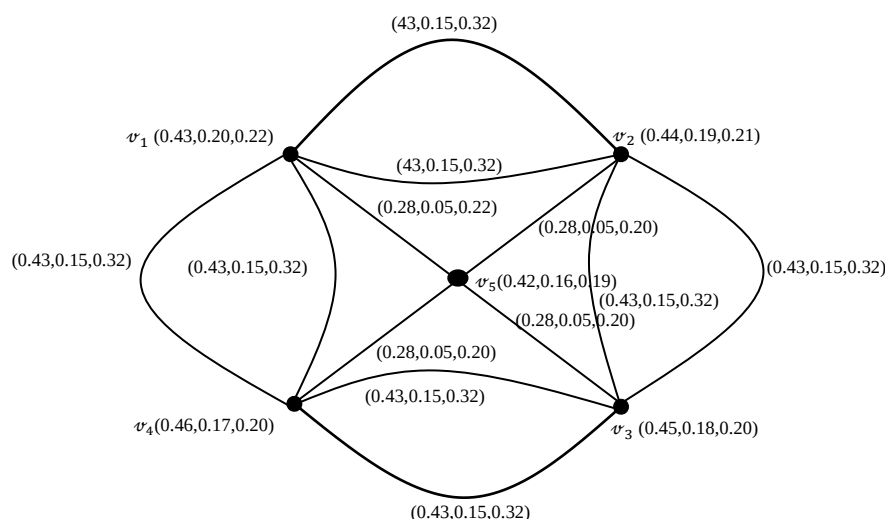


Figure. 14 Example for maximal vertex removal picture fuzzy graph

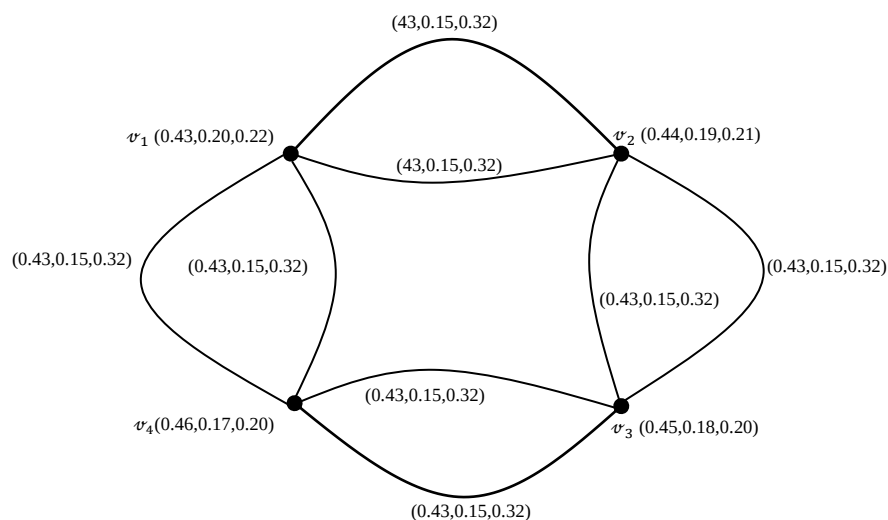


Figure. 14(a) $\mathcal{G} - v_5$

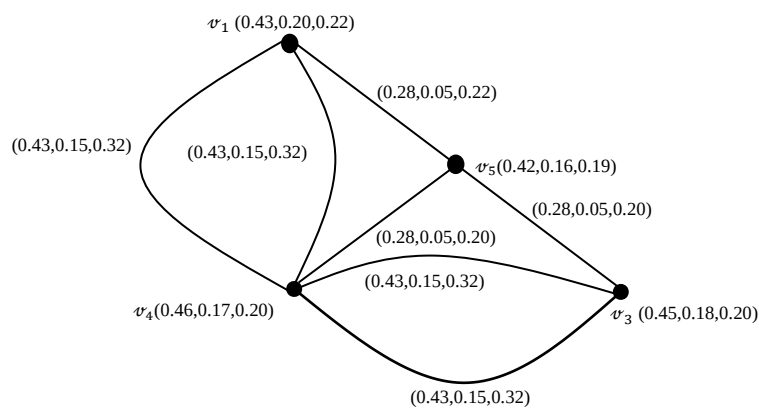


Figure. 14(b) $\mathcal{G} - v_2$

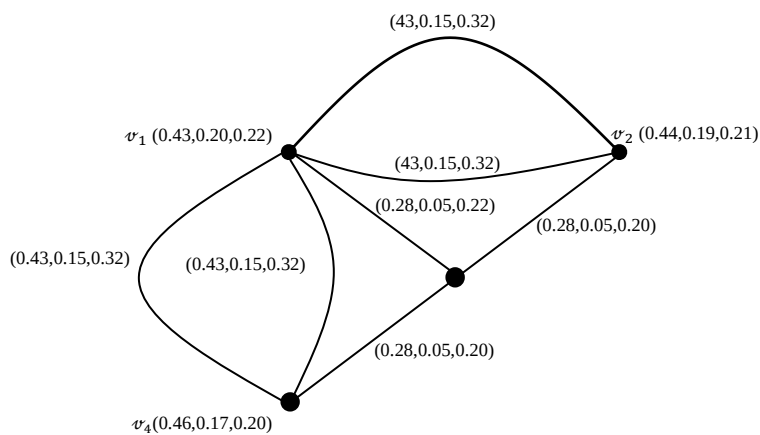


Figure. 14(c) $\mathcal{G} - v_3$

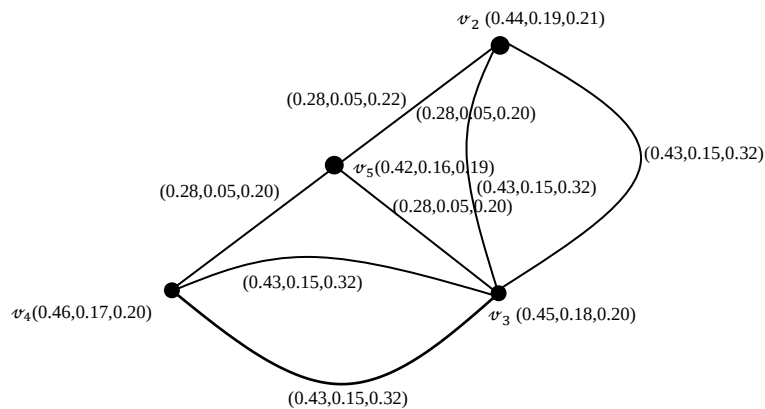


Figure. 14(d) $\mathcal{G} - v_1$

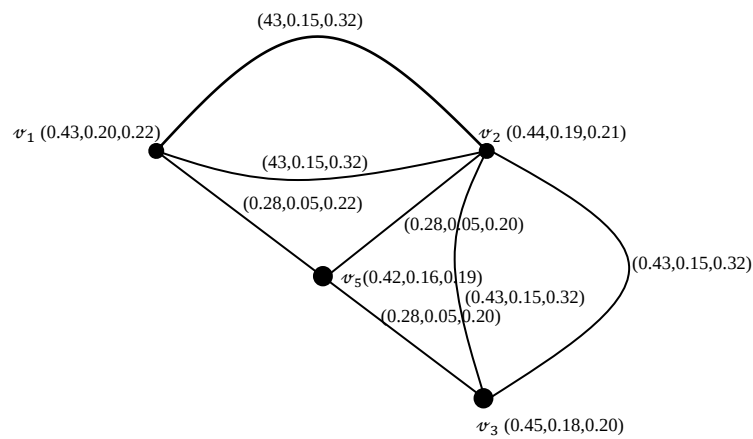


Figure. 14(e) $\mathcal{G} - v_4$

Subset	Vertex removed picture fuzzy outerplanar subgraph	$\mathcal{O}_\mu(\mathcal{G})$	$\mathcal{O}_\eta(\mathcal{G})$	$\mathcal{O}_\nu(\mathcal{G})$	Grade function $\zeta = 3(\xi_\mu + \xi_\eta - \xi_\nu)$	$\mathcal{S}_\mu(\mathcal{G})$	$\mathcal{S}_\eta(\mathcal{G})$	$\mathcal{S}_\nu(\mathcal{G})$	Grade function $\zeta = 3(\xi_\mu + \xi_\eta - \xi_\nu)$
Fig 14(a)	$\mathcal{G} - \{v_1(0.43, 0.20, 0.22)\}$	1.77	0.70	0.80	5.01	1.70	0.45	1.25	2.70
Fig 14(b)	$\mathcal{G} - \{v_2(0.44, 0.19, 0.21)\}$	1.76	0.71	0.81	4.98	1.70	0.45	1.26	2.67
Fig 14(c)	$\mathcal{G} - \{v_3(0.45, 0.18, 0.20)\}$	1.75	0.72	0.82	4.95	1.70	0.45	1.27	2.64
Fig 14(d)	$\mathcal{G} - \{v_4(0.46, 0.17, 0.20)\}$	1.74	0.73	0.82	4.95	1.70	0.45	1.27	2.64
Fig 14(e)	$\mathcal{G} - \{v_5(0.42, 0.16, 0.19)\}$	1.78	0.74	0.83	5.07	1.72	0.60	1.28	3.12

Table 6 Calculation of order and size of these vertex removal picture fuzzy outerplanar subgraphs

All the graphs shown in Figures 14(a)-Fig 14(e) are maximal picture fuzzy outerplanar subgraphs of $\mathcal{G}$. However, Figure 14(e) represents the maximum picture fuzzy outerplanar subgraph of $\mathcal{G}$ according to Table 6. This illustrates that a maximal picture fuzzy outerplanar subgraph does not necessarily coincide with the maximum picture fuzzy outerplanar subgraph of $\mathcal{G}$.

9 Edge removal Picture fuzzy outerplanar subgraph

In this section, the subgraph obtained by removing certain edges from the picture fuzzy graph is discussed, along with the definition of picture fuzzy outerplanar graphs and supporting illustrative examples.

Definition 9.1 Let $\mathcal{G}$ be a picture fuzzy planar graph and let $\mathcal{G}' = (\mathcal{V}, \mathcal{E}', \sigma, \delta)$ be an edge removed subgraph of $\mathcal{G}$. Then $\mathcal{G}'$ is called an edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ if and only if the subgraph obtained after removing edges from $\mathcal{G}$ still satisfies the picture fuzzy outerplanarity property.

