

On the Concept of Outerplanarity in Pythagorean Fuzzy Graph Structures

Deivanai Jaisankar^{1*}, Sujatha Ramalingam²

Received: 03 September 2026/ Accepted: 15 September 2026 / Published online: 05 October

2026 ©Sacred Heart Research Publications 2017

Abstract

This study introduces the concept of Pythagorean fuzzy outerplanar graphs (PFOGs), integrating the theory of Pythagorean fuzzy sets with outerplanar graph structures. In PFOGs, uncertainty is modeled by assigning to each vertex and edge a degree of membership and a degree of non-membership satisfying the Pythagorean condition, where the sum of the squares of these degrees does not exceed one. This representation provides a more flexible and expressive mechanism for handling ambiguity and partial information compared to classical fuzzy outerplanar graphs. Outerplanar graphs, as a significant subclass of planar graphs, possess structural properties that are valuable in simplifying complex network systems. In this work, we generalize crisp outerplanar graphs within the Pythagorean fuzzy framework and examine their structural behavior. We introduce vertex deleted (VD) and edge deleted (ED) Pythagorean fuzzy outerplanar subgraphs, along with the notions of maximum and maximal Pythagorean fuzzy outerplanar subgraphs under vertex and edge deletions. Furthermore, a novel concept of the dual of Pythagorean fuzzy outerplanar graphs is formulated and investigated. Theoretical developments are established through formal definitions, illustrative examples, and rigorous theorems.

Keywords Pythagorean Fuzzy Graphs, Pythagorean Fuzzy Outerplanar Graphs, VD and ED Pythagorean Fuzzy Outerplanar Subgraphs, Maximum and Maximal Pythagorean Fuzzy Outerplanar Subgraphs, Pythagorean Fuzzy Dual Graphs.

¹ Mathematics, School of Science and Humanities, Shiv Nadar University Chennai, Rajiv Gandhi Salai (OMR), Kalavakkam, Chengalpattu, Tamil Nadu 603110, India. *Email: deivanaij932000@gmail.com

² Mathematics, School of Science and Humanities, Shiv Nadar University Chennai, Rajiv Gandhi Salai (OMR), Kalavakkam, Chengalpattu, Tamil Nadu 603110, India. Email: sujathar@snuchennai.edu.in

1 Introduction

The theory of uncertainty modeling originated with the introduction of fuzzy sets by Zadeh [44], which provided a mathematical framework for representing imprecise and vague information. Kaufmann [25] further contributed to the development of fuzzy set theory and its mathematical foundations. Zadeh [45] subsequently introduced similarity relations and fuzzy orderings, which strengthened the theoretical basis for handling imprecise information. Rosenfeld [37] extended fuzzy set theory to graph theory through the concept of fuzzy graphs, thereby providing a framework for representing uncertain relationships and structures in networks. Bhattacharya [12] investigated fundamental properties of fuzzy graphs, while Mordeson and Peng [26] studied various operations on fuzzy graphs, contributing significantly to the development of fuzzy graph theory.

The concept of planar fuzzy graphs emerged as an important extension of fuzzy graph theory. Pramanik et al. [34] investigated special configurations of planar fuzzy graphs. Pal et al. [29] introduced the concept of fuzzy planar graphs, and subsequently developed its theoretical foundations and fundamental properties [30]. The study of dual structures in fuzzy graph environments was further extended by Abdul-Jabbar et al. [1], who introduced fuzzy dual graphs. Nirmala and Dhanabal [28] proposed interval-valued fuzzy planar graphs, thereby incorporating interval-valued uncertainty into planar graph structures. Alshehri and Akram [10] developed intuitionistic fuzzy planar graphs, while Akram et al. [8] investigated the application of bipolar fuzzy sets in planar graphs. These contributions significantly expanded the scope of planar graph theory for representing uncertain and imprecise network structures.

To represent uncertainty more comprehensively, Atanassov [11] introduced intuitionistic fuzzy sets by incorporating both membership and non-membership degrees. Shannon and Atanassov [38] introduced intuitionistic fuzzy graphs, and further investigated their level structures [39]. Parvathi et al. [31] studied several operations on intuitionistic fuzzy graphs. Akram and Davvaz [18] introduced strong intuitionistic fuzzy graphs and investigated their structural properties. Akram and Dudek [6] extended intuitionistic fuzzy theory to hypergraph structures through intuitionistic fuzzy hypergraphs. Dhavudh and Srinivasan [21] introduced intuitionistic fuzzy graphs of the second type, while their structural properties were further investigated in [22]. Rangasamy and Palaniappan [35] also studied operations on intuitionistic fuzzy sets of the second type. These developments demonstrated the increasing capability of intuitionistic fuzzy models to represent uncertainty involving both positive and negative information.

Another significant advancement in uncertainty modeling was the introduction of Pythagorean fuzzy sets by Yager [41]. Unlike intuitionistic fuzzy sets, Pythagorean fuzzy sets allow greater flexibility by imposing the condition that the sum of the squares of the membership and non-membership degrees does not exceed one. Yager and Abbasov [43] investigated Pythagorean membership grades and their applications in decision-making. Yager [42] further developed Pythagorean membership grades for multi-criteria decision-making, while Zhang and Xu [46] extended the TOPSIS method to decision-making problems involving Pythagorean fuzzy information. Peng and Yang [33] established

additional theoretical results for Pythagorean fuzzy sets. Peng and Selvachandran [32] presented a comprehensive state-of-the-art study and discussed future directions of Pythagorean fuzzy set theory. The applicability of Pythagorean fuzzy information in decision-making was further demonstrated by Naz et al. [27], Ren et al. [36], and Garg [23,24].

The integration of Pythagorean fuzzy theory with graph structures has also attracted considerable research attention. Verma et al. [40] introduced Pythagorean fuzzy graphs and established several fundamental results. Akram and Naz [7] investigated the energy of Pythagorean fuzzy graphs and their applications. Akram et al. [3] studied specific types of Pythagorean fuzzy graphs and demonstrated their applicability to decision-making problems. Akram et al. [2] further investigated graphs under the Pythagorean fuzzy environment, while Akram et al. [4] developed additional notions and structural properties of Pythagorean fuzzy graphs. These studies established a strong theoretical foundation for applying Pythagorean fuzzy concepts to graph-based representations of uncertain systems.

In recent years, increasing attention has been devoted to fuzzy outerplanar graph structures because of their ability to model sparse and non-intersecting network systems effectively. Deivanai et al. [13] introduced fuzzy outerplanar graphs and established their fundamental theoretical properties together with practical applications. The framework was subsequently extended to multipolar fuzzy outerplanar graphs for road-network analysis [18]. Bipolar fuzzy outerplanar graphs were developed for image-shrinking applications [14], while interval-valued fuzzy outerplanar graphs were proposed for network-design problems [15]. Furthermore, cubic fuzzy outerplanar graph structures were successfully applied to face-shape recognition [16], and fuzzy soft outerplanar graph models were utilized for image contraction and image-processing applications [17]. A comprehensive review of applications of fuzzy planar graphs and their potential future research directions was subsequently presented by Deivanai et al. [19]. Comprehensive overview of topological indices in fuzzy graphs, focusing on how classical graph-theoretic measures can be extended to fuzzy environments. Fuzzy graph theory incorporates uncertainty and vagueness by assigning membership values to vertices and edges, making it suitable for modeling real-world systems where relationships are not completely precise [20]. Akram et al. [9] introduces planar graphs in a Pythagorean fuzzy environment to represent uncertainty using membership and non-membership degrees. The framework extends classical planar graph concepts and provides a mathematical approach for analyzing uncertain network structures.

Despite these significant developments in fuzzy outerplanar graphs and Pythagorean fuzzy graph theory, the integration of Pythagorean fuzzy sets with outerplanar graph structures remains largely unexplored. This observation motivates the development of a new mathematical framework, namely Pythagorean fuzzy outerplanar graphs (PFOGs), which combines the structural properties of outerplanar graphs with the flexible uncertainty representation of Pythagorean fuzzy sets. In a PFOG, each vertex and edge are characterized by a membership degree and a non-membership degree satisfying the Pythagorean condition. In particular, for every vertex or edge, the sum of the squares of its membership and non-membership degrees does not exceed one. This formulation provides a more flexible and

expressive representation of uncertain information than conventional fuzzy outerplanar graph models.

Outerplanar graphs constitute an important subclass of planar graphs and possess structural characteristics that are particularly suitable for modeling sparse and non-intersecting network systems. Motivated by these properties, the present work extends classical outerplanar graph theory into the Pythagorean fuzzy environment and investigates the resulting structural behavior. Specifically, the notions of vertex deleted (VD) Pythagorean fuzzy outerplanar subgraphs and edge deleted (ED) Pythagorean fuzzy outerplanar subgraphs are introduced and analyzed. Furthermore, the concepts of maximum and maximal Pythagorean fuzzy outerplanar subgraphs arising from vertex and edge deletion operations are formulated and studied in detail. In addition, a novel notion of the dual of a Pythagorean fuzzy outerplanar graph is proposed, and its fundamental properties are investigated. The theoretical framework is supported by rigorous definitions, illustrative examples, and mathematical theorems, thereby contributing to the advancement of fuzzy graph theory and extending the applicability of outerplanar graph structures to uncertain environments. The abbreviations used throughout this study are summarized in Table 1.

Table 1: Summarizes the abbreviations adopted throughout the present study.

Abbreviation	Description
PFS	Pythagorean Fuzzy Set
PFG	Pythagorean Fuzzy Graph
PFMS	Pythagorean Fuzzy Multiset
PFMG	Pythagorean Fuzzy Multigraph
PFPG	Pythagorean Fuzzy Planar Graph
PFF	Pythagorean Fuzzy Face
SPFF	Strong Pythagorean Fuzzy Face
WPFF	Weak Pythagorean Fuzzy Face
PFDG	Pythagorean Fuzzy Dual Graph
PFOG	Pythagorean Fuzzy Outerplanar Graph
PFNOG	Pythagorean Fuzzy Non-Outerplanar Graph
PFOS	Pythagorean Fuzzy Outerplanar Subgraph
VR	Vertex Deletion
ER	Edge Deletion
VR-PFOS	Vertex- Deleted Pythagorean Fuzzy Outerplanar Subgraph
ER-PFOS	Edge- Deleted Pythagorean Fuzzy Outerplanar Subgraph
Max-VR-PFOG	Maximum Vertex- Deleted Pythagorean Fuzzy Outerplanar Subgraph
Maximal-VR-PFOG	Maximal Vertex- Deleted Pythagorean Fuzzy Outerplanar Subgraph
Max-ER-PFOG	Maximum Edge- Deleted Pythagorean Fuzzy Outerplanar Subgraph
Maximal-ER-PFOG	Maximal Edge- Deleted Pythagorean Fuzzy Outerplanar Subgraph
SPFE	Strong Pythagorean Fuzzy Edge
WPFE	Weak Pythagorean Fuzzy Edge

PFD	Pythagorean Fuzzy Degree
PFO	Pythagorean Fuzzy Order
PFS	Pythagorean Fuzzy Size
PFP	Pythagorean Fuzzy Planarity
SPFPG	Strong Pythagorean Fuzzy Planar Graph
PFMD	Pythagorean Fuzzy Membership Degree
PFNMD	Pythagorean Fuzzy Non-membership Degree

2 Motivation of the work

Real-world systems such as transportation networks, communication systems, image-processing systems, and intelligent decision-support environments are often characterized by uncertainty, imprecision, and incomplete information. Classical graph models are effective for representing relationships among objects, but they generally assume precise information about vertices and edges. The development of fuzzy set theory by Zadeh [44] and its incorporation into graph theory through fuzzy graphs [37] provided an effective mathematical framework for representing uncertain relationships by assigning membership values to graph elements. Subsequently, fuzzy planar graph models [29,30,34] were developed to incorporate uncertainty into networks subject to planar structural constraints.

Among planar graph classes, outerplanar graphs possess a particularly useful structure in which all vertices can be embedded on the boundary of the outer face. This structural property makes them suitable for applications involving planar network layouts, transportation systems, communication networks, and other structures where crossing connections are undesirable. The development of fuzzy outerplanar graphs has further demonstrated the usefulness of this structure under uncertainty, with applications involving transportation and road-network design, image processing, face recognition, and related network-analysis problems [13–18]. These developments establish a strong connection between fuzzy graph theory and practical network structures.

However, conventional fuzzy models may not always provide sufficient flexibility for representing situations in which the degrees of membership and non-membership contain a greater level of uncertainty. Pythagorean fuzzy sets, introduced by Yager [41], overcome this limitation by allowing the squared sum of the membership and non-membership degrees to be at most one. This provides a larger feasible domain than intuitionistic fuzzy sets and enables more flexible representation of uncertain information. The subsequent development of Pythagorean fuzzy graph models [2-7,40] has demonstrated the applicability of this framework to graph-theoretic analysis and decision-making problems.

Despite these developments, the combination of Pythagorean fuzzy theory with outerplanar graph structures remains insufficiently explored. In particular, a systematic framework for studying structural properties such as vertex deletion, edge deletion, maximum and maximal subgraphs, and dual structures in a Pythagorean fuzzy outerplanar environment is needed. This gap motivates the present study.

Therefore, the main motivation of this work is to develop a Pythagorean fuzzy outerplanar graph (PFOG) framework that integrates the structural advantages of outerplanar graphs with the enhanced uncertainty-handling capability of Pythagorean fuzzy sets. The proposed framework is intended to provide a systematic basis for investigating vertex deletion and edge deletion operations, maximum and maximal PFOG subgraphs, and dual graph structures. Such a framework can provide a mathematical foundation for modeling and analyzing uncertain planar networks and may support future applications in intelligent transportation systems, communication-network design, image analysis, pattern recognition, and decision-support systems operating under uncertain conditions.

3 Novelty of the article

The novelty of this article can be summarized as follows:

1. Introduction of Pythagorean Fuzzy Outerplanar Graphs (PFOGs). The primary novelty of this work is the introduction of Pythagorean Fuzzy Outerplanar Graphs (PFOGs), which integrate the enhanced uncertainty-representation capability of Pythagorean fuzzy sets with the structural restrictions of outerplanar graphs. Pythagorean fuzzy graph models have been investigated in the literature [2-7,40], and Pythagorean fuzzy planar graphs have also been developed [9]. However, the specific integration of Pythagorean fuzzy information with outerplanar graph structures, together with the structural concepts developed in this article, has not been systematically studied.
2. Development of vertex deletion and edge deletion PFOG subgraphs. The concepts of vertex deleted (VD) and edge deleted (ED) Pythagorean fuzzy outerplanar subgraphs are formulated and investigated. These constructions provide a systematic mechanism for examining how the deletion of vertices or edges affects the underlying outerplanar structure while retaining the associated Pythagorean fuzzy information.
3. Introduction and characterization of maximum and maximal PFOG subgraphs. The notions of maximum and maximal Pythagorean fuzzy outerplanar subgraphs are introduced and characterized. These concepts extend the corresponding structural ideas from fuzzy outerplanar graphs [13–18] to a Pythagorean fuzzy setting and provide a framework for identifying structurally significant subgraphs under uncertainty.
4. Development of a duality framework for PFOGs. A duality concept for Pythagorean fuzzy outerplanar graphs is developed by establishing a correspondence between the faces and edges of the underlying outerplanar structure while preserving the associated Pythagorean fuzzy information. This provides a new structural perspective for analyzing PFOGs and distinguishes the proposed framework from existing fuzzy outerplanar models.
5. Theoretical characterization of the proposed framework. The proposed PFOG model is supported by formal definitions, illustrative examples, and mathematical results

concerning the proposed subgraph and dual structures. Thus, the study establishes a systematic theoretical foundation for extending outerplanar graph analysis to environments in which uncertainty is represented through Pythagorean membership and non-membership degrees.

6. Potential applicability to uncertain network systems. By combining outerplanar structural constraints with Pythagorean fuzzy information, the proposed framework provides a basis for future applications involving uncertain transportation networks, communication systems, image-processing structures, pattern recognition, and intelligent decision-support systems.

4 Objective of the Study

The primary objective of this study is to develop and investigate a theoretical framework for Pythagorean Fuzzy Outerplanar Graphs (PFOGs) by integrating the uncertainty-handling capability of Pythagorean fuzzy sets with the structural properties of outerplanar graphs. Specifically, the study aims to:

1. Define and investigate the fundamental concepts, properties, and structural characteristics of Pythagorean fuzzy outerplanar graphs and relate them to existing fuzzy outerplanar graph models.
2. Develop and analyze vertex deleted (VD) and edge deleted (ED) Pythagorean fuzzy outerplanar subgraphs and investigate the structural changes resulting from vertex and edge deletion.
3. Introduce and characterize maximum and maximal Pythagorean fuzzy outerplanar subgraphs obtained through vertex deletion and edge deletion operations.
4. Formulate and investigate the dual of a Pythagorean fuzzy outerplanar graph and establish the structural relationships between a PFOG and its corresponding dual graph.
5. Establish rigorous theoretical results through definitions, propositions, theorems, corollaries, and illustrative examples to strengthen the mathematical foundation of Pythagorean fuzzy outerplanar graph theory.
6. Develop a mathematical framework for representing and analyzing uncertain planar network systems using Pythagorean fuzzy outerplanar graph structures, thereby providing a foundation for potential applications in transportation, communication, image processing, and decision-support systems.

5 Structure of the Article

The remainder of this article is organized as follows. Section 2 presents the motivation of the study and explains the need for developing Pythagorean fuzzy outerplanar graphs (PFOGs) for representing uncertain network structures. Section 3 highlights the novelty of the proposed work and distinguishes the PFOG framework from existing fuzzy outerplanar and

Pythagorean fuzzy graph models. Section 4 states the objectives of the study and outlines the principal aims of the proposed research.

Section 5 describes the organization of the article. Section 6 provides the necessary preliminaries and fundamental concepts related to Pythagorean fuzzy sets, Pythagorean fuzzy graphs, and outerplanar graphs that are required for the subsequent analysis. Section 7 introduces the concept of Pythagorean fuzzy outerplanar graphs and develops their fundamental properties. Section 7.1 presents the characterization of Pythagorean fuzzy outerplanar graphs and establishes the conditions associated with their structural representation.

Section 8 introduces vertex-deletion Pythagorean fuzzy outerplanar subgraphs and investigates their structural properties. Section 9 focuses on maximal and maximum vertex-deletion Pythagorean fuzzy outerplanar subgraphs and examines their corresponding characteristics. Section 10 introduces edge-deletion Pythagorean fuzzy outerplanar subgraphs and investigates the properties resulting from edge deletion operations. Section 11 presents the maximal and maximum edge-deletion Pythagorean fuzzy outerplanar subgraphs and establishes their relevant structural properties. Finally, Section 12 concludes the article by summarizing the principal findings of the study and highlighting possible directions for future research.

6 Preliminaries

Fundamentals of Pythagorean fuzzy sets, Pythagorean fuzzy graphs, Pythagorean fuzzy planar graphs

We recall several important definitions and properties of Pythagorean fuzzy sets (PFSs), Pythagorean fuzzy graphs (PFGs), and Pythagorean fuzzy planar graphs (PFPGs) that form the basis of this study.

