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Some Remarks on Pairwise Fuzzy Resolvable Spaces

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Abstract

In this paper, the concepts of resolvability of fuzzy bitopological spaces have been studied in detail.

Key words: Pairwise fuzzy open set, pairwise fuzzy dense set, pairwise fuzzy nowhere dense set and pairwise fuzzy irresolvable space.

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1. Introduction

In 1965, L.A.Zadeh [10] introduced the concept of fuzzy set, to accommodate real life situations by giving partial membership to each element of a situation under consideration. The usual notion of set topology was generalized with the introduction of fuzzy topology by C.L.Chang [2] in 1968, based on the concept of fuzzy sets invented by Zadeh. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. In 1989, A.Kandil [5] introduced the concept of fuzzy bitopological spaces

The systematic study of resolvability in classical topology began with the works of E.Hewitt [4] and M.Katetov [6]. The concepts of resolvability and irresolvability in topological spaces were introduced and studied by E.Hewitt [4] in 1943. Since then several mathematicians found interest in the study of resolvable and irresolvable spaces. In 1993, C.Chattopadhyay and C.Bandyopadhyay [3] extended the study of resolvable and irresolvable spaces to the bitopological spaces. The concepts of resolvability and irresolvability of fuzzy bitopological spaces were introduced by

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G.Thangaraj [7]. In this paper, the concepts of resolvability of fuzzy bitopological spaces have been studied extensively.

2. Preliminaries

Now we give some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are two fuzzy topologies on a non-empty set X. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$.

Definition 2.1 [1] Let λ and μ be fuzzy sets in X. Then for all $x \in X$,

(i). $\lambda = \mu \;\Leftrightarrow\; \lambda(x) = \mu(x)$
(ii). $\lambda \leq \mu \;\Leftrightarrow\; \lambda(x) \leq \mu(x)$
(iii). $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = max\{\lambda(x),\ \mu(x)\}$
(iv). $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = min\{\lambda(x),\ \mu(x)\}$
(v). $\eta = \lambda^c \;\Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as

(vi). $\psi(x) = sup_i\{\lambda_i(x) \;/\; x \in X\}$
(vii). $\delta(x) = inf_i\{\lambda_i(x) \;/\; x \in X\}$.

Definition 2.2 [1] Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior $int(\lambda)$ and the closure $cl(\lambda)$ are defined respectively as $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 [1] Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.

Lemma 2.4 [1] For a family $\mathcal{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leq cl\big(\vee(\lambda_\alpha)\big)$. In case $\mathcal{A}$ is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leq int\big(\vee(\lambda_\alpha)\big)$.

Definition 2.5 [8] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set if $\lambda \in T_i,\ (i = 1, 2)$. The complement of pairwise fuzzy open set in (X, T_1, T_2) is called a pairwise fuzzy closed set.

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Definition 2.6 [10] A fuzzy set λ in a set X is a function from X to $[0,1]$, that is., $\lambda : X \rightarrow [0,1]$.

Definition 2.7 [7] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy dense set if $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$.

Definition 2.8 [7] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy nowhere dense set if $int_{T_1}cl_{T_2}(\lambda) = 0 = int_{T_2}cl_{T_1}(\lambda)$.

Theorem 2.9 [9] If $\vee_{k=1}^{N}(\lambda_k) = 1$, where (λ_k)'s are the fuzzy sets defined on X such that $int_{T_i}int_{T_j}(\lambda_k) = 0$ ($i \neq j$ and $i, j = 1, 2$), in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy resolvable space.

3. Pairwise fuzzy resolvable spaces

By means of pairwise fuzzy denseness of fuzzy sets in fuzzy bitopological spaces, the concepts of pairwise fuzzy resolvability and pairwise fuzzy irresolvability of fuzzy bitopological spaces, are defined as follows:

Definition 3.1 A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy resolvable space if there exists a pairwise fuzzy dense set λ in (X, T_1, T_2) such that $1 - \lambda$ is also a pairwise fuzzy dense set in (X, T_1, T_2).

That is, (X, T_1, T_2) is called a pairwise fuzzy resolvable space if there exists a fuzzy set λ defined on X such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}cl_{T_2}(1 - \lambda) = 1 = cl_{T_2}cl_{T_1}(1 - \lambda)$ in (X, T_1, T_2).

Example 3.2 Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ, α, β, γ are defined on X as follows:

$\lambda : X \rightarrow [0,1]$ is defined as $\lambda(a) = 0.3$; $\lambda(b) = 0.4$; $\lambda(c) = 0.5$

$\mu : X \rightarrow [0,1]$ is defined as $\mu(a) = 0.5$; $\mu(b) = 0.7$; $\mu(c) = 0.6$

$\delta : X \rightarrow [0,1]$ is defined as $\delta(a) = 0.8$; $\delta(b) = 0.2$; $\delta(c) = 0.4$

$\alpha : X \rightarrow [0,1]$ is defined as $\alpha(a) = 1$; $\alpha(b) = 1$; $\alpha(c) = 0$

$\beta : X \rightarrow [0,1]$ is defined as $\beta(a) = \frac{1}{6}$; $\beta(b) = 0$; $\beta(c) = \frac{1}{3}$

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$\gamma : X \to [0,1]$ is defined as $\gamma(a) = \frac{2}{3}$; $\gamma(b) = 0$; $\gamma(c) = \frac{1}{2}$.