In other words, removing edges from $\mathcal{G}$ should not destroy the outerplanar structure; the resulting subgraph must remain outerplanar under picture fuzzy conditions.

Note 9.1 An edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ is not necessarily an edge removed picture fuzzy subgraph of $\mathcal{G}$.

Theorem 9.1 In a picture fuzzy outerplanar graph $\mathcal{G}$, every edge removed picture fuzzy subgraph is also an edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Proof Let $\mathcal{G}$ be a picture fuzzy outerplanar graph and let $\mathcal{H}$ be any edge removed subgraph of $\mathcal{G}$. Since all vertices of $\mathcal{G}$ lie on the outer boundary, removing edges does not change the positions of the vertices. Therefore, the subgraph $\mathcal{H}$ still preserves the outerplanarity property. Hence, every edge removed picture fuzzy subgraph of $\mathcal{G}$ is indeed an edge removed picture fuzzy outerplanar subgraph.

Theorem 9.2 Let $\mathcal{W}$ be a subset of edges of a connected picture fuzzy outerplanar graph $\mathcal{G}$, with $\mathcal{W} \subseteq \mathcal{E}$. Then, $\mathcal{G}'$ is an edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ if the resulting graph preserves connectivity with respect to the picture fuzzy dual graph.

Proof The result follows directly by applying the same reasoning as in Theorem 7, which similarly holds for the case of edges.

Example 9.1 Let us consider the picture fuzzy graph (PFG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ shown in Figure 15. Where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$:

$\sigma(v_1) = (0.2, 0.2, 0.4), \sigma(v_2) = (0.2, 0.2, 0.3), \sigma(v_3) = (0.3, 0.3, 0.3), \sigma(v_4) = (0.2, 0.4, 0.2), \sigma(v_5) = (0.4, 0.45, 0.1), \sigma(v_6) = (0.2, 0.4, 0.3), \sigma(v_7) = (0.3, 0.4, 0.2), \sigma(v_8) = (0.4, 0.4, 0.1), \sigma(v_9) = (0.4, 0.2, 0.1)$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$: $\delta(v_1, v_2) = (0.16, 0.13, 0.6), \delta(v_2, v_3) = (0.1, 0.15, 0.6), \delta(v_3, v_4) = (0.1, 0.2, 0.3), \delta(v_4, v_5) = (0.1, 0.2, 0.2), \delta(v_1, v_4) = (0.1, 0.1, 0.4), \delta(v_1, v_5) = (0.2, 0.1, 0.4), \delta(v_2, v_5) = (0.3, 0.2, 0.3), \delta(v_5, v_6) = (0.1, 0.3, 0.3),$

$$\delta(v_6, v_7) = (0.16, 0.26, 0.3), \quad \delta(v_7, v_8) = (0.2, 0.3, 0.2), \quad \delta(v_8, v_9) = (0.3, 0.2, 0.1),$$

$$\delta(v_6, v_9) = (0.1, 0.05, 0.3), \quad \delta(v_6, v_8) = (0.35, 0.25, 0.3).$$