Definition 6.1[33,36] Let $\mathcal{X}$ be a nonempty universe of discourse. A Pythagorean fuzzy set (PFS) $\mathcal{M}$ on $\mathcal{X}$ is expressed as $\mathcal{M} = \{\langle x, \mu_{\mathcal{M}}(x), \nu_{\mathcal{M}}(x) \rangle | x \in \mathcal{X}\}$, where the functions $\mu_{\mathcal{M}}: \mathcal{X} \rightarrow [0,1]$ and $\nu_{\mathcal{M}}: \mathcal{X} \rightarrow [0,1]$ represent the membership and non-membership degrees of $x \in \mathcal{X}$ with respect to $\mathcal{M}$ respectively. These functions satisfy the condition $0 \leq \mu_{\mathcal{M}}^2(x) + \nu_{\mathcal{M}}^2(x) \leq 1, \forall x \in \mathcal{X}$.

The corresponding hesitation degree of x in $\mathcal{M}$ is defined by

$$\pi_{\mathcal{M}}(x) = \sqrt{1 - \mu_{\mathcal{M}}^2(x) - \nu_{\mathcal{M}}^2(x)}$$

Here $\mu_{\mathcal{M}}(x), \nu_{\mathcal{M}}(x)$ and $\pi_{\mathcal{M}}(x)$ denote the membership, non-membership, and hesitation degrees of x respectively. The ordered pair $(\mu_{\mathcal{M}}(x), \nu_{\mathcal{M}}(x))$ is called a Pythagorean fuzzy number (PFN)

Definition 6.2[27] Let $\mathcal{V}$ be a nonempty finite vertex set. A Pythagorean fuzzy graph (PFG) is an ordered pair $\mathfrak{G} = (\mathcal{M}, \mathcal{N})$ where $\mathcal{M}$ is a Pythagorean fuzzy set on $\mathcal{V}$ and $\mathcal{N}$ is a

Pythagorean fuzzy relation on $\mathcal{V} \times \mathcal{V}$. Thus, $\mathcal{M} = \{(\mathfrak{v}, \mu_{\mathcal{M}}(\mathfrak{v}), \nu_{\mathcal{M}}(\mathfrak{v})) | \mathfrak{v} \in \mathcal{V}\}$ and $\mathcal{N} = \{(\mathfrak{uv}, \mu_{\mathcal{N}}(\mathfrak{uv}), \nu_{\mathcal{N}}(\mathfrak{uv})) | \mathfrak{uv} \in \mathcal{E}\}$, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the underlying edge set. The vertex and edge membership and non-membership functions satisfy $0 \leq \mu_{\mathcal{M}}^2(\mathfrak{v}) + \nu_{\mathcal{M}}^2(\mathfrak{v}) \leq 1, \forall \mathfrak{v} \in \mathcal{V}$ and $0 \leq \mu_{\mathcal{N}}^2(\mathfrak{uv}) + \nu_{\mathcal{N}}^2(\mathfrak{uv}) \leq 1, \forall \mathfrak{uv} \in \mathcal{E}$. Moreover, for every edge $\mathfrak{uv} \in \mathcal{E}$, the edge membership degree cannot exceed the membership degree of either of its end vertices, while its non-membership degree cannot be less than the non-membership degree of either end vertex. Hence,

$$\mu_{\mathcal{N}}(\mathfrak{uv}) \leq \min\{\mu_{\mathcal{M}}(\mathfrak{u}), \mu_{\mathcal{M}}(\mathfrak{v})\} \text{ and } \nu_{\mathcal{N}}(\mathfrak{uv}) \geq \max\{\nu_{\mathcal{M}}(\mathfrak{u}), \nu_{\mathcal{M}}(\mathfrak{v})\}.$$

Therefore, the Pythagorean fuzzy graph can be represented as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mu_{\mathcal{M}}, \nu_{\mathcal{M}}, \mu_{\mathcal{N}}, \nu_{\mathcal{N}})$

where $\mu_{\mathcal{M}}, \nu_{\mathcal{M}}$ characterize the Pythagorean fuzzy information associated with vertices and $\mu_{\mathcal{N}}, \nu_{\mathcal{N}}$ characterize the corresponding information associated with edges.

Definition 6.3[9] Let $\mathcal{X}$ be a nonempty set. A Pythagorean fuzzy multiset (PFMS) $\mathcal{M}$ over $\mathcal{X}$ is characterized by two count functions, namely the count-membership function $CM_{\mathcal{M}}$ and the count-non-membership function $CN_{\mathcal{M}}$, defined by $CM_{\mathcal{M}}: \mathcal{X} \rightarrow \mathcal{Q}$, $CN_{\mathcal{M}}: \mathcal{X} \rightarrow \mathcal{Q}$, where $\mathcal{Q}$ denotes the collection of all finite multisets whose elements belong to $[0,1]$. For each $x \in \mathcal{X}$, let the membership sequence and non-membership sequence associated with x be $\mu_{\mathcal{M}}^1(x), \mu_{\mathcal{M}}^2(x), \dots, \mu_{\mathcal{M}}^p(x)$ and $\nu_{\mathcal{M}}^1(x), \nu_{\mathcal{M}}^2(x), \dots, \nu_{\mathcal{M}}^p(x)$ respectively, arranged in non-increasing order. These values satisfy $\mu_{\mathcal{M}}^1(x) \geq \mu_{\mathcal{M}}^2(x) \geq \dots \geq \mu_{\mathcal{M}}^p(x)$ and $\nu_{\mathcal{M}}^1(x) \geq \nu_{\mathcal{M}}^2(x) \geq \dots \geq \nu_{\mathcal{M}}^p(x)$ together with the Pythagorean condition $0 \leq (\mu_{\mathcal{M}}^j(x))^2 + (\nu_{\mathcal{M}}^j(x))^2 \leq 1$, for every $x \in \mathcal{X}$ and $j = 1, 2, \dots, p$. Consequently, a Pythagorean fuzzy multiset $\mathcal{M}$ can be represented as

$$\mathcal{M} = \{ \{x, (\mu_{\mathcal{M}}^1(x), \mu_{\mathcal{M}}^2(x), \dots, \mu_{\mathcal{M}}^p(x)), (\nu_{\mathcal{M}}^1(x), \nu_{\mathcal{M}}^2(x), \dots, \nu_{\mathcal{M}}^p(x))\} \}$$

Here, p denotes the multiplicity of the membership and non-membership information associated with each element of $\mathcal{X}$.

Definition 6.4[9] Let $\mathcal{M} = (\mu_{\mathcal{M}}, \nu_{\mathcal{M}})$ be a Pythagorean fuzzy set on a nonempty vertex set $\mathcal{V}$. A Pythagorean fuzzy multigraph (PFMG) is denoted by $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the underlying multiedge set and $\mathcal{N}$ is a Pythagorean fuzzy multiset defined on $\mathcal{V} \times \mathcal{V}$. For every edge $\mathfrak{uv} \in \mathcal{E}$, let $\mathcal{N}(\mathfrak{uv}) = \{(\mathfrak{uv}, \mu_{\mathcal{N}}(\mathfrak{uv})_j, \nu_{\mathcal{N}}(\mathfrak{uv})_j) | j = 1, 2, \dots, p\}$. The vertex and edge membership and non-membership degrees satisfy $\mu_{\mathcal{N}}(\mathfrak{uv})_j \leq \min\{\mu_{\mathcal{M}}(\mathfrak{u}), \mu_{\mathcal{M}}(\mathfrak{v})\}$ and $\nu_{\mathcal{N}}(\mathfrak{uv})_j \leq \max\{\nu_{\mathcal{M}}(\mathfrak{u}), \nu_{\mathcal{M}}(\mathfrak{v})\}$, for every $\mathfrak{uv} \in \mathcal{E}$ and $j = 1, 2, \dots, p$. In addition,

$$0 \leq (\mu_{\mathcal{N}}(\mathfrak{uv})_j)^2 + (\nu_{\mathcal{N}}(\mathfrak{uv})_j)^2 \leq 1$$

Thus, a Pythagorean fuzzy multigraph permits multiple edges between the same pair of vertices, with each edge carrying its own membership and non-membership information subject to the Pythagorean condition.

Definition 6.5[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph, where $\mathcal{M}$ is the Pythagorean fuzzy vertex set and $\mathcal{N}$ is the Pythagorean fuzzy multiedge set. For each $uv \in \mathcal{E}$, let

$$\mathcal{N}(uv) = \{\langle uv, \mu_{\mathcal{N}}(uv)_j, \nu_{\mathcal{N}}(uv)_j \rangle | j = 1, 2, \dots, p\}$$

Then the following quantities are defined.

1. Order: The order of $\mathfrak{G}$ denoted by $\mathcal{O}(\mathfrak{G})$ is given by

$$\mathcal{O}(\mathfrak{G}) = \left(\sum_{v \in \mathcal{V}} \mu_{\mathcal{M}}(v), \sum_{v \in \mathcal{V}} \nu_{\mathcal{M}}(v) \right)$$

2. Size: The size of $\mathfrak{G}$ denoted by $\mathcal{S}(\mathfrak{G})$ is defined as

$$\mathcal{S}(\mathfrak{G}) = \left(\sum_{uv \in \mathcal{E}} \sum_{j=1}^p \mu_{\mathcal{N}}(uv)_j, \sum_{uv \in \mathcal{E}} \sum_{j=1}^p \nu_{\mathcal{N}}(uv)_j \right)$$

3. Degree: The degree of a vertex $v \in \mathcal{V}$, denoted by $\deg_{\mathfrak{G}}(v)$ is defined by

$$\deg_{\mathfrak{G}}(v) = \left(\sum_{\substack{uv \in \mathcal{E} \\ u \in \mathcal{V}}} \sum_{j=1}^p \mu_{\mathcal{N}}(uv)_j, \sum_{\substack{uv \in \mathcal{E} \\ u \in \mathcal{V}}} \sum_{j=1}^p \nu_{\mathcal{N}}(uv)_j \right)$$

4. Total degree: The total degree of a vertex $v \in \mathcal{V}$, denoted by $\text{td}_{\mathfrak{G}}(v)$ is defined as

$$\text{td}_{\mathfrak{G}}(v) = \left(\mu_{\mathcal{M}}(v) + \sum_{\substack{uv \in \mathcal{E} \\ u \in \mathcal{V}}} \sum_{j=1}^p \mu_{\mathcal{N}}(uv)_j, \nu_{\mathcal{M}}(v) + \sum_{\substack{uv \in \mathcal{E} \\ u \in \mathcal{V}}} \sum_{j=1}^p \nu_{\mathcal{N}}(uv)_j \right)$$

Thus, the order, size, degree, and total degree of a Pythagorean fuzzy multigraph are represented by ordered pairs consisting of the corresponding membership and non-membership information.

Definition 6.6[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph. If every vertex of $\mathfrak{G}$ has the same membership degree and the same non-membership degree in its vertex degree, that is,

$$\deg_{\mathfrak{G}}(u) = \deg_{\mathfrak{G}}(v), \forall u, v \in \mathcal{V},$$

then $\mathfrak{G}$ is called a regular Pythagorean fuzzy multigraph. In other words, a Pythagorean fuzzy multigraph is regular if all of its vertices possess identical Pythagorean fuzzy degree values.

Definition 6.7[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph. For an edge $uv \in \mathcal{E}$, let

$$\mathcal{N}(uv) = \{\langle uv, \mu_{\mathcal{N}}(uv)_j, \nu_{\mathcal{N}}(uv)_j \rangle | j = 1, 2, \dots, p\}$$

The degree of the j -th edge uv , denoted by $d_{\mathfrak{G}}(uv)_j$ is defined as

$$d_{\mathfrak{G}}(uv)_j = (\deg_{\mathfrak{G}}(u)_{\mu} + \deg_{\mathfrak{G}}(v)_{\mu} - 2\mu_{\mathcal{N}}(uv)_j, \deg_{\mathfrak{G}}(u)_{\nu} + \deg_{\mathfrak{G}}(v)_{\nu} - 2\nu_{\mathcal{N}}(uv)_j)$$

where $\deg_{\mathfrak{G}}(u)_{\mu}$ and $\deg_{\mathfrak{G}}(u)_{\nu}$ denote the membership and non-membership components of the degree of u respectively.

The total degree of the j -th edge uv , denoted by $td_{\mathfrak{G}}(uv)_j$ is given by

$$td_{\mathfrak{G}}(uv)_j = (\deg_{\mathfrak{G}}(u)_{\mu} + \deg_{\mathfrak{G}}(v)_{\mu} - \mu_{\mathcal{N}}(uv)_j, \deg_{\mathfrak{G}}(u)_{\nu} + \deg_{\mathfrak{G}}(v)_{\nu} - \nu_{\mathcal{N}}(uv)_j)$$

Here, the subtraction of the edge membership and non-membership values accounts for the contribution of the edge uv that is counted through both of its incident vertices.

Definition 6.8[9] A Pythagorean fuzzy multigraph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ is called an edge-regular Pythagorean fuzzy multigraph if every edge of $\mathfrak{G}$ has the same Pythagorean fuzzy degree; that is,

$$d_{\mathfrak{G}}(uv)_j = d_{\mathfrak{G}}(rs)_k$$

for every pair of edges $uv, rs \in \mathcal{E}$ and for all admissible multiplicity indices j, k . Equivalently, all edges of $\mathfrak{G}$ possess identical membership degree and non-membership degree components. Thus, the edge degrees of $\mathfrak{G}$ are uniform throughout the Pythagorean fuzzy multigraph.

Definition 6.9[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph, and let $\mathcal{N}(uv) = \{\langle uv, \mu_{\mathcal{N}}(uv)_j, \nu_{\mathcal{N}}(uv)_j \rangle | j = 1, 2, \dots, p\}$ be the Pythagorean fuzzy multiedges joining the vertices $u, v \in \mathcal{V}$. A multiedge uv is said to be a strong Pythagorean fuzzy multiedge if, for every $j = 1, 2, \dots, p$,

$$\mu_{\mathcal{N}}(uv)_j \geq \frac{1}{2} \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}$$

and

$$\nu_{\mathcal{N}}(uv)_j \leq \frac{1}{2} \max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}$$

Thus, a Pythagorean fuzzy multiedge is considered strong when its membership information is sufficiently significant and its non-membership information is appropriately bounded relative to the Pythagorean fuzzy information of its incident vertices.

Definition 6.10[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph, and let $\mathcal{N}(uv) = \{\langle uv, \mu_{\mathcal{N}}(uv)_j, \nu_{\mathcal{N}}(uv)_j \rangle | j = 1, 2, \dots, p\}$ be the Pythagorean fuzzy multiedge set corresponding to the pair $u, v \in \mathcal{V}$. A multiedge uv is said to be an effective Pythagorean fuzzy multiedge if, for a fixed j ,

$$\mu_{\mathcal{N}}(uv)_j = \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}$$

and

$$v_{\mathcal{N}}(uv)_j = \max\{v_{\mathcal{M}}(u), v_{\mathcal{M}}(v)\}$$

Thus, an effective Pythagorean fuzzy multiedge attains exactly the limiting membership and non-membership values determined by its two incident vertices. The membership degree of the edge equals the smaller membership degree of its endpoints, while its non-membership degree equals the larger non-membership degree of its endpoints.

Definition 6.11[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph, where $\mathcal{M}$ is the Pythagorean fuzzy vertex set and $\mathcal{N}$ is the Pythagorean fuzzy multiedge set. For every pair of vertices $u, v \in \mathcal{V}$, let $\mathcal{N}(uv) = \{(uv, \mu_{\mathcal{N}}(uv)_j, v_{\mathcal{N}}(uv)_j) | j = 1, 2, \dots, p\}$. The Pythagorean fuzzy multigraph $\mathfrak{G}$ is called a complete Pythagorean fuzzy multigraph if, for every pair of vertices $u, v \in \mathcal{V}$ and for every $j = 1, 2, \dots, p$,

$$\mu_{\mathcal{N}}(uv)_j = \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}$$

and

$$v_{\mathcal{N}}(uv)_j = \max\{v_{\mathcal{M}}(u), v_{\mathcal{M}}(v)\}$$

Thus, a Pythagorean fuzzy multigraph is complete when every pair of vertices is connected by a Pythagorean fuzzy multiedge whose membership and non-membership degrees attain the corresponding minimum and maximum values of its incident vertices, respectively.

Definition 6.12[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph and let $uv \in \mathcal{E}$ be a Pythagorean fuzzy edge with membership degree $\mu_{\mathcal{N}}(uv)_j$ and non-membership degree $v_{\mathcal{N}}(uv)_j$. The strength of the j -th Pythagorean fuzzy edge uv denoted by $\mathcal{S}_{\mathfrak{G}}(uv)_j = (\mathcal{S}_{\mu}(uv)_j, \mathcal{S}_{v}(uv)_j)$ is defined as

$$\mathcal{S}_{\mathfrak{G}}(uv)_j = \left(\frac{\mu_{\mathcal{N}}(uv)_j}{\min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}}, \frac{v_{\mathcal{N}}(uv)_j}{\max\{v_{\mathcal{M}}(u), v_{\mathcal{M}}(v)\}} \right)$$

provided that the denominators are nonzero. The edge uv is called a strong Pythagorean fuzzy edge if $\mathcal{S}_{\mu}(uv)_j \geq 0.5$ and $\mathcal{S}_{v}(uv)_j \leq 0.5$. Otherwise, uv is called a weak Pythagorean fuzzy edge.

Here, $\mathcal{S}_{\mu}(uv)_j$ measures the relative strength of the edge membership with respect to its incident vertices, whereas $\mathcal{S}_{v}(uv)_j$ measures the corresponding relative non-membership strength. Hence, a strong Pythagorean fuzzy edge simultaneously possesses sufficiently high membership strength and sufficiently low non-membership strength.

Definition 6.13[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph. Suppose that two Pythagorean fuzzy edges uv and rs intersect at a point c , where the corresponding edge membership and non-membership values are $\mu_{\mathcal{N}}(uv)_j, v_{\mathcal{N}}(uv)_j$ and $\mu_{\mathcal{N}}(rs)_k, v_{\mathcal{N}}(rs)_k$

respectively, for fixed $j, k \in \{1, 2, \dots, p\}$. The intersection value at the point c , denoted by $J_{\mathfrak{G}}(c) = (J_{\mu}(c), J_{\nu}(c))$ is defined as

$$J_{\mathfrak{G}}(c) = \left(\frac{\mu_{\mathcal{N}}(uv)_j + \mu_{\mathcal{N}}(rs)_k}{2}, \frac{\nu_{\mathcal{N}}(uv)_j + \nu_{\mathcal{N}}(rs)_k}{2} \right)$$

Here, $J_{\mu}(c)$ and $J_{\nu}(c)$ represent the membership and non-membership components of the intersection point, respectively. The intersection value provides a measure of the uncertainty associated with the crossing of two Pythagorean fuzzy edges. In particular, an increase in the number of edge-intersection points generally indicates a more complicated geometric representation and hence a reduction in the planarity of the corresponding Pythagorean fuzzy graph.