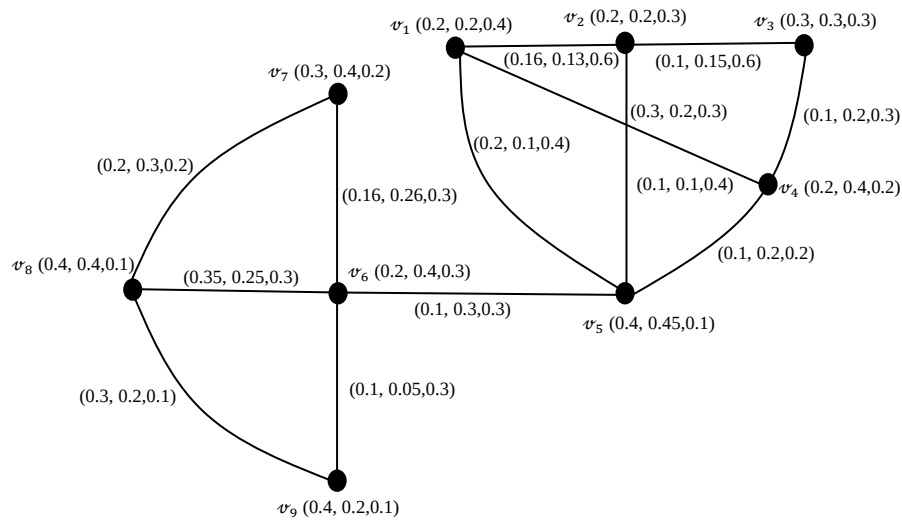


Figure. 15 Picture fuzzy graph $\mathcal{G}$

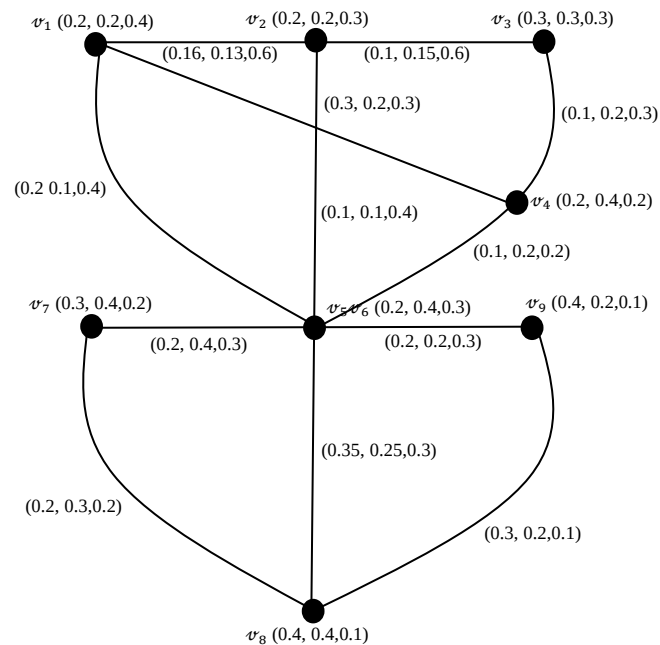


Figure. 15(a) Picture fuzzy edge contraction

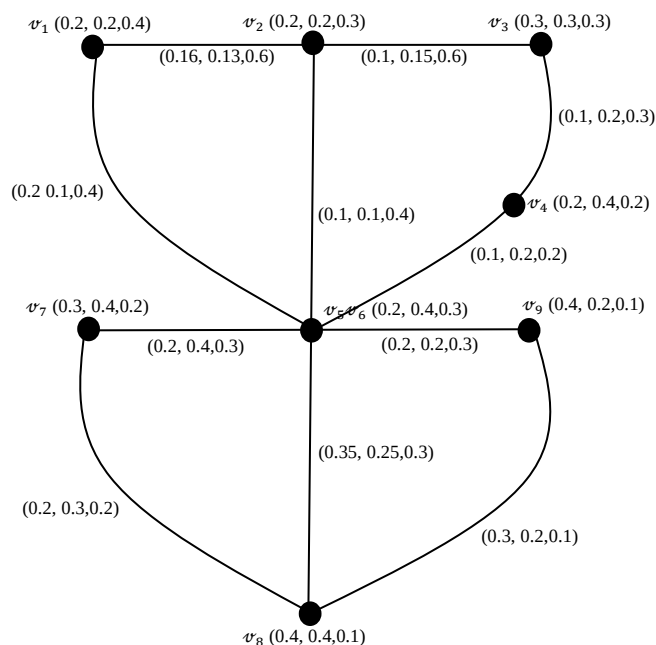


Figure. 15(b) Edge removed picture fuzzy outerplanar graph

Let $\mathcal{G}$ be the picture fuzzy graph, where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is the picture fuzzy set defined on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is the picture fuzzy set defined on the edge set $\mathcal{E}$. The edge contracted picture fuzzy graph, denoted by $\mathcal{G} \setminus v_5 v_6$ is obtained from $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ by contracting the edge $v_5 v_6$.

After contraction, the new vertex set becomes $\mathcal{V}' = [\mathcal{V} \setminus \{v_5, v_6\} \cup \{v_5 v_6\}]$ as shown in the Figure 15(b). For every vertex $x \in \mathcal{V} \setminus \{v_5 v_6\}$, the picture fuzzy membership values $(\mu_\sigma(x), \eta_\sigma(x), \nu_\sigma(x))$ remain unchanged in the contracted graph. For the newly created vertex $w = v_5 v_6$. The adjacency relation in the contracted graph is determined according to the conditions given in Definition (7.2). The resulting edge contracted picture fuzzy graph $\mathcal{G} \setminus v_5 v_6$, represented as $\mathcal{G}_1$ can be seen in Figure 15(a). Similarly, the edge $\mathcal{E} = (v_5, v_6)$ is removed, the picture fuzzy graph $\mathcal{G}_1 - \mathcal{E}$ represented as $\mathcal{G}_2$ is shown in Figure 15(b). It is observed that $\mathcal{G}_2$ an edge removal picture fuzzy outerplanar graph.

10 Maximum and maximal edge removal picture fuzzy outerplanar subgraph

In this section, the concepts of maximum and maximal edge removal picture fuzzy outerplanar subgraphs are presented, and the relationship between these two concepts is examined through appropriate illustrative examples.

Definition 10.1 Maximum edge-removed (ER) picture fuzzy outerplanar subgraph

Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \sigma', \delta')$ be an edge removed picture fuzzy outerplanar subgraph of a non-outerplanar picture fuzzy graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$. Let the size of $\mathcal{G}'$ be $\mathcal{S}(\mathcal{G}') = (\mathcal{S}_\mu(\mathcal{G}'), \mathcal{S}_\eta(\mathcal{G}'), \mathcal{S}_\nu(\mathcal{G}'))$. From the size of $\mathcal{G}'$, the corresponding picture fuzzy planarity values $(\xi_\mu, \xi_\eta, \xi_\nu)$ are determined and the associated grade value is computed using $\zeta = 3(\xi_\mu + \xi_\eta - \xi_\nu)$.

If there exists no other edge removed picture fuzzy outerplanar subgraph $\mathcal{G}''$ of $\mathcal{G}$ having a grade value greater than that of $\mathcal{G}'$, then $\mathcal{G}'$ is called the maximum edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Note 10.1 Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two edges removed picture fuzzy outerplanar subgraphs of picture fuzzy graph $\mathcal{G}$. If their sizes $(\mathcal{S}_\mu(\mathcal{G}_1), \mathcal{S}_\eta(\mathcal{G}_1), \mathcal{S}_\nu(\mathcal{G}_1))$ and $(\mathcal{S}_\mu(\mathcal{G}_2), \mathcal{S}_\eta(\mathcal{G}_2), \mathcal{S}_\nu(\mathcal{G}_2))$. Using these sizes, the corresponding picture fuzzy planarity values are obtained and the grade values $\zeta(\mathcal{G}_1) = \zeta(\mathcal{G}_2)$ and $\mathcal{G}_1 \neq \mathcal{G}_2$, then both $\mathcal{G}_1$ and $\mathcal{G}_2$ are regarded as maximum edge removed picture fuzzy outerplanar subgraphs of $\mathcal{G}$. Consequently, a unique maximum edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$ does not necessarily exist.

Definition 10.2 Maximal edge-removed (ER) picture fuzzy outerplanar subgraph

Let picture fuzzy non outerplanar graph $\mathcal{G}$. If removing an edge $e \in \mathcal{E}$ from $\mathcal{G}$ produces a subgraph $\mathcal{G}' = \mathcal{G} - e$, such that $\mathcal{G}'$ becomes a picture fuzzy outerplanar graph, and any further edge removal destroys the maximal outerplanar structure or reduces the graph below the maximal condition, then $\mathcal{G}'$ is called a maximal edge removed (ER) picture fuzzy outerplanar subgraph of $\mathcal{G}$.

Example 10.1 Let us consider the picture fuzzy graph (PFG) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ shown in Figure. 16. Where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is a picture fuzzy set (PFS) on the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$: $\sigma(v_1) = (0.4, 0.5, 0.1)$, $\sigma(v_2) = (0.3, 0.4, 0.2)$, $\sigma(v_3) = (0.35, 0.3, 0.2)$, $\sigma(v_4) = (0.1, 0.2, 0.3)$, $\sigma(v_5) = (0.1, 0.3, 0.2)$, $\sigma(v_6) = (0.11, 0.12, 0.5)$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is a picture fuzzy set (PFS) on the edge set $\mathcal{E}$: $\delta(e_1) = (0.1, 0.2, 0.3)$, $\delta(e_2) = (0.15, 0.2, 0.4)$, $\delta(e_3) = (0.2, 0.1, 0.5)$, $\delta(e_4) = (0.05, 0.1, 0.4)$, $\delta(e_5) = (0.05, 0.1, 0.6)$, $\delta(e_6) = (0.2, 0.15, 0.6)$, $\delta(e_7) = (0.1, 0.1, 0.4)$, $\delta(e_8) = (0.05, 0.1, 0.6)$, $\delta(e_9) = (0.2, 0.2, 0.3)$.

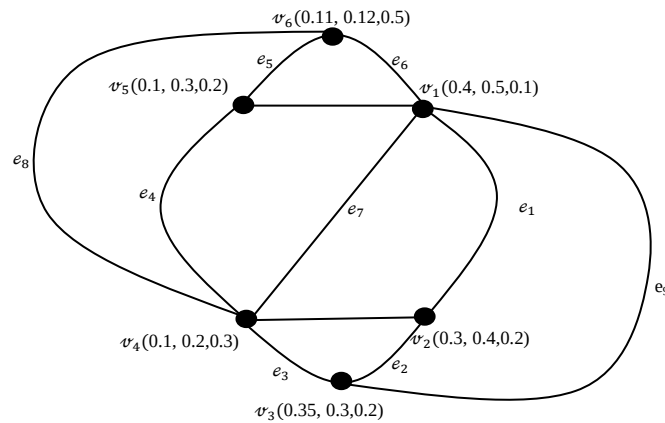


Figure. 16 Picture fuzzy graph $\mathcal{G}$

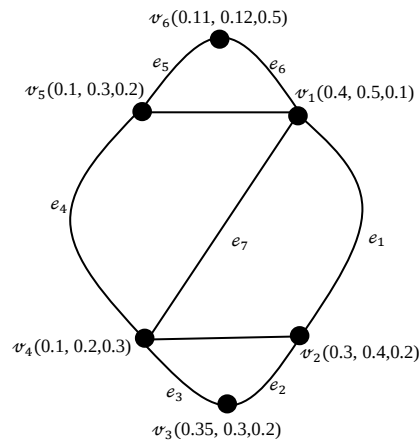


Figure. 16(a) Edge removal from a picture fuzzy subgraph $\mathcal{G} - e_8e_9$

In Fig.19(a), the maximal edge removed picture fuzzy outerplanar subgraph are shown. Thus, a result, $\mathcal{G} - \mathcal{U}$ is the maximum edge removed picture fuzzy outerplanar subgraph of $\mathcal{G}$, and picture fuzzy subgraph is maximal.