Definition 6.14[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy multigraph and suppose that $c_1, c_2, \dots, c_q$ are the intersection points produced by the geometric representation of its edges. Let $J_{\mathfrak{G}}(c_i) = (J_{\mu}(c_i), J_{\nu}(c_i))$, $i = 1, 2, \dots, q$. The Pythagorean fuzzy planarity value of $\mathfrak{G}$ denoted by $\mathcal{P}(\mathfrak{G}) = (\mathcal{P}_{\mu}(\mathfrak{G}), \mathcal{P}_{\nu}(\mathfrak{G}))$ is defined as

$$\mathcal{P}(\mathfrak{G}) = \left(\frac{1}{1 + \sum_{i=1}^q J_{\mu}(c_i)}, \frac{1}{1 + \sum_{i=1}^q J_{\nu}(c_i)} \right)$$

Thus, $0 < \mathcal{P}_{\mu}(\mathfrak{G}) \leq 1, 0 < \mathcal{P}_{\nu}(\mathfrak{G}) \leq 1$. If the geometric representation of $\mathfrak{G}$ contains no intersection points, then $q = 0$, and consequently, $\mathcal{P}(\mathfrak{G}) = (1, 1)$. Hence, the value $(1, 1)$ represents the maximum Pythagorean fuzzy planarity value, corresponding to a crossing-free geometric representation of the underlying graph. As the number and accumulated uncertainty of intersection points increase, the corresponding planarity values decrease.

Definition 6.15[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy planar graph with Pythagorean fuzzy planarity value $\mathcal{P}(\mathfrak{G}) = (\mathcal{P}_{\mu}(\mathfrak{G}), \mathcal{P}_{\nu}(\mathfrak{G}))$. The graph $\mathfrak{G}$ is called a strong Pythagorean fuzzy planar graph if its planarity value satisfies $\mathcal{P}_{\mu}(\mathfrak{G}) > 0.5$ and $\mathcal{P}_{\nu}(\mathfrak{G}) < 0.86$. Thus, a Pythagorean fuzzy planar graph is considered strong when its membership component of planarity exceeds 0.5, while its non-membership component remains below 0.86. These conditions indicate that the geometric representation of the graph possesses a relatively high degree of Pythagorean fuzzy planarity with comparatively low non-planarity.

Definition 6.16[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy graph and let $uv \in \mathcal{E}$ be an edge with membership degree $\mu_{\mathcal{N}}(uv)$ and non-membership degree $\nu_{\mathcal{N}}(uv)$. Let $\mathfrak{C}$ be a rational number satisfying $0 < \mathfrak{C} < 0.5$. The edge uv is called a considerable Pythagorean fuzzy edge if

$$\frac{\mu_{\mathcal{N}}(uv)}{\min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}} \geq \mathfrak{C} \text{ and } \frac{\nu_{\mathcal{N}}(uv)}{\max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}} \leq \mathfrak{C}$$

If an edge does not satisfy both of the above conditions, it is called a nonconsiderable Pythagorean fuzzy edge.

For a Pythagorean fuzzy multigraph, a multiedge uv is called considerable if its strength components satisfy $\mathcal{S}_\mu(uv)_j \geq \mathfrak{C}$ and $\mathcal{S}_\nu(uv)_j \leq \mathfrak{C}$, for every $j = 1, 2, \dots, p$, where $\mathcal{S}_\mu(uv)_j$ and $\mathcal{S}_\nu(uv)_j$ denote the membership and non-membership strength components of the j -th multiedge, respectively. This definition provides a threshold-based criterion for distinguishing edges that contribute significantly to the Pythagorean fuzzy structure from those having comparatively weaker influence.

Definition 6.17[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy planar graph having planarity value $(1,1)$. Let $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{E}$ be the collection of edges forming a bounded region in a planar representation of $\mathfrak{G}$. The region enclosed by the edges in $\mathcal{E}_{\mathcal{F}}$ is called a Pythagorean fuzzy face of $\mathfrak{G}$.

The membership value of the Pythagorean fuzzy face $\mathcal{F}$, denoted by $\mu_{\mathcal{F}}(\mathcal{F})$ is defined as

$$\mu_{\mathcal{F}}(\mathcal{F}) = \min \left\{ \frac{\mu_{\mathcal{N}}(uv)_j}{\min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}} : uv \in \mathcal{E}_{\mathcal{F}}, j = 1, 2, \dots, p \right\},$$

and its non-membership value, denoted by $\nu_{\mathcal{F}}(\mathcal{F})$ is defined as

$$\nu_{\mathcal{F}}(\mathcal{F}) = \max \left\{ \frac{\nu_{\mathcal{N}}(uv)_j}{\max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}} : uv \in \mathcal{E}_{\mathcal{F}}, j = 1, 2, \dots, p \right\}.$$

Hence, the Pythagorean fuzzy face can be represented by $\mathcal{F} = (\mu_{\mathcal{F}}(\mathcal{F}), \nu_{\mathcal{F}}(\mathcal{F}))$. The membership value reflects the minimum relative membership strength among the edges bounding the face, whereas the non-membership value represents the maximum relative non-membership strength of those edges.

Definition 6.18[9] Let $\mathcal{F}$ be a Pythagorean fuzzy face of $\mathfrak{G}$, with membership value $\mu_{\mathcal{F}}(\mathcal{F})$ and non-membership value $\nu_{\mathcal{F}}(\mathcal{F})$. The face $\mathcal{F}$ is called a strong Pythagorean fuzzy face if $\mu_{\mathcal{F}}(\mathcal{F}) \geq 0.5$ and $\nu_{\mathcal{F}}(\mathcal{F}) \leq 0.5$. Otherwise, $\mathcal{F}$ is called a weak Pythagorean fuzzy face.

Furthermore, in the planar representation of a Pythagorean fuzzy planar graph, the unbounded region is called the outer Pythagorean fuzzy face, while every bounded region is called an inner Pythagorean fuzzy face.

Definition 6.19[9] Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy planar graph with a planar representation having planarity value $(1,1)$. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{\mathcal{K}}$ be the strong Pythagorean fuzzy faces of $\mathfrak{G}$. The Pythagorean fuzzy dual graph of $\mathfrak{G}$ denoted by $\mathfrak{G}^* = (\mathcal{V}^*, \mathcal{E}^*, \mathcal{M}^*, \mathcal{N}^*)$ is constructed by associating a dual vertex $v_i^* \in \mathcal{V}^*$ with each strong Pythagorean fuzzy face $\mathcal{F}_i$, for $i = 1, 2, \dots, \mathcal{K}$. The membership and non-membership values of the dual vertex v_i^* are determined from the edges surrounding the corresponding face $\mathcal{F}_i$. Specifically,

$$\mu_{\mathcal{M}^*}(v_i^*) = \max\{\mu_{\mathcal{N}}(uv)_j : uv \text{ is incident with } \mathcal{F}_i, j = 1, 2, \dots, p\}$$

and

$$v_{\mathcal{M}^*}(\mathfrak{v}_i^*) = \min\{v_{\mathcal{N}}(uv) : uv \text{ is incident with } \mathcal{F}_i, j = 1, 2, \dots, p\}$$

Two dual vertices $\mathfrak{v}_i^*$ and $\mathfrak{v}_j^*$ are joined by a dual edge whenever the corresponding faces $\mathcal{F}_i$ and $\mathcal{F}_j$ have a common edge in $\mathfrak{G}$. If two faces share more than one edge, the corresponding dual vertices may be connected by multiple dual edges. Suppose $\mathcal{F}_i$ and $\mathcal{F}_j$ have q common edges, $e_1, e_2, \dots, e_q$. Then the membership and non-membership values of the corresponding dual edges are inherited from the common edges. For the s -th common edge, the dual edge $\mathfrak{v}_i^* \mathfrak{v}_j^*$ has $\mu_{\mathcal{N}^*}^s(\mathfrak{v}_i^* \mathfrak{v}_j^*) = \mu_{\mathcal{N}}(e_s)$ and $v_{\mathcal{N}^*}^s(\mathfrak{v}_i^* \mathfrak{v}_j^*) = v_{\mathcal{N}}(e_s), s = 1, 2, \dots, q$.

Thus, the Pythagorean fuzzy dual graph preserves the adjacency relationships between the strong Pythagorean fuzzy faces of $\mathfrak{G}$, while transferring the corresponding fuzzy information from the original graph to its dual structure. Under an appropriate planar representation, the dual graph admits a crossing-free representation and consequently has Pythagorean fuzzy planarity value (1,1).

7 Pythagorean fuzzy Outerplanar Graphs

In this section, introduced the concept of Pythagorean fuzzy outerplanar graphs, which extend the classical notion of outerplanarity to graphs with Pythagorean fuzzy values. The focus is on defining the graph structure and understanding its properties in the context of Pythagorean fuzziness.

Definition 7.1 A Pythagorean fuzzy outerplanar graph (PFOG) is a Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ that admits a planar embedding satisfied the following conditions:

1. All vertices of $\mathfrak{G}$ lie on the boundary of the outer face.
2. Each vertex $\mathfrak{v} \in \mathcal{V}$, the Pythagorean fuzzy information is given by $(\mu_{\mathcal{M}}(\mathfrak{v}), v_{\mathcal{M}}(\mathfrak{v}))$ and it satisfies

$$0 \leq \mu_{\mathcal{M}}^2(\mathfrak{v}) + v_{\mathcal{M}}^2(\mathfrak{v}) \leq 1$$

3. Each edge $e = uv \in \mathcal{E}$ the Pythagorean fuzzy information is given by

$$0 \leq \mu_{\mathcal{N}}^2(uv) + v_{\mathcal{N}}^2(uv) \leq 1$$

4. where $\mu_{\mathcal{N}}(uv) \leq \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}$ and $v_{\mathcal{N}}(uv) \geq \max\{v_{\mathcal{M}}(u), v_{\mathcal{M}}(v)\}$.
5. No two edges cross in the planar embedding.

Let $i(\mathfrak{G})$ represent the number of vertices in a planar embedding that do not depend on the boundary of the outer region. Therefore, $i(\mathfrak{G}) = 0$ for a Pythagorean fuzzy outerplanar graph.

The graphs given in Fig. 1 and Fig. 2 are examples of Pythagorean fuzzy outerplanar graphs.

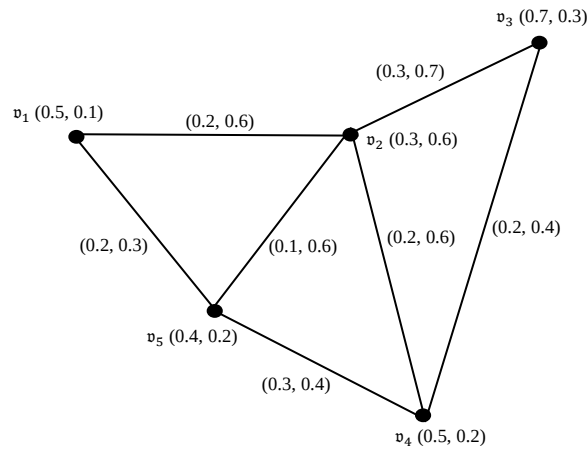


Figure 1. Pythagorean fuzzy outerplanar graph with cycle

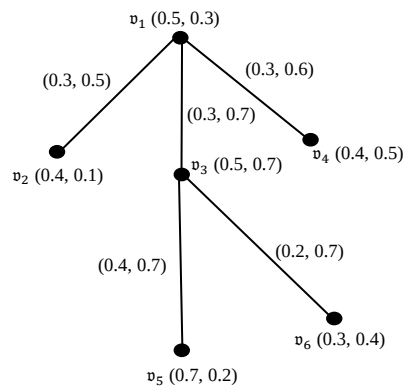


Figure 2. Pythagorean fuzzy outerplanar graph without cycle

Definition 7.2 A Pythagorean fuzzy non-outerplanar graph (PFNOG) is a Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ that admits a planar embedding satisfied the following conditions:

1. All vertices of $\mathfrak{G}$ does not lie on the boundary of the outer face.
2. Each vertex $v \in \mathcal{V}$, the Pythagorean fuzzy information is given by $(\mu_{\mathcal{M}}(v), \nu_{\mathcal{M}}(v))$ and it satisfies

$$0 \leq \mu_{\mathcal{M}}^2(v) + \nu_{\mathcal{M}}^2(v) \leq 1$$

3. Each edge $e = uv \in \mathcal{E}$ the Pythagorean fuzzy information is given by

$$0 \leq \mu_{\mathcal{N}}^2(uv) + \nu_{\mathcal{N}}^2(uv) \leq 1$$

4. where $\mu_{\mathcal{N}}(uv) \leq \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}$ and $\nu_{\mathcal{N}}(uv) \geq \max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}$.
5. The graph admits a planar embedding in which at least one vertex lies in the interior of the embedding, that is not all vertices lie on the boundary of the outer face.

Let $i(\mathfrak{G})$ denote the number of vertices in a planar embedding of $\mathfrak{G}$ that do not lie on the boundary of the outer face. Then $i(\mathfrak{G}) \neq 0$ for a Pythagorean fuzzy non- outerplanar graph.

The graphs given in Fig. 3 and Fig. 4 are examples of Pythagorean fuzzy non outerplanar graphs.

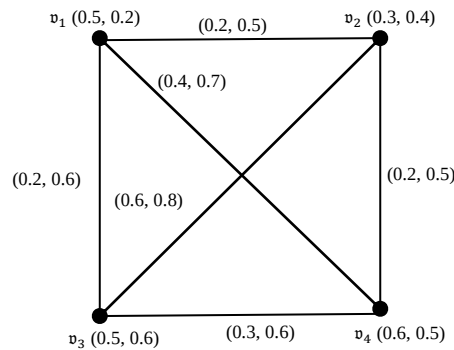


Figure 3. Pythagorean fuzzy graph $\mathcal{K}_4$

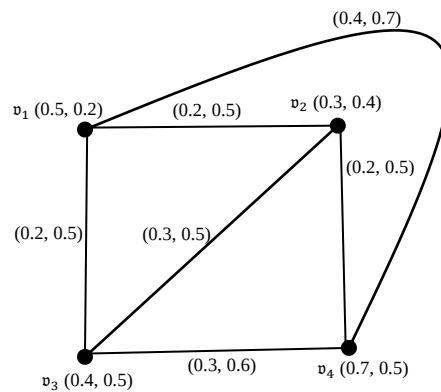


Figure 4. Pythagorean fuzzy planar embedding of $\mathcal{K}_4$

7.1 Characterization of Pythagorean fuzzy Outerplanar Graphs

In this section, we explore the defining theorems and example criteria that characterize Pythagorean fuzzy outerplanar graphs.

Definition 7.1.1 Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy graph where the vertex Pythagorean fuzzy function is $\mathcal{M}: \mathcal{V} \rightarrow [0, 1] \times [0, 1]$, $\mathcal{M}(v) = (\mu_{\mathcal{M}}(v), \nu_{\mathcal{M}}(v))$ with $0 \leq \mu_{\mathcal{M}}^2(v) + \nu_{\mathcal{M}}^2(v) \leq 1, \forall v \in \mathcal{V}$; the edge Pythagorean fuzzy function is $\mathcal{N}: \mathcal{E} \rightarrow [0, 1] \times [0, 1]$, $\mathcal{N}(uv) = (\mu_{\mathcal{N}}(uv), \nu_{\mathcal{N}}(uv))$ with $0 \leq \mu_{\mathcal{N}}^2(uv) + \nu_{\mathcal{N}}^2(uv) \leq 1, \forall uv \in \mathcal{E}$ and admissibility: $\mu_{\mathcal{N}}(uv) \leq \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}$ and $\nu_{\mathcal{N}}(uv) \geq \max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}$. Let $uv \in \mathcal{E}$. A Pythagorean fuzzy subdivision of the edge uv is a Pythagorean fuzzy graph $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{M}', \mathcal{N}')$ obtained by deleting the edge uv and introducing a new vertex $w \notin \mathcal{V}$ and the two edges uw and wv . $\mathcal{V}' = \mathcal{V} \cup \{w\}$, $\mathcal{E}' = (\mathcal{E} \setminus \{uv\}) \cup \{uw, wv\}$.

Define $\mathcal{M}'(w) = (\mu_{\mathcal{M}'}(w), \nu_{\mathcal{M}'}(w))$ by arithmetic mean component wise:

$$\mu_{\mathcal{M}'}(\mathfrak{w}) = \frac{\mu_{\mathcal{M}}(\mathfrak{u}) + \mu_{\mathcal{M}}(\mathfrak{v})}{2}, v_{\mathcal{M}'}(\mathfrak{w}) = \frac{v_{\mathcal{M}}(\mathfrak{u}) + v_{\mathcal{M}}(\mathfrak{v})}{2}$$

$$0 \leq \mu_{\mathcal{M}'}^2(\mathfrak{w}) + v_{\mathcal{M}'}^2(\mathfrak{w}) \leq 1$$

For all original vertices $x \in \mathcal{V}$, $\mathcal{M}'(x) = \mathcal{M}(x)$.

For edge $\mathfrak{uw}$, choose $\mathcal{N}'(\mathfrak{uw}) = (\mu_{\mathcal{N}'}(\mathfrak{uw}), v_{\mathcal{N}'}(\mathfrak{uw}))$ such that

$$\mu_{\mathcal{N}'}(\mathfrak{uw}) \leq \min\{\mu_{\mathcal{M}'}(\mathfrak{u}), \mu_{\mathcal{M}'}(\mathfrak{w})\}$$

$$v_{\mathcal{N}'}(\mathfrak{uw}) \geq \max\{v_{\mathcal{M}'}(\mathfrak{u}), v_{\mathcal{M}'}(\mathfrak{w})\}$$

and

$$0 \leq \mu_{\mathcal{N}'}^2(\mathfrak{uw}) + v_{\mathcal{N}'}^2(\mathfrak{uw}) \leq 1$$

For edge $\mathfrak{wv}$, choose $\mathcal{N}'(\mathfrak{wv}) = (\mu_{\mathcal{N}'}(\mathfrak{wv}), v_{\mathcal{N}'}(\mathfrak{wv}))$ such that

$$\mu_{\mathcal{N}'}(\mathfrak{wv}) \leq \min\{\mu_{\mathcal{M}'}(\mathfrak{w}), \mu_{\mathcal{M}'}(\mathfrak{v})\}$$

$$v_{\mathcal{N}'}(\mathfrak{wv}) \geq \max\{v_{\mathcal{M}'}(\mathfrak{w}), v_{\mathcal{M}'}(\mathfrak{v})\}$$

and

$$0 \leq \mu_{\mathcal{N}'}^2(\mathfrak{wv}) + v_{\mathcal{N}'}^2(\mathfrak{wv}) \leq 1$$

For all original edges $xy \in \mathcal{E}\{\mathfrak{uv}\}$, $\mathcal{N}'(xy) = \mathcal{N}(xy)$.

Theorem 7.1.1 A Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ is Pythagorean fuzzy outer planar if and only if it contains no Pythagorean fuzzy subgraph homomorphic from $\mathcal{K}_4$ or $\mathcal{K}_{2,3}$.