11 Application of picture fuzzy outerplanar graphs to urban traffic flow and intersection risk analysis

Urban transportation systems in developing regions are characterized by highly dynamic and uncertain environments, where traffic conditions continuously fluctuate due to the interaction of numerous complex factors. These factors include rapidly changing vehicle densities, inconsistent compliance with traffic regulations, irregular driving patterns, varying road infrastructures, unpredictable pedestrian movement, weather-related disturbances, road maintenance activities, and temporary construction works. In densely populated urban areas, the transportation network often experiences congestion during peak hours, while traffic flow may suddenly improve or deteriorate because of accidents, signal failures, or unauthorized parking. Such uncertainties make the analysis and management of transportation systems extremely challenging. Traditional graph-theoretic models and classical fuzzy graph structures generally rely on binary or limited-valued representations, which are insufficient to

capture the simultaneous presence of supportive, uncertain, and opposing traffic conditions within a transportation network. Consequently, there is a strong need for advanced mathematical tools capable of representing ambiguity, inconsistency, and incomplete information more effectively. Picture fuzzy graphs (PFGs) provide a powerful and flexible framework for handling such multidimensional uncertainty in transportation systems. Unlike classical fuzzy graphs, picture fuzzy graphs assign to every vertex and edge three independent membership degrees: positive membership, neutral membership, and negative membership. This tri-component representation allows picture fuzzy graphs to model traffic networks with greater realism by simultaneously incorporating agreement, uncertainty, and disagreement within a single mathematical structure. As a result, PFGs are particularly suitable for representing urban transportation scenarios where precise information is rarely available and decision-making must often rely on partially uncertain data.

Picture fuzzy positive membership values (μ): Higher positive membership values are assigned to intersections experiencing severe congestion, large vehicle accumulation, low average speed, and continuous traffic inflow. **Picture fuzzy neutral membership values (η):** Neutral membership values represent hesitation or uncertainty in traffic conditions caused by fluctuating vehicle movement, irregular driver behavior, temporary signal variations, or changing environmental conditions. Vertices with moderate traffic fluctuations are therefore assigned intermediate neutral values. **Picture fuzzy negative membership values (ν):** Negative membership values indicate the absence of congestion or relatively smooth traffic flow. Roads or intersections with less vehicle accumulation and better mobility receive comparatively larger negative membership values.

In the directed road network illustrated in Figure 20, an edge (v_i, v_j) represents a one-way roadway extending from city u to city v . In the event of accidents, blockages, or sudden obstructions on a one-way road, vehicles are redirected to alternate routes. This diversion places added strain on neighboring roads, often leading to secondary congestion and the formation of new high-risk traffic clusters. Empirical research indicates that convergence zone's locations where multiple heavily trafficked roads intersect are critical sources of accidents. Under picture fuzzy modeling, these intersections typically show elevated positive membership values (reflecting heavy traffic), moderate hesitation values (suggesting irregular traffic patterns), and low negative membership values (indicating low safety levels), highlighting the significant impact of congestion across interconnected roadways.

In the picture fuzzy sense, each vertex v_i is described by $\sigma(v_i) = (\mu_\sigma(v_i), \eta_\sigma(v_i), \nu_\sigma(v_i))$ where:

- $\mu_\sigma(v_i)$ represents the degree of current traffic congestion at the junction,
- $\eta_\sigma(v_i)$ indicates the neutral/hesitation level mixed traffic behaviour,
- $\nu_\sigma(v_i)$ represents the degree of absence of congestion at that point or free flow evidence.

Vertices with high positive membership and low negative membership are interpreted as critical intersection zones because they exhibit strong interactions with several neighbouring edges is shown in the Table 7.

Similarly, each directed road (edge) (v_i, v_j) is assigned a picture fuzzy membership $\delta(v_i, v_j) = (\mu_\delta(v_i, v_j), \eta_\delta(v_i, v_j), \nu_\delta(v_i, v_j))$. Which captures the traffic density and

traffic behavior of the corresponding roadway. The positive membership degree represents heavy traffic flow and congestion intensity, the neutral membership degree reflects temporary fluctuations and uncertain traffic conditions, while the negative membership degree denotes smooth or non-congested traffic movement. The corresponding picture fuzzy edge membership values are presented in Table 8.

The edge membership values are assigned in accordance with the traffic characteristics of the adjacent intThese conditions ensure that the traffic condition of a road segment cannot exceed the congestion influence of its adjacent intersections. In particular, the positive membership value of an edge is bounded by the minimum congestion level of its incident vertices, the neutral membership value reflects the shared uncertainty between connected intersections, and the negative membership value represents the maximum degree of smooth traffic flow associated with the adjacent junctions. Thus, the assigned edge membership values preserve structural consistency within the picture fuzzy urban traffic network.

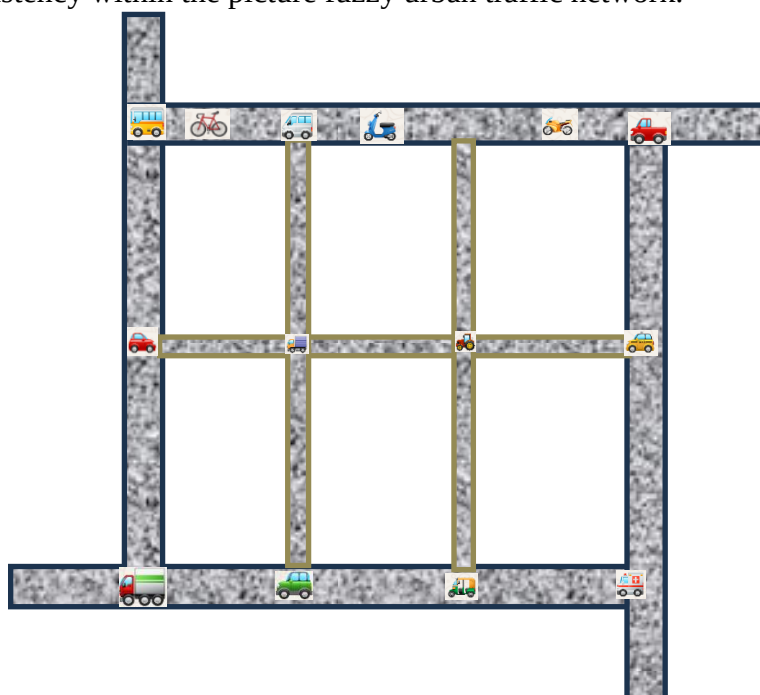


Figure 17. Urban Road network

Table 7 Picture fuzzy vertex membership values $(\mu_{\sigma}(v_i), \eta_{\sigma}(v_i), \nu_{\sigma}(v_i))$

Picture fuzzy vertex membership values $\sigma(v_i)$	Positive membership $(\mu_{\sigma}(v_i))$	Neutral membership $(\eta_{\sigma}(v_i))$	Negative membership $(\nu_{\sigma}(v_i))$
v_1	0.80	0.10	0.10
v_2	0.75	0.15	0.05
v_3	0.70	0.15	0.10
v_4	0.65	0.20	0.10
v_5	0.60	0.20	0.15
v_6	0.70	0.10	0.15
v_7	0.55	0.25	0.15
v_8	0.75	0.10	0.10

v_9	0.65	0.15	0.15
v_{10}	0.60	0.25	0.10
v_{11}	0.85	0.10	0.05
v_{12}	0.70	0.15	0.10
v_{13}	0.50	0.30	0.15
v_{14}	0.65	0.15	0.15
v_{15}	0.55	0.25	0.15
v_{16}	0.60	0.20	0.15

Table 8 Picture fuzzy edge membership values ($\mu_\sigma(v_i, v_j), \eta_\sigma(v_i, v_j), \nu_\sigma(v_i, v_j)$)

Picture fuzzy edge membership values $\delta(v_i, v_j)$	Positive membership $\mu_\sigma(v_i, v_j)$	Neutral membership $\eta_\sigma(v_i, v_j)$	Negative membership $\nu_\sigma(v_i, v_j)$
(v_1, v_2)	0.65	0.08	0.10
(v_2, v_3)	0.60	0.10	0.10
(v_3, v_4)	0.55	0.10	0.10
(v_4, v_{14})	0.55	0.12	0.15
(v_4, v_5)	0.55	0.15	0.15
(v_5, v_6)	0.50	0.08	0.15
(v_6, v_7)	0.50	0.08	0.16
(v_6, v_{15})	0.45	0.08	0.15
(v_7, v_8)	0.50	0.08	0.17
(v_8, v_9)	0.55	0.08	0.18
(v_9, v_{16})	0.50	0.12	0.19
(v_9, v_{10})	0.50	0.12	0.20
(v_{10}, v_1)	0.50	0.08	0.21
(v_1, v_{13})	0.45	0.08	0.22
(v_{10}, v_{11})	0.55	0.08	0.23
(v_{11}, v_{12})	0.60	0.08	0.24
(v_{12}, v_5)	0.50	0.12	0.25
(v_2, v_{11})	0.60	0.10	0.26
(v_3, v_{12})	0.55	0.10	0.27
(v_{11}, v_8)	0.60	0.08	0.28
(v_{12}, v_7)	0.50	0.12	0.29

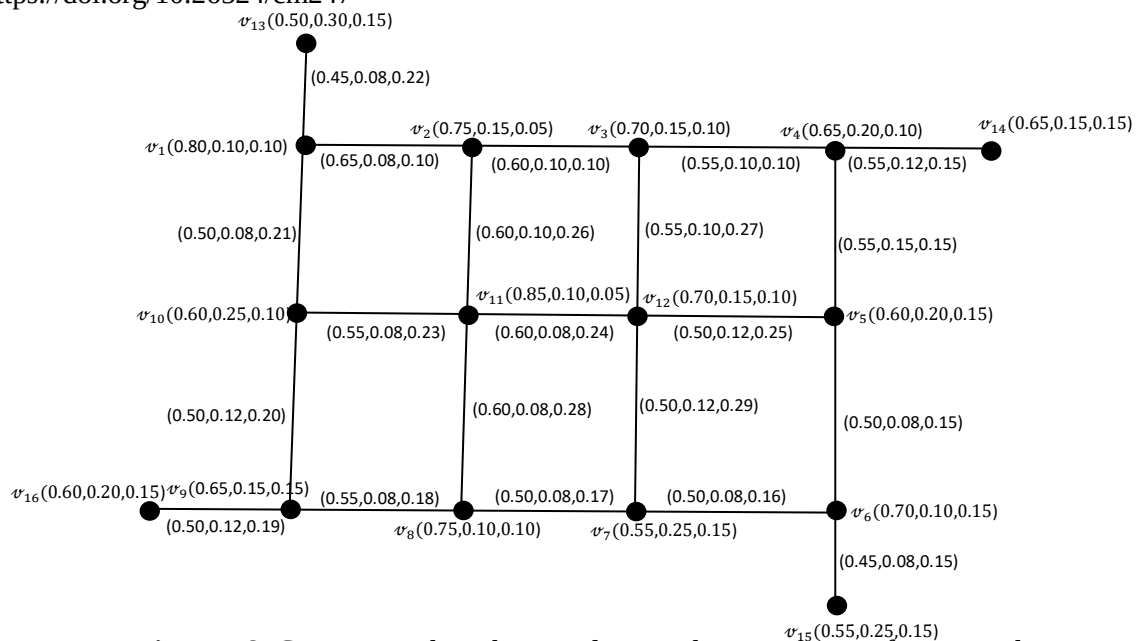


Figure 18. Converting the urban road network into a picture fuzzy graph

The transformation of the traffic system into a picture fuzzy graph is illustrated in Figure 18. This representation enables the simultaneous analysis of traffic density, accident probability, uncertainty in traffic behavior, congestion propagation, and structural interaction among neighboring intersections. A major challenge in urban traffic management is that congestion at one intersection rapidly spreads to adjacent intersections, producing interconnected high-risk clusters. To simplify these complex local structures while preserving critical risk information, picture fuzzy edge contraction is employed. When two adjacent intersections are connected by a road segment that consistently experiences: Heavy traffic density, persistent congestion, and a high probability of accidents, both intersections begin to operate as a single functional risk zone. By contracting the edge between them, the two vertices merge into a super-vertex whose picture fuzzy membership values collectively represent the congestion behavior of both intersections. Their interactions become so pronounced that they essentially function as a single risk unit. By contracting the edge between them, these two vertices merge into a single super-vertex, whose picture fuzzy membership values reflect the aggregated congestion behaviors of both intersections. This approach aids in identifying dominant traffic clusters, minimizes graphical complexity, and reveals the fundamental structural sources of congestion within the network.