Proof. It is known that $\mathcal{K}_4$ or $\mathcal{K}_{2,3}$ are not Pythagorean fuzzy outerplanar. Thus, no Pythagorean fuzzy graph having a Pythagorean fuzzy subgraph homomorphic from $\mathcal{K}_4$ or $\mathcal{K}_{2,3}$ is Pythagorean fuzzy outerplanar graphs shown in the Figs. 5 and 6.

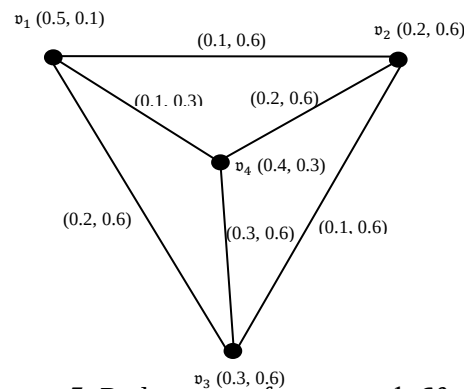


Figure.5. Pythagorean fuzzy graph $\mathcal{K}_4$

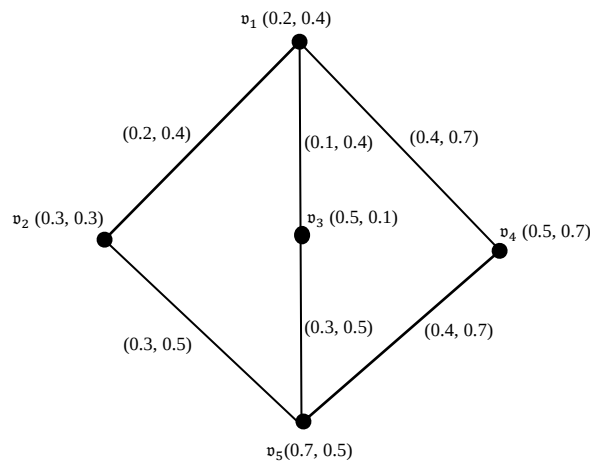


Figure.6. Pythagorean fuzzy graph $\mathcal{K}_{2,3}$

Conversely, suppose $\mathfrak{G}$ is a Pythagorean fuzzy graph containing no Pythagorean fuzzy subgraph homomorphic from $\mathcal{K}_4$ or $\mathcal{K}_{2,3}$. By Kuratowski's theorem, $\mathfrak{G}$ is planar. Assume $\mathfrak{G}$ is not Pythagorean fuzzy outerplanar graph. Thus, without loss of generality, we assume that $\mathfrak{G}$ is a cyclic block which is not Pythagorean fuzzy outerplanar graph. Further we assume that $\mathfrak{G}$ is so embedded in the Pythagorean fuzzy plane that the exterior region contains maximum number of vertices.

The exterior region is bounded by a cycle $\mathcal{C}$. Since not all vertices of $\mathfrak{G}$, lie on $\mathcal{C}$, there are one or more vertices lying interior to $\mathcal{C}$.

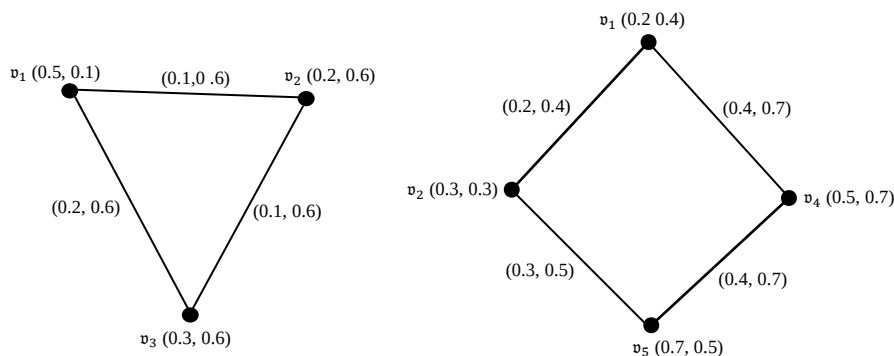


Figure.7. Pythagorean fuzzy outerplanar graphs

If there exist a vertex v_3 interior on $\mathcal{C}$ and three mutually disjoint paths between v_3 and three distinct vertices of $\mathcal{C}$, then $\mathfrak{G}$ contains Pythagorean fuzzy subgraph homomorphic from $\mathcal{K}_4$.

Otherwise, since $\mathfrak{G}$ is a cyclic block, there must exist a vertex v_3 and two disjoint paths between v_3 and two disjoint vertices v_1 and v_5 on $\mathcal{C}$.

Moreover, from the above choice of $\mathcal{C}$, the edge v_1v_5 does not belong to $\mathcal{C}$. This implies $\mathfrak{G}$ contains a Pythagorean fuzzy subgraph homomorphic from $\mathcal{K}_{2,3}$. This is a contradiction. Therefore, Fig. 7 $\mathfrak{G}$ is a Pythagorean fuzzy outerplanar graph. This completes the proof.

Theorem 7.1.2 If $\mathfrak{G}$ is a connected Pythagorean fuzzy outerplanar graph, then it has a Pythagorean fuzzy dual $\mathfrak{G}^*$ there exist a vertex v such that $\mathfrak{G}^* - v$ contains no Pythagorean fuzzy cycle.

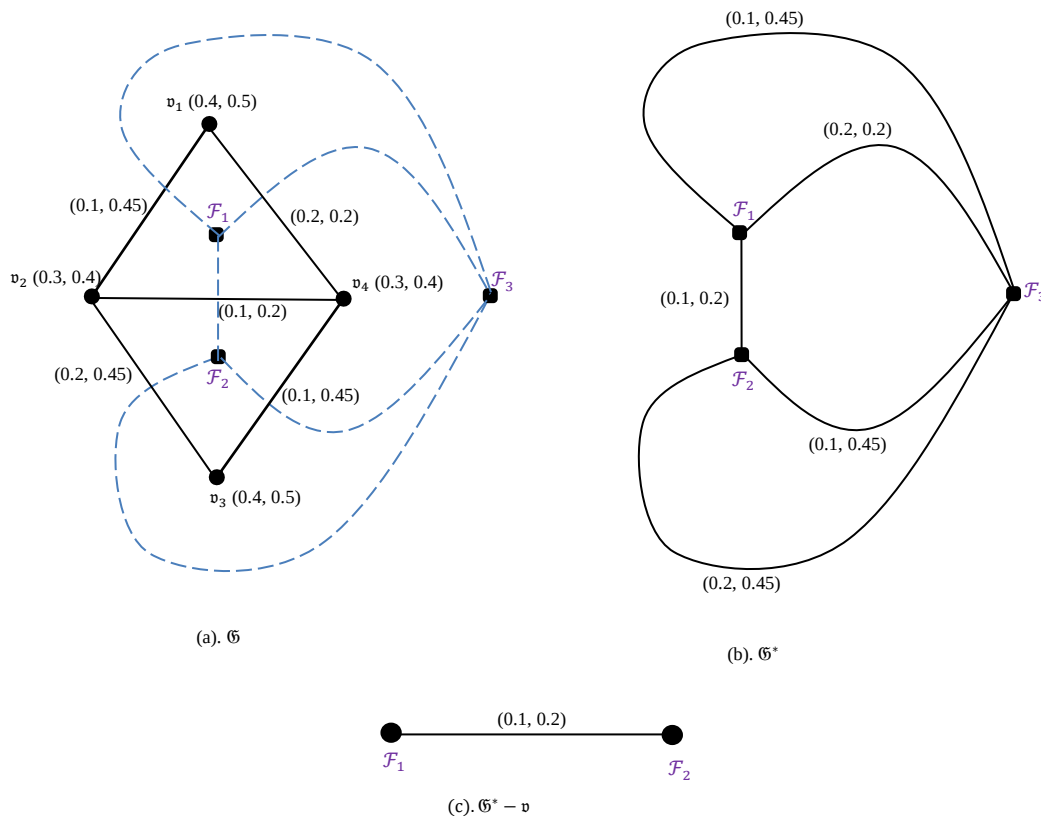


Figure 8. (a)-(c) Example for Pythagorean fuzzy outerplanar graph $\mathfrak{G}$ has Pythagorean fuzzy dual graph $\mathfrak{G}^*$ then there exist v such that $\mathfrak{G}^* - v$ contains no Pythagorean fuzzy cycle.

Proof. Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a connected Pythagorean fuzzy outerplanar graph. Therefore, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{E}$ be the collection of edges forming a bounded region in a planar representation of $\mathcal{G}$. The region enclosed by the edges in $\mathcal{E}_{\mathcal{F}}$ is called a Pythagorean fuzzy face of $\mathcal{G}$.

The membership value of the Pythagorean fuzzy face $\mathcal{F}$, denoted by $\mu_{\mathcal{F}}(\mathcal{F})$ is defined as

$$\mu_{\mathcal{F}}(\mathcal{F}) = \min \left\{ \frac{\mu_{\mathcal{N}}(uv)_j}{\min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}} : uv \in \mathcal{E}_{\mathcal{F}}, j = 1, 2, \dots, p \right\},$$

and its non-membership value, denoted by $\nu_{\mathcal{F}}(\mathcal{F})$ is defined as

$$\nu_{\mathcal{F}}(\mathcal{F}) = \max \left\{ \frac{\nu_{\mathcal{N}}(uv)_j}{\max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}} : uv \in \mathcal{E}_{\mathcal{F}}, j = 1, 2, \dots, p \right\}.$$

Hence, the Pythagorean fuzzy face can be represented by $\mathcal{F} = (\mu_{\mathcal{F}}(\mathcal{F}), \nu_{\mathcal{F}}(\mathcal{F}))$.

When the Pythagorean fuzzy graph divided into regions with Pythagorean fuzzy face and values, it becomes very easy to Pythagorean fuzzy dual graph $\mathfrak{G}^*$ such that each face of $\mathfrak{G}$ corresponds to a vertex and each edge of $\mathfrak{G}$ corresponds to edge in $\mathfrak{G}^* = (\mathcal{V}', \mathcal{A}', \mathcal{B}')$.

Consequently, the Pythagorean fuzzy dual graph $\mathfrak{G}^*$ exists for the Pythagorean fuzzy outerplanar graph.

Since $\mathfrak{G}$ is Pythagorean fuzzy outerplanar graph. Its outer face is a Pythagorean fuzzy cycle. Therefore, the graph $\mathfrak{G}^* - v$ that results from deleting the vertex v from $\mathfrak{G}^*$, the resulting graph $\mathfrak{G}^* - v$ contains no Pythagorean fuzzy cycle.

Example 7.1.1 Consider the Pythagorean fuzzy outerplanar graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ illustrated in the Figure. 8(a). If the dotted lines in figure. 8(b) are superimposed Pythagorean fuzzy dual graph of $\mathfrak{G}^*$ and separated diagram. The membership value is given in the Table 2. Then the Pythagorean fuzzy faces $(\mu_{\mathcal{F}}(\mathcal{F}_1), \nu_{\mathcal{F}}(\mathcal{F}_1)) = (0.3333, 0.9)$, $(\mu_{\mathcal{F}}(\mathcal{F}_2), \nu_{\mathcal{F}}(\mathcal{F}_2)) = (0.3333, 0.9)$, $(\mu_{\mathcal{F}}(\mathcal{F}_3), \nu_{\mathcal{F}}(\mathcal{F}_3)) = (0.3333, 0.9)$. Each Pythagorean fuzzy face of $\mathfrak{G}$ corresponds to a vertex and each edge of $\mathfrak{G}$ corresponds to edge in $\mathfrak{G}^* = (\mathcal{V}', \mathcal{E}', \mathcal{M}', \mathcal{N}')$. Then v exists in such a way that $\mathfrak{G}^* - v$ contains no Pythagorean fuzzy cycle.

Table 2. Calculation of Pythagorean fuzzy faces

S.No	Pythagorean fuzzy Faces	Membership $(\mu_{\mathcal{F}}(\mathcal{F}_i))$	Non-membership $(\nu_{\mathcal{F}}(\mathcal{F}_i))$
1.	$(\mu_{\mathcal{F}}(\mathcal{F}_1), \nu_{\mathcal{F}}(\mathcal{F}_1))$	0.3333	0.9
2.	$(\mu_{\mathcal{F}}(\mathcal{F}_2), \nu_{\mathcal{F}}(\mathcal{F}_2))$	0.3333	0.9
3.	$(\mu_{\mathcal{F}}(\mathcal{F}_3), \nu_{\mathcal{F}}(\mathcal{F}_3))$	0.3333	0.9

Definition 7.3 A Pythagorean fuzzy outerplanar graph $\mathfrak{G}$ is maximal Pythagorean fuzzy outerplanar if it is outerplanar and the addition of any edge (satisfying Pythagorean fuzzy membership constraints) destroys its outerplanarity.

The Pythagorean fuzzy graph given in Fig.9 are examples of Pythagorean fuzzy maximal outerplanar graphs.

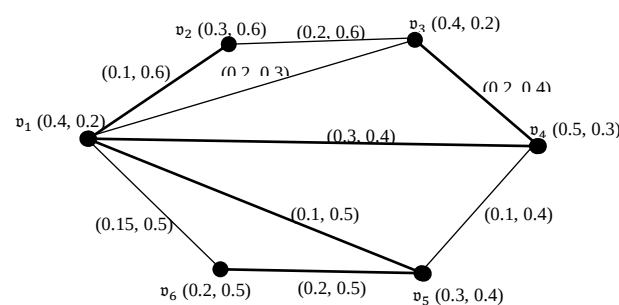


Figure 9. Pythagorean fuzzy maximal outerplanar graph

Definition 7.4 A Pythagorean fuzzy planar graph $\mathfrak{G}$ is said to be minimally Pythagorean fuzzy non-outerplanar if $i(\mathfrak{G}) \neq 0$ and there exists at most one vertex $v \in \mathcal{V}(\mathfrak{G})$ lying in the interior of the planar embedding such that $\mathcal{V}(v) > 0$.

The graphs given in Fig.11 and Fig.12 are examples of minimally Pythagorean fuzzy non-outerplanar graphs.

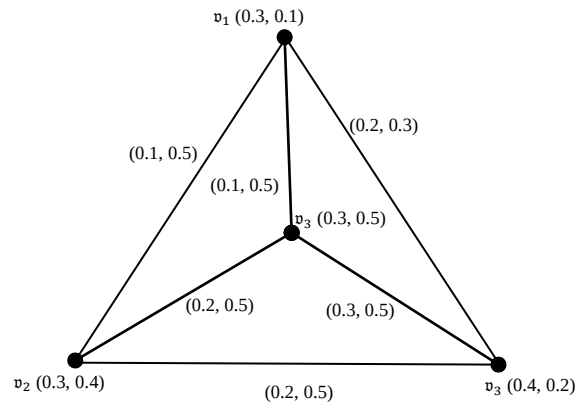


Figure 11. Pythagorean fuzzy graph $\mathcal{K}_4$

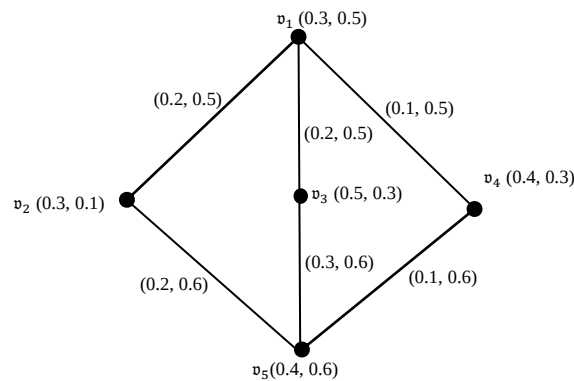


Figure 12. Pythagorean fuzzy graph $\mathcal{K}_{2,3}$

8 Vertex Deletion Pythagorean fuzzy Outerplanar Subgraphs

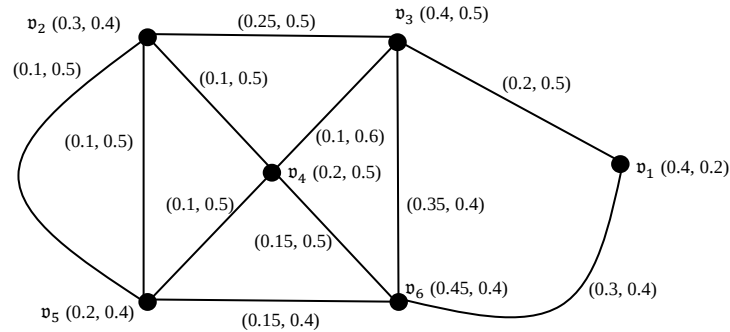
In this section, the subgraph formed by deleting specific vertices from the Pythagorean fuzzy, while also defining Pythagorean fuzzy outerplanar graphs and offering relevant illustrations.