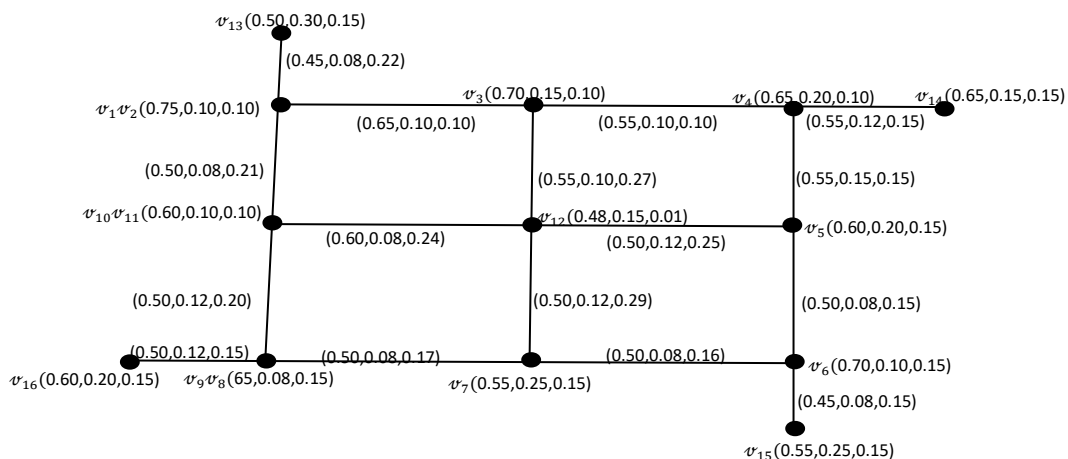


Figure 19. Picture fuzzy edge contraction

Let $\mathcal{G}$ be the picture fuzzy graph, where $\sigma = (\mu_\sigma, \eta_\sigma, \nu_\sigma)$ is the picture fuzzy set defined on the vertex set $\mathcal{V}$ and $\delta = (\mu_\delta, \eta_\delta, \nu_\delta)$ is the picture fuzzy set defined on the edge set $\mathcal{E}$. The edge contracted picture fuzzy graph, denoted by $\mathcal{G} \setminus v_{10}v_{11}$, is obtained from $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \delta)$ by contracting the edge $v_{10}v_{11}$. After contraction, the new vertex set becomes $\mathcal{V}' = [\mathcal{V} \setminus \{v_{10}v_{11}\} \cup \{v_{10}v_{11}\}]$ as illustrated in Figure 19. For every vertex $x \in \mathcal{V} \setminus \{v_{10}v_{11}\}$, the picture fuzzy membership values remain unchanged in the contracted graph. For the newly formed vertex $w = v_{10}v_{11}$, the membership values are defined by

$$\mu_\sigma(v_{10}v_{11}) = \min(\mu_\sigma(v_{10}), \mu_\sigma(v_{11}))$$

$$\eta_\sigma(v_{10}v_{11}) = \min(\eta_\sigma(v_{10}), \eta_\sigma(v_{11}))$$

$$\nu_\sigma(v_{10}v_{11}) = \max(\nu_\sigma(v_{10}), \nu_\sigma(v_{11}))$$

Using the values from Table 7, $v_{10} = (0.60, 0.25, 0.10)$, $v_{11} = (0.85, 0.10, 0.05)$, the contracted vertex becomes $v_{10}v_{11} = (0.60, 0.10, 0.10)$. The adjacency relation in the contracted graph is determined according to Definition 7.2. The resulting edge contracted picture fuzzy graph $\mathcal{G} \setminus v_{10}v_{11}$ represented as $\mathcal{G}_1$ is shown in Figure 19. Similarly, the picture fuzzy $\mathcal{G} \setminus v_1v_2$ and $\mathcal{G} \setminus v_8v_9$ can also be obtained by contracting the edges v_1v_2 and v_8v_9 respectively.

From the application point of view, the contraction process demonstrates how congestion behavior changes when two neighboring intersections operate as a single traffic zone. Initially, vertex v_{11} possesses a comparatively high positive membership value 0.85, indicating severe congestion, continuous vehicle accumulation, and higher accident probability. However, after merging with the adjacent intersection v_{10} the positive membership value decreases to 0.60. This reduction indicates that the heavy traffic concentration at v_{10} becomes partially redistributed across the merged region.

Thus, the contraction process reflects the sharing of traffic load between adjacent intersections rather than treating v_{10} as a congestion point. The resulting super-vertex still represents an important congestion zone, but with comparatively reduced traffic intensity. In practical urban transportation analysis, this indicates that neighboring road interactions can help balance traffic pressure and reduce localized congestion severity. After the edge contraction operation, the transportation model becomes structurally simpler and easier to analyze mathematically. The reduction in the positive membership value from 0.85 to 0.60 indicates that congestion intensity decreases after combining the neighboring intersections into a unified traffic region. This shows that traffic flow is redistributed through adjacent connected roads, thereby reducing localized pressure at a single intersection. Consequently, the contracted graph helps identify how congestion propagates across interconnected urban roads and how neighboring intersections collectively influence traffic behavior.

In the proposed urban traffic model, the edge contraction process provides several important outcomes for traffic analysis and congestion management. When neighboring highly congested intersections are merged, the severe traffic concentration at overloaded junctions becomes comparatively reduced because traffic pressure is redistributed across the contracted region.

As a result, accident risk and congestion density no longer remain isolated at a single intersection but become shared within the merged traffic cluster. The contraction process also enables neighboring intersections to function as a unified operational traffic zone, thereby simplifying the overall network structure while preserving the essential congestion characteristics of the original graph. This simplification helps in identifying dominant congestion clusters and structurally critical road segments more effectively. Furthermore, the merged traffic regions allow inner-city congestion behavior to be analyzed in a clearer and more organized manner. Consequently, transportation planners and traffic management authorities can better study congestion propagation, traffic balancing, accident-prone intersections, and alternative routing strategies under uncertain urban traffic conditions.