Definition 8.1 Let $\mathfrak{G}$ be a Pythagorean fuzzy planar graph. If $\mathfrak{G}'$ is a Pythagorean fuzzy subgraph obtained from $\mathfrak{G}$ by deleting a vertex (and all edges incident to it) such that $\mathfrak{G}'$ admits an outerplanar embedding, then $\mathfrak{G}'$ is called a vertex-deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$

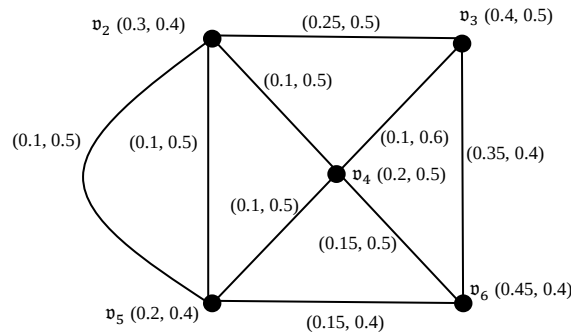
Example 8.1 Let us consider the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ shown in figure 13(a). The set of Pythagorean fuzzy vertices in $\mathfrak{G}$ is $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and their

membership values of these are as follows: $\mathcal{V}$ is $\mu_{\mathcal{M}}(v_1) = 0.4, \nu_{\mathcal{N}}(v_1) = 0.2, \mu_{\mathcal{M}}(v_2) = 0.3, \nu_{\mathcal{N}}(v_2) = 0.4, \mu_{\mathcal{M}}(v_3) = 0.4, \nu_{\mathcal{N}}(v_3) = 0.5, \mu_{\mathcal{M}}(v_4) = 0.2, \nu_{\mathcal{N}}(v_4) = 0.5, \mu_{\mathcal{M}}(v_5) = 0.2, \nu_{\mathcal{N}}(v_5) = 0.4, \mu_{\mathcal{M}}(v_6) = 0.45, \nu_{\mathcal{N}}(v_6) = 0.4$ and the Pythagorean fuzzy edges $\mathcal{E}$ is $\mu_{\mathcal{M}}(v_1, v_6) = 0.3, \nu_{\mathcal{N}}(v_1, v_6) = 0.4, \mu_{\mathcal{M}}(v_1, v_3) = 0.2, \nu_{\mathcal{N}}(v_1, v_3) = 0.5, \mu_{\mathcal{M}}(v_2, v_3) = 0.25, \nu_{\mathcal{N}}(v_2, v_3) = 0.5, \mu_{\mathcal{M}}(v_2, v_5) = 0.1, \nu_{\mathcal{N}}(v_2, v_5) = 0.5, \mu_{\mathcal{M}}(v_5, v_2) = 0.1,$

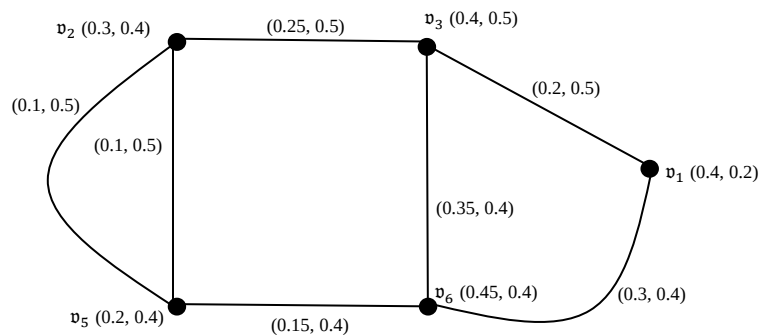
$\nu_{\mathcal{N}}(v_5, v_2) = 0.5, \mu_{\mathcal{M}}(v_3, v_6) = 0.35, \nu_{\mathcal{N}}(v_3, v_6) = 0.4, \mu_{\mathcal{M}}(v_5, v_6) = 0.15, \nu_{\mathcal{N}}(v_5, v_6) = 0.4, \mu_{\mathcal{M}}(v_2, v_4) = 0.1, \nu_{\mathcal{N}}(v_2, v_4) = 0.5, \mu_{\mathcal{M}}(v_3, v_4) = 0.1, \nu_{\mathcal{N}}(v_3, v_4) = 0.6, \mu_{\mathcal{M}}(v_4, v_5) = 0.1, \nu_{\mathcal{N}}(v_4, v_5) = 0.5, \mu_{\mathcal{M}}(v_4, v_6) = 0.15, \nu_{\mathcal{N}}(v_4, v_6) = 0.5.$



(a). $\mathcal{G}$



(b). $\mathcal{G}_1$



(c). $\mathcal{G}_2$

Figure 13: (a). Pythagorean fuzzy graph (b). Vertex deleted Pythagorean fuzzy subgraph & (c). Vertex deletion Pythagorean fuzzy outerplanar subgraph

The edges $\mu_{\mathcal{M}}(v_2, v_6), \nu_{\mathcal{N}}(v_2, v_6)$ and $\mu_{\mathcal{M}}(v_4, v_5), \nu_{\mathcal{N}}(v_4, v_5)$ intersect in the Pythagorean fuzzy graph $\mathcal{G}$. Let us define two subsets of the vertex set $\mathcal{V}$ as

$$\mathcal{X} = \{v_1(0.4, 0.2)\} \text{ and } \mathcal{Y} = \{v_4(0.2, 0.5)\}$$

The Pythagorean fuzzy subgraph $\mathfrak{G} - \mathcal{X}$, denoted by $\mathfrak{G}_1$, is illustrated in Fig. 13(b). Similarly, the Pythagorean fuzzy subgraph $\mathfrak{G} - \mathcal{Y}$, denoted by $\mathfrak{G}_2$, is shown in Fig. 13(c). It is observed that the Pythagorean fuzzy subgraph $\mathfrak{G}_1$ is a vertex-deleted Pythagorean fuzzy subgraph, whereas $\mathfrak{G}_2$ is a vertex-deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Note 8.1 It is not necessary for the vertex deletion Pythagorean fuzzy subgraph of $\mathfrak{G}$ to be the vertex deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$. The Pythagorean fuzzy graphs in Figs 13(b) and 13(c) allow for the observation of this.

Theorem 8.1 A Pythagorean fuzzy outerplanar graph $\mathfrak{G}$ always has a vertex deleted Pythagorean fuzzy outerplanar subgraph that is also a vertex deleted Pythagorean fuzzy subgraph of $\mathfrak{G}$.

Proof. Let the Pythagorean fuzzy outerplanar graph be $\mathfrak{G}$ and $\mathcal{H}$ be the vertex deleted Pythagorean fuzzy subgraph of $\mathfrak{G}$. The vertices of the Pythagorean fuzzy graph $\mathfrak{G}$ will all be in the outer region since it is outerplanar. As a result, the Pythagorean fuzzy subgraph that was created by deleting vertices still has outerplanarity. Any vertex deleted Pythagorean fuzzy subgraph in $\mathfrak{G}$ is therefore a vertex deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Theorem 8.2 Let $\mathfrak{G}$ be a Pythagorean fuzzy outerplanar graph which is connected and let $\mathcal{W}$ be a subset of its vertices, such that $\mathcal{W} \subseteq \mathcal{V}$. The Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$ with a vertex deleted is $\mathfrak{G}' = \mathfrak{G} \setminus \mathcal{W}$. Its Pythagorean fuzzy dual graph is $\mathfrak{G}''$.

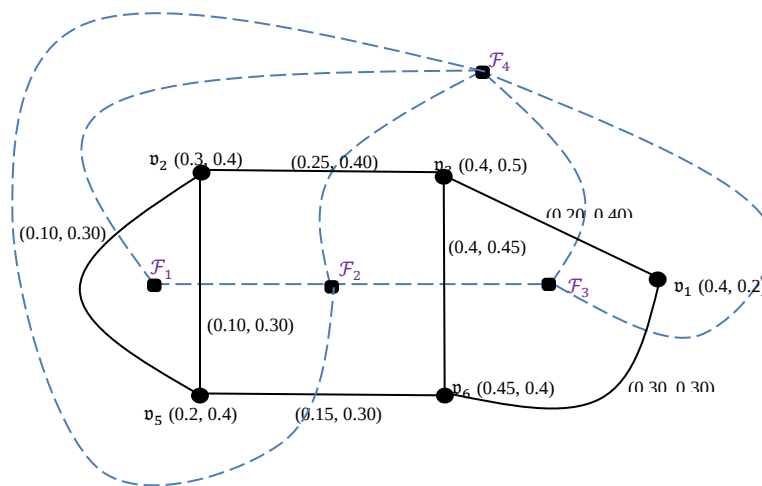


Figure 14(a). Pythagorean fuzzy outerplanar graph

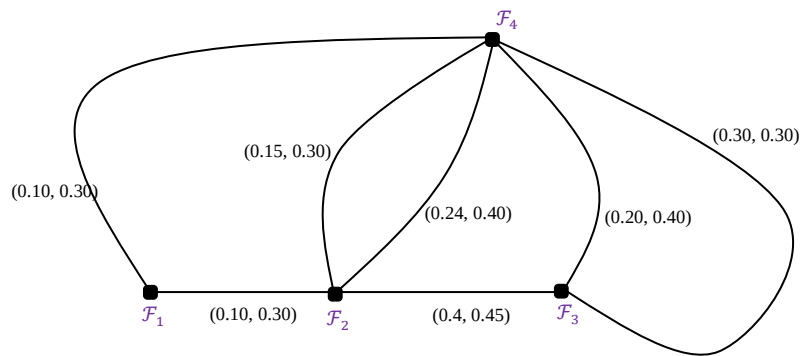


Figure 14(b). Pythagorean fuzzy dual graph

Figure 14(a), (b). Example for vertex deletion Pythagorean fuzzy outerplanar subgraph and its Pythagorean fuzzy dual graph.

Proof. Let us consider a connected Pythagorean fuzzy outerplanar subgraph $\mathfrak{G}$. Therefore, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{E}$ be the collection of edges forming a bounded region in a planar representation of $\mathcal{G}$. The region enclosed by the edges in $\mathcal{E}_{\mathcal{F}}$ is called a Pythagorean fuzzy face of $\mathcal{G}$.

The membership value of the Pythagorean fuzzy face $\mathcal{F}$, denoted by $\mu_{\mathcal{F}}(\mathcal{F})$ is defined as

$$\mu_{\mathcal{F}}(\mathcal{F}) = \min \left\{ \frac{\mu_{\mathcal{N}}(uv)_j}{\min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}} : uv \in \mathcal{E}_{\mathcal{F}}, j = 1, 2, \dots, p \right\},$$

and its non-membership value, denoted by $\nu_{\mathcal{F}}(\mathcal{F})$ is defined as

$$\nu_{\mathcal{F}}(\mathcal{F}) = \max \left\{ \frac{\nu_{\mathcal{N}}(uv)_j}{\max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}} : uv \in \mathcal{E}_{\mathcal{F}}, j = 1, 2, \dots, p \right\}.$$

Hence, the Pythagorean fuzzy face can be represented by $\mathcal{F} = (\mu_{\mathcal{F}}(\mathcal{F}), \nu_{\mathcal{F}}(\mathcal{F}))$.

Using the fact that each Pythagorean fuzzy graph with planarity value as 1 has Pythagorean fuzzy dual graph, $\mathfrak{G} - \mathcal{W}$ has a Pythagorean fuzzy dual graph $\mathfrak{G}''$. Consequently, the Pythagorean fuzzy dual graph exists for the vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Example 8.2 From Theorem 8.2, it is evident that the Pythagorean fuzzy subgraph $\mathfrak{G}''$ is the vertex deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$. The dotted lines represent the superimposed Pythagorean fuzzy dual graph of $\mathfrak{G}''$ in Figure 14(b). The Pythagorean fuzzy dual graph is associated with Pythagorean fuzzy face values, with all Pythagorean fuzzy faces having membership values is given in the Table 3. Then the Pythagorean fuzzy faces $(\mu_{\mathcal{F}}(\mathcal{F}_1), \nu_{\mathcal{F}}(\mathcal{F}_1)) = (0.50, 0.75)$, $(\mu_{\mathcal{F}}(\mathcal{F}_2), \nu_{\mathcal{F}}(\mathcal{F}_2)) = (0.50, 0.80)$, $(\mu_{\mathcal{F}}(\mathcal{F}_3), \nu_{\mathcal{F}}(\mathcal{F}_3)) = (0.50, 0.80)$ and $(\mu_{\mathcal{F}}(\mathcal{F}_4), \nu_{\mathcal{F}}(\mathcal{F}_4)) = (0.50, 0.80)$. Therefore, the vertex deletion Pythagorean fuzzy outerplanar subgraph $\mathfrak{G}''$ indeed possesses a Pythagorean fuzzy dual graph.

Table 3. Calculation of Pythagorean fuzzy faces

S.N o	Pythagorean fuzzy Faces	Membership ($\mu_{\mathcal{F}}(\mathcal{F}_i)$)	Non-membership ($\nu_{\mathcal{F}}(\mathcal{F}_i)$)
1.	$(\mu_{\mathcal{F}}(\mathcal{F}_1), \nu_{\mathcal{F}}(\mathcal{F}_1))$	0.50	0.75
2.	$(\mu_{\mathcal{F}}(\mathcal{F}_2), \nu_{\mathcal{F}}(\mathcal{F}_2))$	0.50	0.80
3.	$(\mu_{\mathcal{F}}(\mathcal{F}_3), \nu_{\mathcal{F}}(\mathcal{F}_3))$	0.50	0.80
4.	$(\mu_{\mathcal{F}}(\mathcal{F}_4), \nu_{\mathcal{F}}(\mathcal{F}_4))$	0.50	0.80

Definition 8.2 Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ be a Pythagorean fuzzy graph (PFG), where each vertex $x \in \mathcal{V}$ has Pythagorean fuzzy membership $0 \leq \mu_{\mathcal{M}}^2(x) + \nu_{\mathcal{M}}^2(x) \leq 1$ and each edge $uv \in \mathcal{E}$ has Pythagorean fuzzy membership $(\mu_{\mathcal{M}}(uv), \nu_{\mathcal{M}}(uv))$. Let $uv \in \mathcal{E}$ be an edge. A Pythagorean fuzzy edge contraction of $\mathfrak{G}$ with respect to the edge uv . Introduce a new vertex w (the subdivision vertex) and define $\mathcal{V}' = \mathcal{V} \cup \{w\} \setminus \emptyset$. Replace the edge uv by two edges uw and wv : $\mathcal{E}' = (\mathcal{E} \setminus \{uv\}) \cup \{uw, wv\}$. Assign to the new vertex w :

$$\mu_{\mathcal{N}}(w) = \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\} \text{ and } \nu_{\mathcal{N}}(w) = \max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}.$$

Set the Pythagorean fuzzy data on edges uw and wv by the same “endpoint-bounding” principle as in Pythagorean fuzzy graphs:

$$\mu_{\mathcal{N}}(uw) = \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(w)\} \text{ and } \nu_{\mathcal{N}}(uw) = \max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(w)\}.$$

$$\mu_{\mathcal{N}}(wv) = \min\{\mu_{\mathcal{M}}(w), \mu_{\mathcal{M}}(v)\} \text{ and } \nu_{\mathcal{N}}(wv) = \max\{\nu_{\mathcal{M}}(w), \nu_{\mathcal{M}}(v)\}.$$

So, the new edge values are determined consistently from the original edge uv and the Pythagorean fuzzy endpoints.

Example 8.3. Let us consider the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ shown in figure 15(a). The set of Pythagorean fuzzy vertices in $\mathfrak{G}$ is $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and their membership values of these are as follows: $\mathcal{V}$ is $\mu_{\mathcal{M}}(v_1) = 0.2, \nu_{\mathcal{N}}(v_1) = 0.4, \mu_{\mathcal{M}}(v_2) = 0.3, \nu_{\mathcal{N}}(v_2) = 0.2, \mu_{\mathcal{M}}(v_3) = 0.5, \nu_{\mathcal{N}}(v_3) = 0.5, \mu_{\mathcal{M}}(v_4) = 0.2, \nu_{\mathcal{N}}(v_4) = 0.4, \mu_{\mathcal{M}}(v_5) = 0.3, \nu_{\mathcal{N}}(v_5) = 0.6, \mu_{\mathcal{M}}(v_6) = 0.5, \nu_{\mathcal{N}}(v_6) = 0.2, \mu_{\mathcal{M}}(v_7) = 0.85, \nu_{\mathcal{N}}(v_7) = 0.5, \mu_{\mathcal{M}}(v_8) = 0.3, \nu_{\mathcal{N}}(v_8) = 0.4, \mu_{\mathcal{M}}(v_9) = 0.4, \nu_{\mathcal{N}}(v_9) = 0.5, \mu_{\mathcal{M}}(v_{10}) = 0.4, \nu_{\mathcal{N}}(v_{10}) = 0.5$ and the Pythagorean fuzzy edges $\mathcal{E}$ is $\mu_{\mathcal{M}}(v_1, v_2) = 0.1, \nu_{\mathcal{N}}(v_1, v_2) = 0.2, \mu_{\mathcal{M}}(v_2, v_3) = 0.2, \nu_{\mathcal{N}}(v_2, v_3) = 0.5, \mu_{\mathcal{M}}(v_3, v_4) = 0.15, \nu_{\mathcal{N}}(v_3, v_4) = 0.5, \mu_{\mathcal{M}}(v_4, v_5) = 0.15, \nu_{\mathcal{N}}(v_4, v_5) = 0.6, \mu_{\mathcal{M}}(v_5, v_6) = 0.25, \nu_{\mathcal{N}}(v_5, v_6) = 0.6, \mu_{\mathcal{M}}(v_6, v_7) = 0.25, \nu_{\mathcal{N}}(v_6, v_7) = 0.5, \mu_{\mathcal{M}}(v_7, v_8) = 0.1, \nu_{\mathcal{N}}(v_7, v_8) = 0.5, \mu_{\mathcal{M}}(v_8, v_1) = 0.2, \nu_{\mathcal{N}}(v_8, v_1) = 0.4, \mu_{\mathcal{M}}(v_1, v_9) = 0.1, \nu_{\mathcal{N}}(v_1, v_9) = 0.5, \mu_{\mathcal{M}}(v_2, v_9) = 0.1, \nu_{\mathcal{N}}(v_2, v_9) = 0.5, \mu_{\mathcal{M}}(v_2, v_{10}) = 0.2, \nu_{\mathcal{N}}(v_2, v_{10}) = 0.5, \mu_{\mathcal{M}}(v_3, v_{10}) = 0.1, \nu_{\mathcal{N}}(v_3, v_{10}) = 0.5, \mu_{\mathcal{M}}(v_8, v_9) = 0.2, \nu_{\mathcal{N}}(v_8, v_9) = 0.5, \mu_{\mathcal{M}}(v_9, v_{10}) = 0.1, \nu_{\mathcal{N}}(v_9, v_{10}) = 0.5, \mu_{\mathcal{M}}(v_{10}, v_4) = 0.2, \nu_{\mathcal{N}}(v_{10}, v_4) = 0.5, \mu_{\mathcal{M}}(v_7, v_9) = 0.3, \nu_{\mathcal{N}}(v_7, v_9) = 0.5, \mu_{\mathcal{M}}(v_6, v_9) = 0.1, \nu_{\mathcal{N}}(v_6, v_9) = 0.5, \mu_{\mathcal{M}}(v_6, v_{10}) = 0.3, \nu_{\mathcal{N}}(v_6, v_{10}) = 0.5, \mu_{\mathcal{M}}(v_5, v_{10}) = 0.2, \nu_{\mathcal{N}}(v_5, v_{10}) = 0.6.$