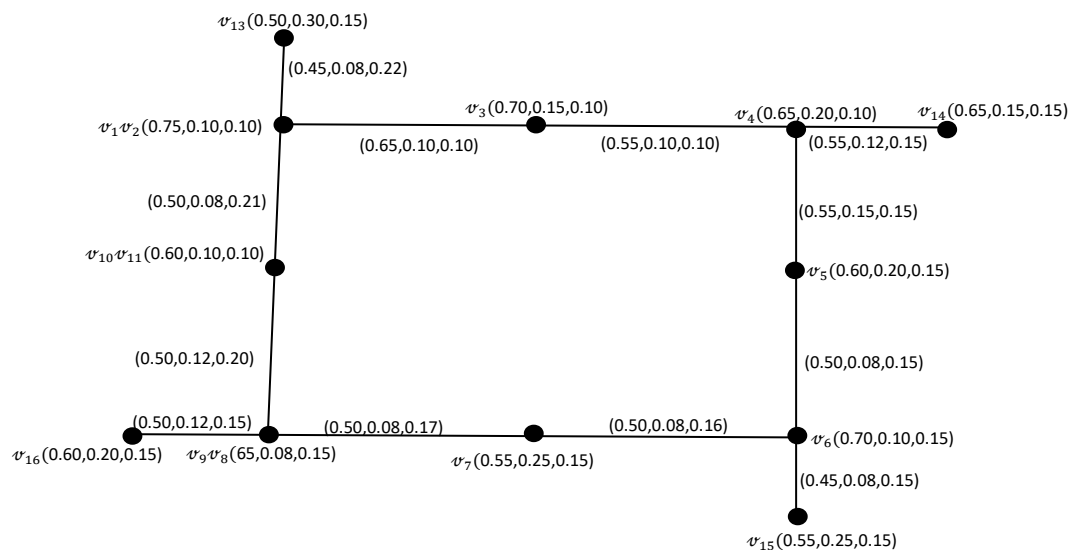


Figure 23. Picture fuzzy outerplanar graph

The vertex $\mathcal{V} = v_{12}$ is removed from the contracted picture fuzzy graph $\mathcal{G}_1$, and the resulting graph $\mathcal{G}_1 - \mathcal{V}$ represented as $\mathcal{V}$, is shown in Figure 23. It is observed that $\mathcal{V}$ forms a vertex deletion picture fuzzy outerplanar graph. In the proposed urban traffic model, the vertex deletion process represents the removal of a structurally critical and highly congested intersection from the transportation network in order to study the behavior of the remaining traffic system. After performing picture fuzzy edge contraction, the merged vertex v_{12} acts as a major congestion cluster with significant traffic accumulation and accident probability. When this congested super-vertex is deleted, all edges incident to it are also removed, causing the neighboring intersections to reorganize into a reduced and simplified traffic structure. The resulting graph preserves the picture fuzzy outerplanar property, allowing the remaining transportation network to be analyzed more effectively with respect to congestion redistribution, traffic stability, and structural connectivity.

From the application perspective, the vertex deletion process isolates highly dangerous traffic zones from the network and reduces the influence of severe congestion on adjacent intersections. The removal of the overloaded traffic cluster enables researchers and transportation planners to examine whether congestion propagation decreases after eliminating a critical junction. It also helps identify alternative traffic routes and comparatively safer roadway structures within the remaining network. Since the graphical complexity becomes reduced after deletion, the transportation system becomes easier to analyze while still preserving the essential traffic characteristics of the original urban network. In practical transportation systems, this process may

represent permanent closure of accident-prone intersections, temporary road blocking during emergencies, traffic diversion due to maintenance activities, or redesigning congested urban junctions. Furthermore, the resulting picture fuzzy outerplanar graph demonstrates that inner-city intersections contribute more heavily to congestion accumulation when compared to outer-road regions. After removing the highly congested vertex, the remaining outer regions exhibit comparatively smoother traffic movement and improved structural balance. Consequently, the vertex deletion stage provides a safer, simplified, and more manageable transportation model for studying congestion control, traffic redistribution, and accident risk under uncertain urban traffic conditions.

The results obtained from the picture fuzzy edge contraction and vertex deletion processes also provide useful guidance for future urban transportation planning. The analysis indicates that highly congested inner-city intersections and structurally overloaded road segments should be avoided in future road network designs because they significantly increase traffic accumulation, accident probability, and congestion propagation. Therefore, transportation authorities may introduce alternative traffic management strategies such as constructing flyovers, bypass roads, ring roads, elevated corridors, additional service lanes, and smart signal control systems in highly congested regions. The development of alternative routes can help distribute traffic more uniformly and reduce excessive pressure on critical intersections. Furthermore, the study suggests that heavily loaded convergence zones should be redesigned using modern infrastructure planning techniques, including multi-level junctions, one-way traffic regulation, intelligent traffic monitoring systems, and adaptive signal synchronization. By identifying congestion-prone intersections through picture fuzzy graph analysis, planners can prioritize the expansion of urban transportation infrastructure in critical regions before severe traffic conditions emerge. Consequently, the picture fuzzy outerplanar graph model not only assists in present traffic risk analysis but also supports long-term urban planning strategies for developing safer, smoother, and more efficient transportation systems under uncertain traffic environments.

Methodology clarification: It should be noted that the traffic-flow application presented in this study is intended solely as a conceptual illustration of the proposed picture fuzzy outerplanar graph framework. The picture fuzzy membership values assigned to the vertices and edges are illustrative values selected to demonstrate the mathematical representation and graph operations under uncertain traffic conditions. These values are not derived from traffic measurements, simulation models, or an automated membership assignment algorithm. Accordingly, the application serves to explain the theoretical applicability of the proposed model rather than to provide a validated traffic prediction or decision-support system.

Future work and limitation: The present study does not include empirical validation, simulation experiments, or comparisons with existing traffic analysis models, as its primary objective is to establish the theoretical foundations of picture fuzzy outerplanar graphs. Future work will focus on developing reproducible procedures for assigning picture fuzzy membership values from real traffic parameters, constructing quantitative traffic risk indices, validating the proposed framework using real-world traffic datasets and simulation studies, and comparing its performance with existing fuzzy graph-based traffic models. Therefore, the practical traffic-management implications presented in this section should be interpreted as potential applications of the proposed framework rather than experimentally validated results.

12 Conclusion

In conclusion, this work investigates the concept of picture fuzzy outerplanar graphs, which extend the frameworks of fuzzy and intuitionistic fuzzy graphs. The study explains how the deletion of vertices or edges from picture fuzzy graphs can generate subgraphs of picture fuzzy outerplanar graphs. In addition, the notions of maximal and maximum picture fuzzy outerplanar subgraphs associated with vertex and edge removal are examined with the support of illustrative examples. Several theorems and results are established to describe the relationships among these concepts. Furthermore, the paper introduces the concept of picture fuzzy dual graphs, which are closely connected to picture fuzzy outerplanar graphs, and discusses their important properties. Further, an application in the transportation planning using edge contraction of the picture fuzzy graph was demonstrated. We believe, picture fuzzy outerplanar graphs provide a powerful mathematical framework for detecting high-risk crossings and modeling uncertainty in urban traffic systems, thereby supporting efficient traffic planning, congestion control, and safer transportation management. Future research directions may focus on extending outerplanar graph theory into multi-valued fuzzy environments, thereby creating new opportunities for the investigation of m-polar bipolar and interval-valued picture fuzzy outerplanar graphs for practical and illustrative implementations. Additional research can emphasize the development of efficient computational and optimization algorithms for picture fuzzy outerplanar structures, together with sensitivity and robustness analysis in uncertain and dynamically changing environments. Moreover, wider applications in intelligent transportation systems, communication networks, social networks, biological systems, and artificial intelligence-based decision support frameworks can be explored. Comparative studies involving hypergraphs, neutrosophic graphs, and other generalized fuzzy graph models may also provide deeper theoretical insights and support broader practical applicability.

Author Contributions All authors contributed to the study conception and design. Conceptualization, Formal Analysis, Investigation, Data Curation, Writing-Original Draft were performed by DJ and SR; Resources and Supervision, SR. Methodology, DJ and SR; Writing-Review &Editing, Visualization, SR. All authors have read and agreed to the published version of the manuscript.

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Data Availability All data generated or analysed during this study are included in this published article.

Declarations

Conflict of interest All the authors declare that they have no conflict of interest.

Ethical approval Not applicable

Consent for publication All the authors agree with this manuscript's publication.

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