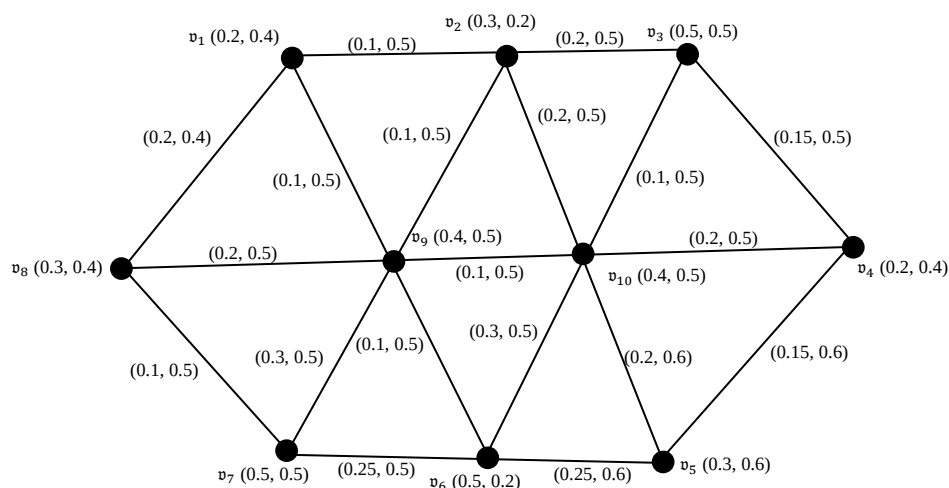


Figure 15(a). Pythagorean fuzzy graph

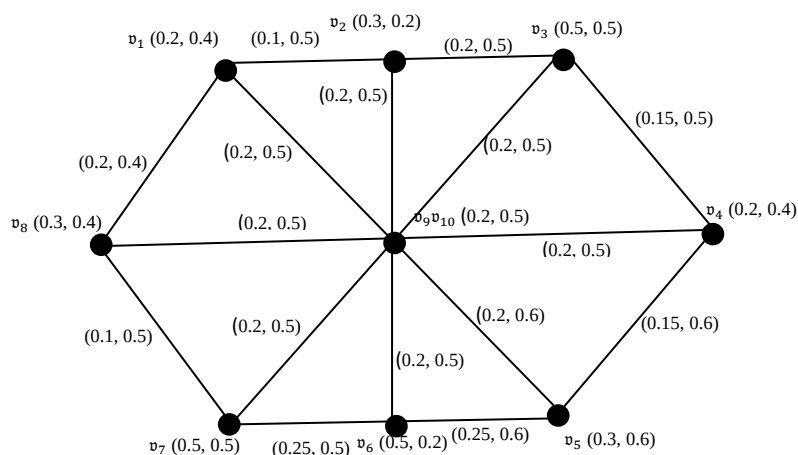


Figure 15(b). Pythagorean fuzzy edge contraction

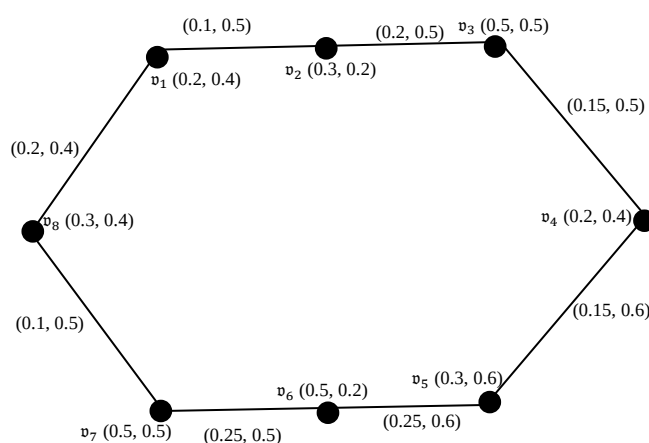


Figure 15(c). Vertex deleted Pythagorean fuzzy outerplanar graph

Let $\mathfrak{G}$ be the Pythagorean fuzzy graph. then the edge contracted Pythagorean fuzzy graph, denoted as $\mathfrak{G}/v_9v_{10}$ derived from the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ by contracting the edge v_9v_{10} . In this Fig.15(b) the vertex set $\mathcal{V}' = \left\{ \frac{\mathcal{V}}{\{v_9, v_{10}\}} \cup \{w\} \right\}$, where $(\mu_{\mathcal{M}}, \nu_{\mathcal{N}}) \left(\frac{\mathfrak{G}}{v_9v_{10}} \right) = (\mu_{\mathcal{M}}, \nu_{\mathcal{N}})(\mathfrak{G})$ for all vertices $x \in \mathcal{V}$. The membership value of $\mathfrak{G}/v_9v_{10}$ remains the same in $\mathfrak{G}$ for all vertices $x \in \mathcal{V}$, and the membership value of w is calculated by

$$(\mu_{\mathcal{N}})_{\frac{\mathfrak{G}}{v_9v_{10}}}(xy) = \min\{\mu_{\mathcal{M}}(v_9), \mu_{\mathcal{M}}(v_{10})\}, (\nu_{\mathcal{N}})_{\frac{\mathfrak{G}}{v_9v_{10}}}(xy) = \max\{\nu_{\mathcal{M}}(v_9), \nu_{\mathcal{M}}(v_{10})\}$$

The adjacency between two vertices x and y in $\mathfrak{G}/v_9v_{10}$ is determined by Definition (8.2) conditions.

The edge contracted Pythagorean fuzzy graph $\mathfrak{G}/v_9v_{10}$, represented as $\mathfrak{G}_1$ can be seen in Fig.15(b). Similarly, the Pythagorean fuzzy graph $\mathfrak{G}_1 - \mathcal{V}$ represented as $\mathfrak{G}_2$ is shown in Fig.15(c). It can be noted that, is categorized as a vertex deletion Pythagorean fuzzy outerplanar graph.

9 Maximal and Maximum Vertex Deletion Pythagorean fuzzy Outerplanar Subgraphs

In this section, defined the concepts of maximum and maximal vertex deletion Pythagorean fuzzy outerplanar subgraphs and explores the relationship between these two notions with appropriate examples.

Definition 9.1 If $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{M}', \mathcal{N}')$ is the vertex deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$ such that there is no other vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$ with maximum order and size compared to the Pythagorean fuzzy subgraph $\mathfrak{G}'$, then $\mathfrak{G}'$ is said to be the maximum vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Note 9.1 Let truth and falsity order $\mathcal{O}' = (\mathcal{O}'_{\mathcal{M}}, \mathcal{O}'_{\mathcal{N}})$, $\mathcal{O}'' = (\mathcal{O}''_{\mathcal{M}}, \mathcal{O}''_{\mathcal{N}})$ and truth and falsity size $\mathcal{S}' = (\mathcal{S}'_{\mathcal{M}}, \mathcal{S}'_{\mathcal{N}})$, $\mathcal{S}'' = (\mathcal{S}''_{\mathcal{M}}, \mathcal{S}''_{\mathcal{N}})$ of the vertex deleted Pythagorean fuzzy outerplanar subgraphs $\mathfrak{G}'$ and $\mathfrak{G}''$ of $\mathfrak{G}$ be $\mathcal{O}'_{\mathcal{M}} > \mathcal{O}''_{\mathcal{M}}$ then $|\mathcal{O}'_{\mathcal{N}}| > |\mathcal{O}''_{\mathcal{N}}|$ and $\mathcal{S}'_{\mathcal{M}} > \mathcal{S}''_{\mathcal{M}}$ then $|\mathcal{S}'_{\mathcal{N}}| > |\mathcal{S}''_{\mathcal{N}}|$ respectively. The Pythagorean fuzzy subgraph $\mathfrak{G}'$ is said to be maximum vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$ if $(\mathcal{O}'_{\mathcal{M}}, \mathcal{S}'_{\mathcal{M}}) > (\mathcal{O}''_{\mathcal{M}}, \mathcal{S}''_{\mathcal{M}})$ and $(\mathcal{O}'_{\mathcal{N}}, \mathcal{S}'_{\mathcal{N}}) > (\mathcal{O}''_{\mathcal{N}}, \mathcal{S}''_{\mathcal{N}})$. Suppose two vertices deleted Pythagorean fuzzy outerplanar subgraphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are obtained in the Pythagorean fuzzy graph $\mathfrak{G}$. If $\mathfrak{G}_1 \neq \mathfrak{G}_2$ but the order and size of two Pythagorean fuzzy subgraphs are equal (i.e.), $(\mathcal{O}'_{\mathcal{M}}, \mathcal{S}'_{\mathcal{M}}) = (\mathcal{O}''_{\mathcal{M}}, \mathcal{S}''_{\mathcal{M}})$ and $(\mathcal{O}'_{\mathcal{N}}, \mathcal{S}'_{\mathcal{N}}) > (\mathcal{O}''_{\mathcal{N}}, \mathcal{S}''_{\mathcal{N}})$, then the two Pythagorean fuzzy subgraphs are maximum vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Example 9.1 Let us consider the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ shown in figure 16. The set of Pythagorean fuzzy vertices in $\mathfrak{G}$ is $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and their membership values of these are as follows: $\mathcal{V}$ is $\mu_{\mathcal{M}}(v_1) = 0.25$, $\nu_{\mathcal{N}}(v_1) = 0.30$, $\mu_{\mathcal{M}}(v_2) = 0.35$, $\nu_{\mathcal{N}}(v_2) = 0.50$, $\mu_{\mathcal{M}}(v_3) = 0.40$, $\nu_{\mathcal{N}}(v_3) = 0.50$, $\mu_{\mathcal{M}}(v_4) =$

$0.15, v_{\mathcal{N}}(v_4) = 0.10, \mu_{\mathcal{M}}(v_5) = 0.15, v_{\mathcal{N}}(v_5) = 0.10, \mu_{\mathcal{M}}(v_6) = 0.45, v_{\mathcal{N}}(v_6) = 0.50$ and the Pythagorean fuzzy edges $\mathcal{E}$ is $\mu_{\mathcal{M}}(v_1, v_2) = 0.10,$

$v_{\mathcal{N}}(v_1, v_2) = 0.50, \mu_{\mathcal{M}}(v_1, v_6) = 0.10,$
 $v_{\mathcal{N}}(v_2, v_6) = 0.50, \mu_{\mathcal{M}}(v_4, v_6) = 0.10,$
 $v_{\mathcal{N}}(v_5, v_6) = 0.39, \mu_{\mathcal{M}}(v_1, v_5) = 0.15,$
 $v_{\mathcal{N}}(v_2, v_3) = 0.60, \mu_{\mathcal{M}}(v_3, v_4) = 0.20,$
 $v_{\mathcal{N}}(v_4, v_5) = 0.30.$

$v_{\mathcal{N}}(v_1, v_6) = 0.20, \mu_{\mathcal{M}}(v_2, v_6) = 0.20,$
 $v_{\mathcal{N}}(v_4, v_6) = 0.50, \mu_{\mathcal{M}}(v_5, v_6) = 0.10,$
 $v_{\mathcal{N}}(v_1, v_5) = 0.35, \mu_{\mathcal{M}}(v_2, v_3) = 0.10,$
 $v_{\mathcal{N}}(v_3, v_4) = 0.50, \mu_{\mathcal{M}}(v_4, v_5) = 0.10,$

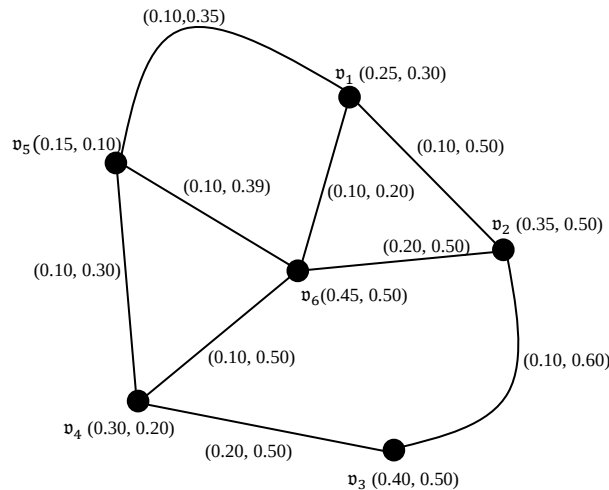


Figure 16. Pythagorean fuzzy graph

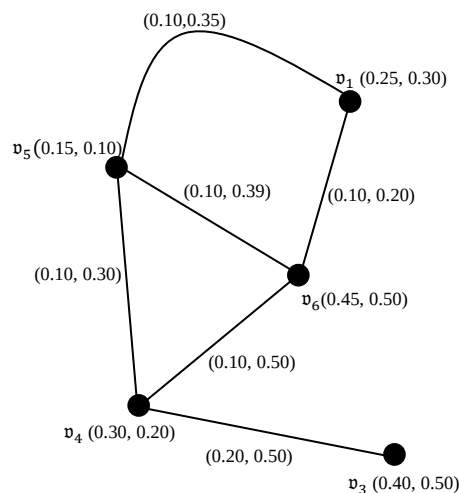


Figure 16 (a). $\mathcal{G} - \mathcal{W}_2$

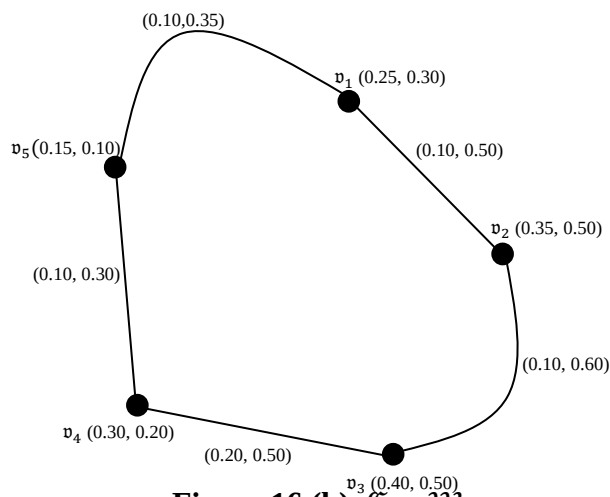


Figure 16 (b). $\mathcal{G} - \mathcal{W}_6$

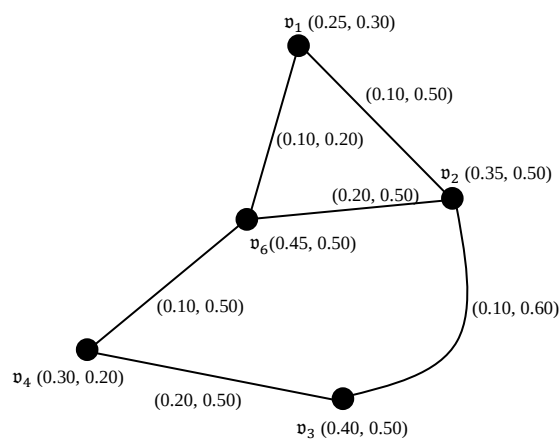


Figure 16 (c). $\mathcal{G} - \mathcal{W}_5$

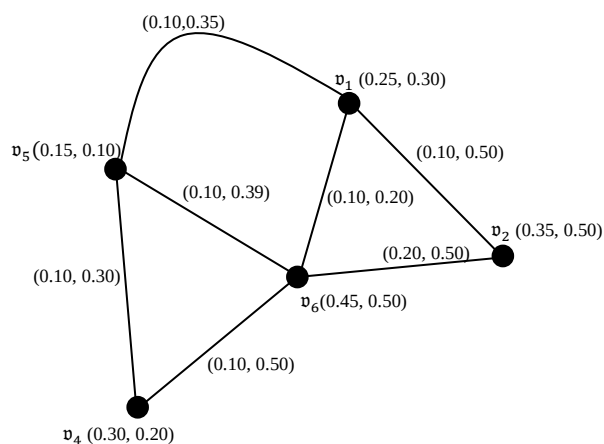


Figure 16 (d). $\mathcal{G} - \mathcal{W}_3$

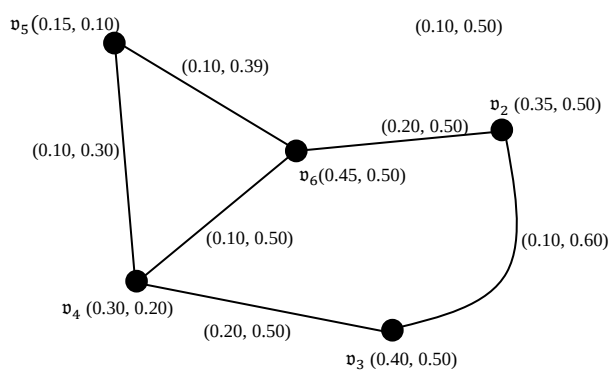


Figure 16 (e). $\mathfrak{G} - \mathcal{W}_1$

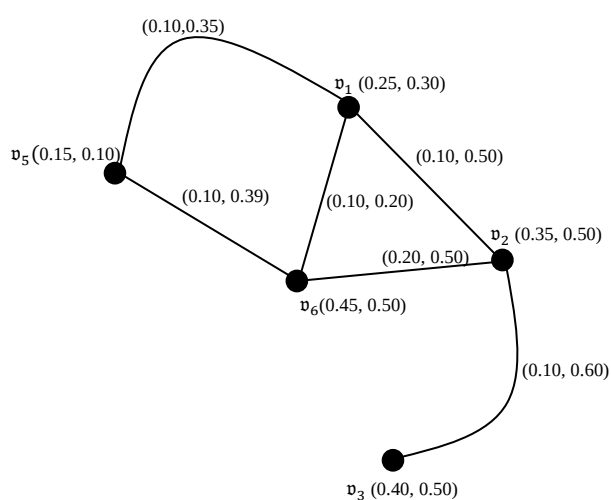


Figure. 16 (f). $\mathfrak{G} - \mathcal{W}_4$

Figure 16(a)-16(f). Maximum vertex deletion Pythagorean fuzzy outerplanar subgraph

Table 4. Calculation of order and size of these vertex deletion Pythagorean fuzzy outerplanar subgraphs

Subset	Vertex deleted Pythagorean fuzzy outerplanar subgraph	Membership Order $\mathcal{O}_{\mathcal{M}}(\mathfrak{G})$	Non-Membership Order $\mathcal{O}_{\mathcal{N}}(\mathfrak{G})$	Membership Size $\mathcal{S}_{\mathcal{M}}(\mathfrak{G})$	Non-Membership Size $\mathcal{S}_{\mathcal{N}}(\mathfrak{G})$
$\mathcal{W}_2 = v_2(0.35, 0.50)$	$\mathfrak{G} - \mathcal{W}_2$	1.55	1.60	0.65	2.19
$\mathcal{W}_6 = v_6(0.45, 0.50)$	$\mathfrak{G} - \mathcal{W}_6$	1.45	1.60	0.65	2.14
$\mathcal{W}_5 = v_5(0.15, 0.10)$	$\mathfrak{G} - \mathcal{W}_5$	1.75	2.00	0.80	2.80
$\mathcal{W}_3 = v_3(0.40, 0.50)$	$\mathfrak{G} - \mathcal{W}_3$	1.50	1.60	0.75	2.74
$\mathcal{W}_1 = v_1(0.25, 0.30)$	$\mathfrak{G} - \mathcal{W}_1$	1.65	1.80	0.25	2.79
$\mathcal{W}_4 = v_4(0.30, 0.20)$	$\mathfrak{G} - \mathcal{W}_4$	1.60	1.90	0.75	2.54

The vertex deleted of Pythagorean fuzzy outerplanar subgraphs of $\mathfrak{G}$ are shown in Fig. 16(a)-(f). From the Table 4 note that Fig. 16(e) is the maximum vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Theorem 9.1 The Pythagorean fuzzy outerplanar subgraph of maximum vertex deletion has a Pythagorean fuzzy dual graph, but the converse need not be true.

Proof. we can easily conclude that maximum vertex deletion Pythagorean fuzzy outerplanar subgraph has a Pythagorean fuzzy dual and that the converse need not be true based on the proof of Theorem 8.2.

Definition 9.2 The maximal vertex deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}'$ is $\mathfrak{G}$. This is because if a vertex deletion Pythagorean fuzzy outerplanar subgraph $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{M}', \mathcal{N}')$ induced on $\mathfrak{G}$ and all other Pythagorean fuzzy subgraphs of $\mathfrak{G}$ induced by the vertex set $\mathcal{V}'' = \mathcal{V}' \cup \{v\}$ (where $v \in \mathcal{V} \setminus \mathcal{V}'$) does not satisfy the outerplanarity property.

Theorem 9.2 The Pythagorean fuzzy outerplanar subgraph of each maximum vertex deletion is the maximal vertex deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Proof. Let $\mathfrak{G}'$ denote the maximum Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$ resulting from vertex deletion. A subset $\mathcal{W} \neq \emptyset \subseteq \mathcal{V}$ exists, in which some of the $\mathfrak{G}$ vertices have been deleted. Choose a vertex u in $\mathcal{W}$, and consider $\mathfrak{G}'' = \mathfrak{G}' \cup \{u\}$ where $\mathfrak{G}'' = \{x \in \mathcal{V}' \cup \frac{\{u\}}{(x_{\mathfrak{G}}(u,v), \mu_{\mathfrak{G}}(u,v))} \neq \emptyset, x, y \in \mathcal{V} \setminus \mathcal{W} \cup \{u\}\}$. Since $\mathfrak{G}'$ is a maximum induced Pythagorean fuzzy outerplanar subgraph, adding any of the vertex's results u in $\mathcal{F}_{\mathfrak{G}} \neq 1$. Hence, $\mathfrak{G}'$ is the maximal induced vertex elimination Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$. Therefore, every maximum induced vertex deleted Pythagorean fuzzy outerplanar graph is a maximal induced vertex deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Theorem 9.3 The answer to Theorem 9.2 is not necessarily true.

Proof. Let $\mathfrak{G}$ be the Pythagorean fuzzy graph with $\mathcal{F}_{\mathfrak{G}} \neq 1$ and, $\mathfrak{G}'$ and $\mathfrak{G}''$ are the vertex deletion Pythagorean fuzzy outerplanar subgraphs of $\mathfrak{G}$. Let $\mathcal{W}_1, \mathcal{W}_2$ be the subsets of vertex set $\mathcal{V}$ in $\mathfrak{G}$. $\mathcal{W}_1$ and $\mathcal{W}_2$ contains the deleted vertices from $\mathfrak{G}$ respectively. Then $\mathfrak{G}'$ and $\mathfrak{G}''$ are said to be maximal Pythagorean fuzzy outerplanar subgraph if $\mathfrak{G}' \cup \{u\}$ where $u \in \mathcal{W}_1$ and $\mathfrak{G}'' \cup \{v\}$ where $v \in \mathcal{W}_2$ are Pythagorean fuzzy non outerplanar graph.

Consequently, these Pythagorean fuzzy subgraphs are considered Pythagorean fuzzy outerplanar subgraphs with maximal vertex deletion of $\mathfrak{G}$. However, since Pythagorean fuzzy subgraphs can vary in order and size, a vertex deletion Pythagorean fuzzy outerplanar subgraph is defined as a maximal vertex deletion Pythagorean fuzzy outerplanar subgraph that has the highest order and size. Thus, either $\mathfrak{G}'$ or $\mathfrak{G}''$ qualifies as a maximal vertex deletion Pythagorean fuzzy outerplanar subgraph, but Pythagorean fuzzy graphs are maximal. As a result, the Pythagorean fuzzy outerplanar subgraph of any maximal vertex deletion does not necessarily have to be a maximum vertex deletion Pythagorean fuzzy outerplanar subgraph.

Note 9.2 Only when both Pythagorean fuzzy subgraphs are maximum vertex deletion Pythagorean fuzzy outerplanar subgraphs as defined by Note 9.1 are two vertex deletion Pythagorean fuzzy outerplanar subgraphs maximal.

Example 9.2 Let us consider the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ shown in figure 17(a). The set of Pythagorean fuzzy vertices in $\mathfrak{G}$ is $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and their membership values of these are as follows: $\mathcal{V}$ is $\mu_{\mathcal{M}}(v_1) = 0.35, v_{\mathcal{N}}(v_1) = 0.40, \mu_{\mathcal{M}}(v_2) = 0.45, v_{\mathcal{N}}(v_2) = 0.50, \mu_{\mathcal{M}}(v_3) = 0.10, v_{\mathcal{N}}(v_3) = 0.50, \mu_{\mathcal{M}}(v_4) = 0.20, v_{\mathcal{N}}(v_4) = 0.50, \mu_{\mathcal{M}}(v_5) = 0.15, v_{\mathcal{N}}(v_5) = 0.40$ and the Pythagorean fuzzy edges $\mathcal{E}$ is $\mu_{\mathcal{M}}(v_1, v_5) = 0.10, v_{\mathcal{N}}(v_1, v_5) = 0.40, \mu_{\mathcal{M}}(v_2, v_5) = 0.10, v_{\mathcal{N}}(v_2, v_5) = 0.50, \mu_{\mathcal{M}}(v_3, v_5) = 0.10, v_{\mathcal{N}}(v_3, v_5) = 0.50, \mu_{\mathcal{M}}(v_4, v_5) = 0.10, v_{\mathcal{N}}(v_4, v_5) = 0.50, \mu_{\mathcal{M}}(v_1, v_2) = 0.10, v_{\mathcal{N}}(v_1, v_2) = 0.50, \mu_{\mathcal{M}}(v_2, v_3) = 0.20, v_{\mathcal{N}}(v_2, v_3) = 0.50, \mu_{\mathcal{M}}(v_3, v_4) = 0.10, v_{\mathcal{N}}(v_3, v_4) = 0.50, \mu_{\mathcal{M}}(v_4, v_1) = 0.20, v_{\mathcal{N}}(v_4, v_1) = 0.50.$

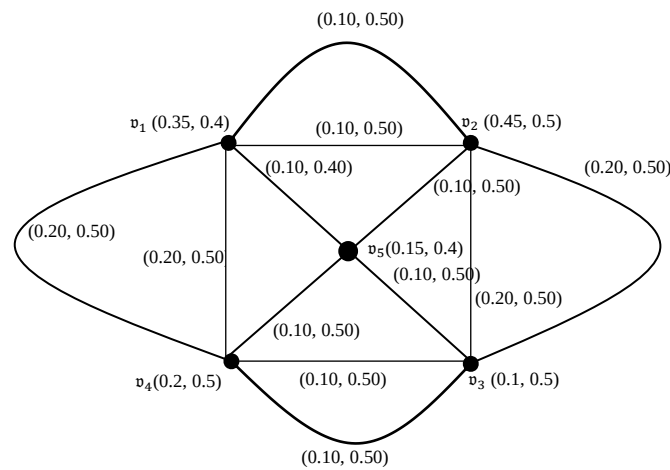


Figure 17. Example for maximal vertex deletion Pythagorean fuzzy graph

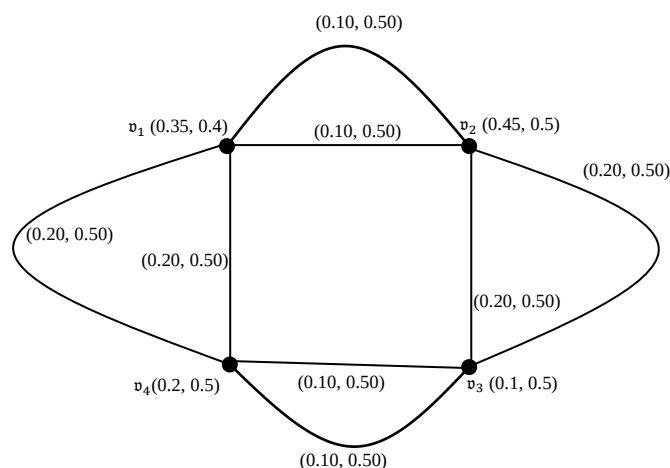


Figure 17(a). $\mathfrak{G} - \{v_5(0.15, 0.4)\}$

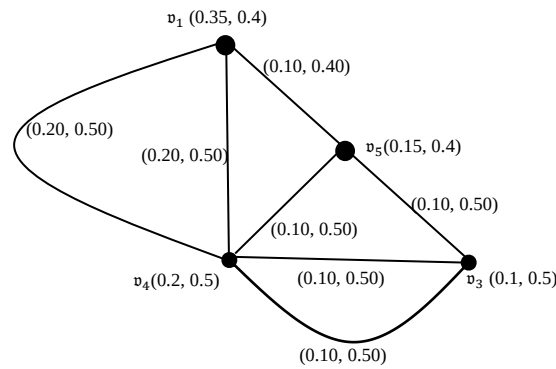


Figure 17(b). $\mathcal{G} - \{v_2(0.45, 0.5)\}$

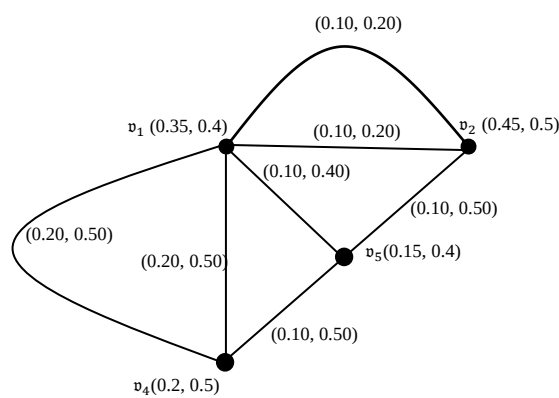


Figure 17(c). $\mathcal{G} - \{v_3(0.1, 0.5)\}$

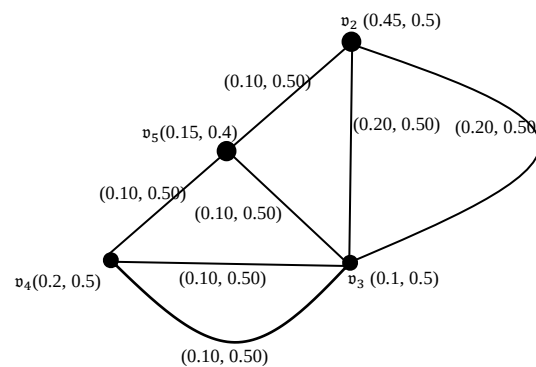


Figure 17(d). $\mathcal{G} - \{v_1(0.35, 0.4)\}$

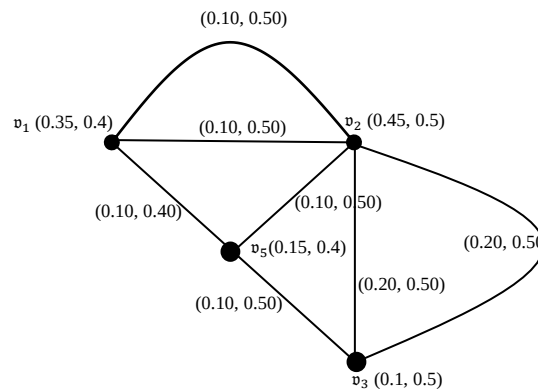


Figure 17(e). $\mathcal{G} - \{v_4 (0.2, 0.5)\}$

Figure 17(a)-17(e). Maximal vertex deletion Pythagorean fuzzy outerplanar subgraphs

Table 5. Calculation of order and size of these vertex deletion Pythagorean fuzzy outerplanar subgraphs

Figures	Vertex deleted Pythagorean fuzzy outerplanar subgraph	Membership Order $\mathcal{O}_{\mathcal{M}}(\mathcal{G})$	Non- Membership Order $\mathcal{O}_{\mathcal{N}}(\mathcal{G})$	Membership Size $\mathcal{S}_{\mathcal{M}}(\mathcal{G})$	Non- Membership Size $\mathcal{S}_{\mathcal{N}}(\mathcal{G})$
Fig.17(a)	$\mathcal{G} - \{v_5 (0.15, 0.40)\}$	1.18	1.91	1.10	4.00
Fig.17(b)	$\mathcal{G} - \{v_2 (0.45, 0.50)\}$	0.80	1.80	0.90	3.40
Fig.17(c)	$\mathcal{G} - \{v_3 (0.10, 0.50)\}$	1.15	1.80	0.90	2.95
Fig.17(d)	$\mathcal{G} - \{v_1 (0.35, 0.40)\}$	0.90	1.90	0.80	3.05
Fig.17(e)	$\mathcal{G} - \{v_4 (0.20, 0.50)\}$	1.05	1.80	0.80	2.95

All the graphs in Fig.17(a)-Fig.17(e) are maximal Pythagorean fuzzy outerplanar subgraph of $\mathcal{G}$. But Fig. 17(a) is the maximum Pythagorean fuzzy outerplanar subgraph of $\mathcal{G}$ in the Table 5. Then it can be observed that maximal Pythagorean fuzzy outerplanar subgraph need not be maximum Pythagorean fuzzy outerplanar subgraph of $\mathcal{G}$.

10 Edge Deletion Pythagorean fuzzy Outerplanar Subgraphs

In this section, discussed the subgraph created by deleting specific edges from the Pythagorean fuzzy graph, while also defining Pythagorean fuzzy outerplanar graphs and providing relevant illustrations.

Definition 10.1 If $\mathcal{G}$ is a Pythagorean fuzzy planar graph and $\mathcal{G}'$ is its edge deletion Pythagorean fuzzy subgraph, then $\mathcal{G}'$ is the edge deletion Pythagorean fuzzy outerplanar subgraph of $\mathcal{G}$ if and only if it preserves a Pythagorean fuzzy outerplanar graph.

Note 10.1 There is no need for an edge deletion Pythagorean fuzzy outerplanar subgraph of $\mathcal{G}$ to be an edge deletion Pythagorean fuzzy subgraph of $\mathcal{G}$.

Theorem 10.1 In a Pythagorean fuzzy outerplanar graph $\mathfrak{G}$, every edge deletion Pythagorean fuzzy subgraph is always an edge deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Proof. Let $\mathfrak{G}$ be a Pythagorean fuzzy outerplanar graph and let $\mathcal{H}$ be an edge deletion Pythagorean fuzzy subgraph of $\mathfrak{G}$. Since a Pythagorean fuzzy graph $\mathfrak{G}$ is outerplanar, all its vertices will be in the outer region. Thus, the subgraph obtained by edge deletion still possess outerplanarity property. Therefore, any edge deleted Pythagorean fuzzy subgraphs in $\mathfrak{G}$ are edge deleted Pythagorean fuzzy outerplanar subgraphs of $\mathfrak{G}$.

Theorem 10.2 Let $\mathcal{W}$ be the subset of edges of a connected Pythagorean fuzzy outerplanar graph such that $\mathcal{W} \subseteq \mathcal{E}$. If $\mathfrak{G}'$ is an edge deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$ if the resulting graph connects $\mathfrak{G}'$ to the Pythagorean fuzzy dual graph.

Proof. By using the same proof as Theorem 10.1 and which is obviously true for the case of edges.

Example 10.1 Let us consider the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ shown in Figure 18(a). The set of Pythagorean fuzzy vertices in $\mathfrak{G}$ is $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and their membership values of these are as follows: $\mathcal{V}$ is $\mu_{\mathcal{M}}(v_1) = 0.2, v_{\mathcal{N}}(v_1) = 0.1, \mu_{\mathcal{M}}(v_2) = 0.3, v_{\mathcal{N}}(v_2) = 0.2, \mu_{\mathcal{M}}(v_3) = 0.4, v_{\mathcal{N}}(v_3) = 0.3, \mu_{\mathcal{M}}(v_4) = 0.5, v_{\mathcal{N}}(v_4) = 0.4, \mu_{\mathcal{M}}(v_5) = 0.6, v_{\mathcal{N}}(v_5) = 0.5, \mu_{\mathcal{M}}(v_6) = 0.4, v_{\mathcal{N}}(v_6) = 0.6, \mu_{\mathcal{M}}(v_7) = 0.3, v_{\mathcal{N}}(v_7) = 0.6, \mu_{\mathcal{M}}(v_8) = 0.5, v_{\mathcal{N}}(v_8) = 0.4, \mu_{\mathcal{M}}(v_9) = 0.2, v_{\mathcal{N}}(v_9) = 0.7$ and the Pythagorean fuzzy edges $\mathcal{E}$ is $\mu_{\mathcal{M}}(v_1, v_2) = 0.1, v_{\mathcal{N}}(v_1, v_2) = 0.3, \mu_{\mathcal{M}}(v_1, v_4) = 0.1, v_{\mathcal{N}}(v_1, v_4) = 0.5, \mu_{\mathcal{M}}(v_1, v_5) = 0.1, v_{\mathcal{N}}(v_1, v_5) = 0.6, \mu_{\mathcal{M}}(v_2, v_3) = 0.2, v_{\mathcal{N}}(v_2, v_3) = 0.5, \mu_{\mathcal{M}}(v_3, v_4) = 0.2, v_{\mathcal{N}}(v_3, v_4) = 0.4, \mu_{\mathcal{M}}(v_4, v_5) = 0.4, v_{\mathcal{N}}(v_4, v_5) = 0.5, \mu_{\mathcal{M}}(v_2, v_5) = 0.2, v_{\mathcal{N}}(v_2, v_5) = 0.6, \mu_{\mathcal{M}}(v_5, v_7) = 0.1, v_{\mathcal{N}}(v_5, v_7) = 0.6, \mu_{\mathcal{M}}(v_6, v_7) = 0.3, v_{\mathcal{N}}(v_6, v_7) = 0.6, \mu_{\mathcal{M}}(v_7, v_8) = 0.2, v_{\mathcal{N}}(v_7, v_8) = 0.6, \mu_{\mathcal{M}}(v_6, v_9) = 0.1, v_{\mathcal{N}}(v_6, v_9) = 0.7, \mu_{\mathcal{M}}(v_7, v_9) = 0.15, v_{\mathcal{N}}(v_7, v_9) = 0.7, \mu_{\mathcal{M}}(v_8, v_9) = 0.2, v_{\mathcal{N}}(v_8, v_9) = 0.7.$

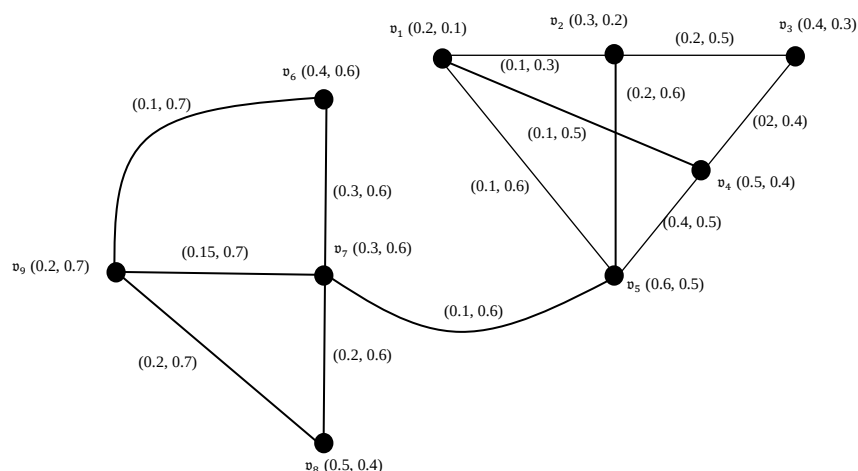


Figure 18(a). Pythagorean fuzzy graph

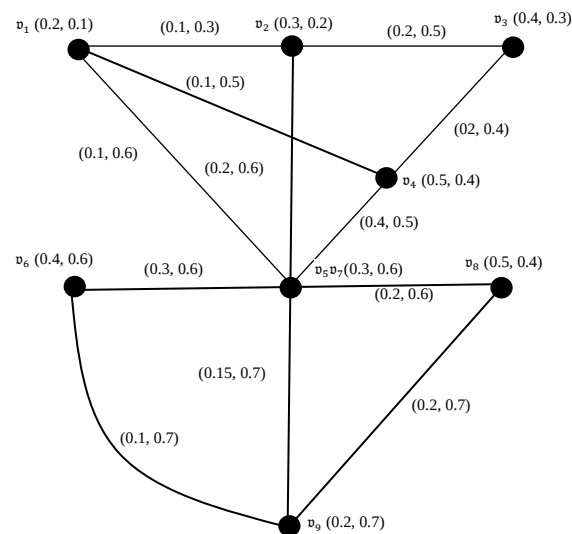


Figure 18(b). Pythagorean fuzzy edge contraction

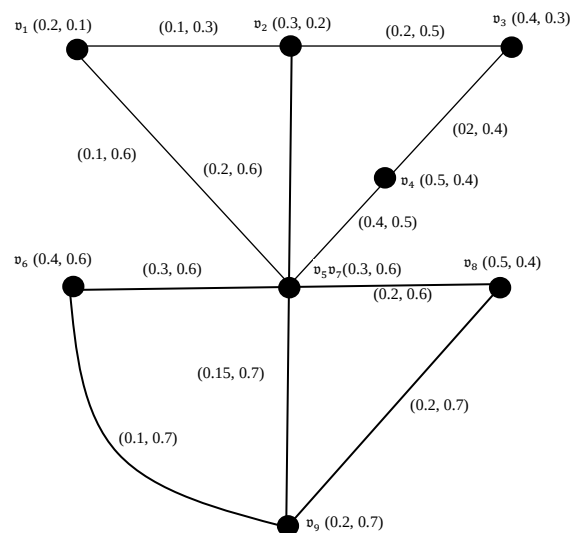


Figure 18(c). Edge deleted Pythagorean fuzzy outerplanar graph

Let $\mathcal{G}$ be the Pythagorean fuzzy graph. then the edge contracted Pythagorean fuzzy graph, denoted as $\mathcal{G} \setminus v_5 v_7$ derived from the Pythagorean fuzzy graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ by contracting the edge $v_5 v_7$. In this Figure 18(b) the vertex set $\mathcal{V}' = \left\{ \frac{\mathcal{V}}{\{v_5, v_7\}} \cup \{\mathfrak{w}\} \right\}$, where

$(\mu_{\mathcal{M}}, \nu_{\mathcal{M}})\left(\frac{\mathfrak{G}}{v_5 v_7}\right) = (\mu_{\mathcal{M}}, \nu_{\mathcal{M}})(\mathfrak{G})$ for all vertices $x \in \mathcal{V}$. The membership value of $\mathfrak{G}/v_5 v_7$ remains the same as τ in $\mathfrak{G}$ for all vertices $x \in \mathcal{V}$, and the membership value of $\mathfrak{w}$ is calculated by

$$(\mu_{\mathcal{N}})_{\frac{\mathfrak{G}}{v_5 v_7}}(xy) = \min\{\mu_{\mathcal{M}}(u), \mu_{\mathcal{M}}(v)\}, (\nu_{\mathcal{N}})_{\frac{\mathfrak{G}}{v_5 v_7}}(xy) = \max\{\nu_{\mathcal{M}}(u), \nu_{\mathcal{M}}(v)\}.$$

The adjacency between two vertices x and y in $\mathfrak{G}/v_5 v_7$ is determined by Definition (8.2) conditions.

The edge contracted Pythagorean fuzzy graph $\mathfrak{G} \setminus v_5 v_7$, represented as $\mathfrak{G}_1$ can be seen in Fig 18(b). Similarly, the Pythagorean fuzzy graph $\mathfrak{G}_1 - \mathcal{E}$ represented as $\mathfrak{G}_2$ is shown in Fig 18(c). It can be noted that, is categorized as an edge deletion Pythagorean fuzzy outerplanar graph.

11. Maximal and Maximum Edge Deletion Pythagorean fuzzy Outerplanar Subgraphs

In this section, outlined the concepts of maximum and maximal edge deletion Pythagorean fuzzy outerplanar subgraphs and investigates the relationship between these two concepts, illustrated with suitable examples.

Definition 11.1 Maximum ED- Pythagorean fuzzy outerplanar subgraph

If $\mathfrak{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{M}', \mathcal{N}')$ with positive and negative size $\mathcal{S}' = (\mathcal{S}'_{\mathcal{M}}, \mathcal{S}'_{\mathcal{N}})$ is the edge deletion Pythagorean fuzzy outerplanar subgraph of non-outerplanar Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ such that there is no other edge deletion Pythagorean fuzzy outerplanar subgraph $\mathfrak{G}''$ of size $\mathcal{S}'' = (\mathcal{S}''_{\mathcal{M}}, \mathcal{S}''_{\mathcal{N}})$ of $\mathfrak{G}$ such that $\mathcal{S}'' > \mathcal{S}'$, then $\mathfrak{G}'$ is the maximum Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Note 11.1 Suppose two edge deletion Pythagorean fuzzy outerplanar subgraphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is obtained from the Pythagorean fuzzy graph $\mathfrak{G}$. If the sizes of the two Pythagorean fuzzy subgraphs are equal but $\mathfrak{G}_1 \neq \mathfrak{G}_2$. (i.e.) $\mathcal{S}'' = \mathcal{S}'$, then the two Pythagorean fuzzy subgraphs are maximum edge deletion Pythagorean fuzzy outerplanar subgraphs of $\mathfrak{G}$. This means that there is no unique maximum edge deletion Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$.

Example 11.1 Let us consider the Pythagorean fuzzy graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{M}, \mathcal{N})$ shown in Figure 19(a). The set of Pythagorean fuzzy vertices in $\mathfrak{G}$ is $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and their membership values of these are as follows: $\mathcal{V}$ is $\mu_{\mathcal{M}}(v_1) = 0.3, \nu_{\mathcal{N}}(v_1) = 0.5, \mu_{\mathcal{M}}(v_2) = 0.15, \nu_{\mathcal{N}}(v_2) = 0.4, \mu_{\mathcal{M}}(v_3) = 0.25, \nu_{\mathcal{N}}(v_3) = 0.4, \mu_{\mathcal{M}}(v_4) = 0.6, \nu_{\mathcal{N}}(v_4) = 0.5, \mu_{\mathcal{M}}(v_5) = 0.5, \nu_{\mathcal{N}}(v_5) = 0.4, \mu_{\mathcal{M}}(v_6) = 0.2, \nu_{\mathcal{N}}(v_6) = 0.6$ and the Pythagorean fuzzy edges $\mathcal{E}$ is $\mu_{\mathcal{M}}(e_1) = 0.15, \nu_{\mathcal{N}}(e_1) = 0.5, \mu_{\mathcal{M}}(e_2) = 0.15, \nu_{\mathcal{N}}(e_2) = 0.4, \mu_{\mathcal{M}}(e_3) = 0.25, \nu_{\mathcal{N}}(e_3) = 0.5, \mu_{\mathcal{M}}(e_4) = 0.4, \nu_{\mathcal{N}}(e_4) = 0.5, \mu_{\mathcal{M}}(e_5) = 0.2, \nu_{\mathcal{N}}(e_5) = 0.6, \mu_{\mathcal{M}}(e_6) = 0.3, \nu_{\mathcal{N}}(e_6) = 0.6, \mu_{\mathcal{M}}(e_7) = 0.3, \nu_{\mathcal{N}}(e_7) = 0.5, \mu_{\mathcal{M}}(e_8) = 0.1, \nu_{\mathcal{N}}(e_8) = 0.5, \mu_{\mathcal{M}}(e_9) = 0.1, \nu_{\mathcal{N}}(e_9) = 0.6, \mu_{\mathcal{M}}(e_{10}) = 0.3, \nu_{\mathcal{N}}(e_{10}) = 0.5, \mu_{\mathcal{M}}(e_{11}) = 0.1, \nu_{\mathcal{N}}(e_{11}) = 0.5.$

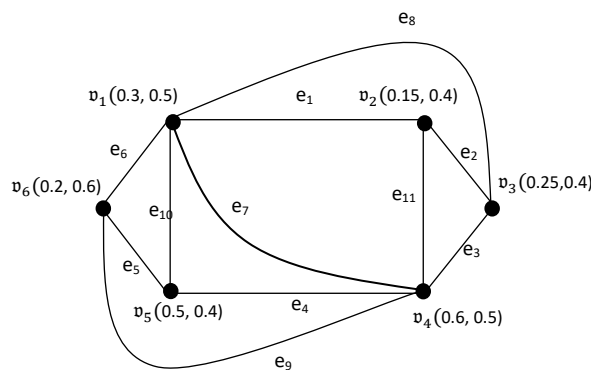


Figure 19(a). Pythagorean fuzzy graphs $\mathfrak{G}$

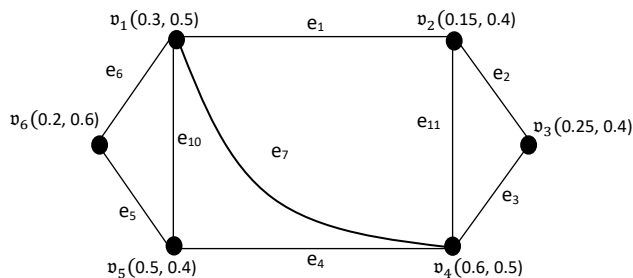


Figure 19(b). Maximum and maximal edge deletion from a Pythagorean fuzzy subgraph of $\mathfrak{G}$

In Fig.19(b), the maximal edge deleted Pythagorean fuzzy outerplanar subgraph are shown. Thus, a result, $\mathfrak{G} - \mathcal{U}$ is the greatest edge deleted Pythagorean fuzzy outerplanar subgraph of $\mathfrak{G}$, and Pythagorean fuzzy subgraph is maximal.

12 Conclusion

In this study, we introduced the concept of Pythagorean fuzzy outerplanar graphs (PFOGs) by integrating Pythagorean fuzzy set theory with the structural framework of outerplanar graphs. The proposed model extends classical and fuzzy outerplanar graphs by incorporating both membership and non-membership degrees that satisfy the Pythagorean condition, thereby providing a more flexible and effective representation of uncertainty and imprecision in network structures. We developed the fundamental theory of PFOGs and investigated several of their structural properties. In particular, we defined and analyzed vertex deleted (VD) and edge deleted (ED) Pythagorean fuzzy outerplanar subgraphs, along with the concepts of maximum and maximal Pythagorean fuzzy outerplanar subgraphs obtained through vertex and edge deletions. Furthermore, we introduced the notion of the dual of a Pythagorean fuzzy outerplanar graph and established its key characteristics. The theoretical results presented in this work were supported by rigorous proofs and illustrative examples, demonstrating the consistency and applicability of the proposed framework.

The study provides a foundation for further research on Pythagorean fuzzy graph structures and their applications in uncertain network analysis, communication systems, transportation

networks, and decision-making problems. Future work may focus on extending these concepts to other classes of planar and non-planar Pythagorean fuzzy graphs, developing efficient algorithms for their characterization, and exploring practical applications in real-world systems involving complex and uncertain relationships.

Author Contributions All authors contributed to the study conception and design. Conceptualization, Formal Analysis, Investigation, Data Curation, Writing-Original Draft were performed by DJ and SR; Resources and Supervision, SR. Methodology, DJ and SR; Writing-Review & Editing, Visualization, SR. All authors have read and agreed to the published version of the manuscript.

Acknowledgments The authors would like to express our sincere thanks to the editor and the anonymous reviewers for their timely and competent reviews. The research work of J. Deivanai and R. Sujatha is supported by Shiv Nadar University Chennai.

Data Availability All data generated or analysed during this study are included in this published article.

Declarations

Conflict of interest All the authors declare that they have no conflict of interest.

Ethical approval Not applicable

Consent for publication All the authors agree with this manuscript's publication.

References

- [1] Abdul-Jabbar N., Naoom J.H. and Ouda E.H., Fuzzy dual graphs, Journal of Al-Nahrain University, 12, 2009, 168–171.
- [2] Akram M., Dar J.M. and Naz S., Certain graphs under Pythagorean fuzzy environment, Complex & Intelligent Systems, In Press.
- [3] Akram M., Habib A., Ilyas F. and Dar J.M., Specific types of Pythagorean fuzzy graphs and application to decision-making, Mathematical and Computational Applications, 23, 2018, 42.
- [4] Akram M., Ilyas F. and Borumand Saeid A., Certain notions of Pythagorean fuzzy graphs, Journal of Intelligent & Fuzzy Systems, 36, 2019, 5857–5874.
- [5] Akram M. and Davvaz B., Strong intuitionistic fuzzy graphs, Filomat, 26, 2012, 177–196.
- [6] Akram M. and Dudek W.A., Intuitionistic fuzzy hypergraphs with applications, Information Sciences, 218, 2013, 182–193.
- [7] Akram M. and Naz S., Energy of Pythagorean fuzzy graphs with applications, Mathematics, 6, 2018, 136.
- [8] Akram M., Samanta S. and Pal M., Application of bipolar fuzzy sets in planar graph, International Journal of Applied and Computational Mathematics, 3, 2017, 773–785.
- [9] Akram, M., Mohsan Dar, J. and Farooq, A., Planar graphs under Pythagorean fuzzy environment, Mathematics, 6, 2018, 278.

- [10] Alshehri N. and Akram M., Intuitionistic fuzzy planar graphs, *Discrete Dynamics in Nature and Society*, 2014, 2014, 397823.
- [11] Atanassov K.T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 1986, 87–96.
- [12] Bhattacharya P., Some remarks on fuzzy graphs, *Pattern Recognition Letters*, 5, 1987, 297–302.
- [13] Deivanai J., Sujatha R., Nagarajan D. and Tadesse W., Fuzzy outerplanar graphs and its applications, *International Journal of Computational Intelligence Systems*, 17, 2024, 231.
- [14] Deivanai J., Sujatha R., Nagarajan D. and Tadesse W., Bipolar fuzzy outerplanar graphs in image shrinking, *Scientific Reports*, 15, 2025, 24587.
- [15] Deivanai J., Sujatha R., Nagarajan D. and Tadesse W., Network design using interval-valued fuzzy outerplanar graphs, *Scientific Reports*, 15, 2025, 34018.
- [16] Deivanai J., Sujatha R. and Bayou Z.G., A fuzzy graph theoretic approach to face shape recognition using cubic outerplanar structures, *Scientific Reports*, 15, 2025, 44072.
- [17] Deivanai J., Sujatha R. and Bayou Z.G., Image contraction through fuzzy soft outerplanar graph structures, *Scientific Reports*, 16, 2026, 9779.
- [18] Deivanai J. and Sujatha R., Multipolar fuzzy outerplanar graph approach to road networks, *Journal of Computational Analysis and Applications*, 33, 2025, 463–500.
- [19] Deivanai J. and Sujatha R., Topological indices in fuzzy graphs: an overview, *Journal of Computational Analysis and Applications*, 34(11), 2025, 710–722.
- [20] Deivanai J., Sujatha R. and Nagarajan D., A comprehensive review of applications of fuzzy planar graphs, in *Data-Driven Decision Making and Soft Computing*, 1st ed., CRC Press, Boca Raton, FL, USA, 2026, Chapter 3.
- [21] Dhavudh S.S. and Srinivasan R., Intuitionistic fuzzy graphs of second type, *Advances in Fuzzy Mathematics*, 12, 2017, 197–204.
- [22] Dhavudh S.S. and Srinivasan R., Properties of intuitionistic fuzzy graphs of second type, *International Journal of Computational Applications and Mathematics*, 12, 2017, 815–823.
- [23] Garg H., Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment, *International Journal of Intelligent Systems*, 33, 2018, 687–712.
- [24] Garg H., Linguistic Pythagorean fuzzy sets and its applications in multi-attribute decision making process, *International Journal of Intelligent Systems*, 33, 2018, 1234–1263.
- [25] Kaufmann A., *Introduction à la Théorie des Sous-Ensembles Flous*, Masson et Cie, Paris, France, 1973.
- [26] Mordeson J.N. and Peng C.S., Operations on fuzzy graphs, *Information Sciences*, 79, 1994, 159–170.
- [27] Naz S., Ashraf S. and Akram M., A novel approach to decision-making with Pythagorean fuzzy information, *Mathematics*, 6, 2018, 1–28.
- [28] Nirmala G. and Dhanabal K., Interval-valued fuzzy planar graphs, *International Journal of Machine Learning and Cybernetics*, 7, 2016, 653–664.
- [29] Pal A., Samanta S. and Pal M., Concept of fuzzy planar graphs, *Proceedings of the Science and Information Conference*, London, UK, 14 November 2013, 557–563.
- [30] Pal A., Samanta S. and Pal M., New concept of fuzzy planar graphs, *International Journal of Advanced Research in Artificial Intelligence*, 3, 2014, 52–59.

- [31] Parvathi R., Karunambigai M.G. and Atanassov K.T., Operations on intuitionistic fuzzy graphs, Proceedings of the IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 20–24 August 2009, 1396–1401.
- [32] Peng X. and Selvachandran G., Pythagorean fuzzy set: State of the art and future directions, *Artificial Intelligence Review*, 2017, 1–55.
- [33] Peng X. and Yang Y., Some results for Pythagorean fuzzy sets, *International Journal of Intelligent Systems*, 30, 2015, 1133–1160.
- [34] Pramanik T., Samanta S. and Pal M., Special planar fuzzy graph configurations, *International Journal of Scientific Research and Publications*, 2, 2012, 1–4.
- [35] Rangasamy P. and Palaniappan N., Some operations on intuitionistic fuzzy sets of second type, *Notes on Intuitionistic Fuzzy Sets*, 10, 2003, 1–19.
- [36] Ren P., Xu Z. and Gou X., Pythagorean fuzzy TODIM approach to multi-criteria decision making, *Applied Soft Computing*, 42, 2016, 246–259.
- [37] Rosenfeld A., Fuzzy graphs, in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, Zadeh L.A., Fu K.S., Shimura M. (Eds.), Academic Press, Cambridge, MA, USA, 1975, 77–95.
- [38] Shannon A. and Atanassov K.T., A first step to a theory of intuitionistic fuzzy graphs, Proceedings of the Fuzzy Based Expert Systems, Sofia, Bulgaria, 26–29 June 1994, 59–61.
- [39] Shannon A. and Atanassov K.T., Intuitionistic fuzzy graphs from a-, b-, (a,b)-levels, *Notes on Intuitionistic Fuzzy Sets*, 1, 1995, 32–35.
- [40] Verma R., Merigó J.M. and Sahni M., Pythagorean fuzzy graphs: Some results, arXiv, 2018, arXiv:1806.06721.
- [41] Yager R.R., Pythagorean fuzzy subsets, Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013, 57–61.
- [42] Yager R.R., Pythagorean membership grades in multi-criteria decision making, *IEEE Transactions on Fuzzy Systems*, 22, 2014, 958–965.
- [43] Yager R.R. and Abbasov A.M., Pythagorean membership grades, complex numbers, and decision making, *International Journal of Intelligent Systems*, 28, 2013, 436–452.
- [44] Zadeh L.A., Fuzzy sets, *Information and Control*, 8, 1965, 338–353.
- [45] Zadeh L.A., Similarity relations and fuzzy orderings, *Information Sciences*, 3, 1971, 177–200.
- [46] Zhang X. and Xu Z., Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, *International Journal of Intelligent Systems*, 29, 2014, 1061–1078.