Clearly, $T_1 = \{0, \lambda, \mu, 1\}$ and $T_2 = \{0, \beta, \gamma, 1\}$ are fuzzy topologies on X. On computation, one can see that α, δ, μ, γ, $1-\alpha$, $1-\delta$, $1-\beta$, $1-\gamma$ are the pairwise fuzzy dense sets in (X, T_1, T_2). Hence there exist pairwise fuzzy dense sets α, δ and γ in (X, T_1, T_2) such that $1-\alpha$, $1-\delta$ and $1-\gamma$ are also pairwise fuzzy dense sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Remark: In example 3.2, the fuzzy topological spaces (X, T_1) and (X, T_2) are fuzzy resolvable spaces.

The fuzzy dense sets in (X, T_1) are μ, δ, α, γ, $1-\delta$, $1-\alpha$, $1-\beta$ and $1-\gamma$. For the fuzzy dense sets δ, α and γ in (X, T_1), $1-\delta$, $1-\alpha$ and $1-\gamma$ are fuzzy dense sets in (X, T_1). This implies that (X, T_1) is a fuzzy resolvable space.

The fuzzy dense sets in (X, T_2) are α and $1-\alpha$. Thus, there exists a fuzzy dense set α in (X, T_2) such that $1-\alpha$ is also a fuzzy dense set in (X, T_2). Therefore (X, T_2) is a fuzzy resolvable space.

Example 3.3 Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ, α, β, γ are defined on X as follows:

$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.2$; $\lambda(b) = 0.3$; $\lambda(c) = 0.4$

$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.5$; $\mu(b) = 0.4$; $\mu(c) = 0.7$

$\delta : X \to [0,1]$ is defined as $\delta(a) = 0.4$; $\delta(b) = 0.5$; $\delta(c) = 0.6$

$\alpha : X \to [0,1]$ is defined as $\alpha(a) = 0.4$; $\alpha(b) = 0$; $\alpha(c) = 0$

$\beta : X \to [0,1]$ is defined as $\beta(a) = 0.7$; $\beta(b) = 0.2$; $\beta(c) = 0$

$\gamma : X \to [0,1]$ is defined as $\gamma(a) = 0.5$; $\gamma(b) = 0$; $\gamma(c) = 0.3$.

Clearly, $T_1 = \{0, \lambda, \mu, \delta, \mu \vee \delta, \mu \wedge \delta, 1\}$ and $T_2 = \{0, \alpha, \beta, \gamma, \beta \vee \gamma, \beta \wedge \gamma, 1\}$ are fuzzy topologies on X. On computation, one can see that $cl_{T_1} cl_{T_2}(\beta \vee \gamma) = 1 = cl_{T_2} cl_{T_1}(\beta \vee \gamma)$ and $cl_{T_1} cl_{T_2}[1-(\beta \vee \gamma)] = 1 = cl_{T_2} cl_{T_1}[1-(\beta \vee \gamma)]$ in (X, T_1, T_2). Hence there exists a pairwise fuzzy dense set $\beta \vee \gamma$ in (X, T_1, T_2) such that $1-(\beta \vee \gamma)$ is also a pairwise fuzzy dense set in (X, T_1, T_2). Therefore the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space.

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Remark: If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then it does not imply that (X, T_1) and (X, T_2) are fuzzy resolvable spaces. For, consider the fuzzy topological spaces (X, T_1) and (X, T_2) in the example 3.3. On computation, one can see that μ, $\mu \vee \delta$, $1-\alpha$ are the fuzzy dense sets in (X, T_1) and $cl(1-\mu) = 1-\mu \neq 1$; $cl\,(1-(\mu \vee \delta)) = 1-(\mu \vee \delta) \neq 1$ and $cl\,(1-(1-\alpha)) = cl(\alpha) = 1-(\mu \vee \delta) \neq 1$, in (X, T_1). Hence (X, T_1) is not a fuzzy resolvable space.

Also, the fuzzy dense sets in (X, T_2) are β, $\beta \vee \gamma$, $1-\lambda$ and $cl(1-\beta) = 1-\beta \neq 1$; $cl\,(1-(\beta \vee \gamma)) = 1-(\beta \vee \gamma) \neq 1$; $cl\,(1-(1-\lambda)) = cl(\lambda) = 1-(\beta \vee \gamma) \neq 1$, in (X, T_2). Hence (X, T_2) is not a fuzzy resolvable space. But, in the fuzzy bitopological space (X, T_1, T_2), there exists a pairwise fuzzy dense set $\beta \vee \gamma$ such that $1-(\beta \vee \gamma)$ is also a pairwise fuzzy dense set in (X, T_1, T_2). Thus, (X, T_1, T_2) is a pairwise fuzzy resolvable space, eventhough (X, T_1) and (X, T_2) are not fuzzy resolvable spaces.

Example 3.4 Let $X = \{a, b, c\}$. The fuzzy sets

$\lambda_1 : X \to [0,1]$ is defined as $\lambda_1(a) = 0.2$; $\lambda_1(b) = 0.3$; $\lambda_1(c) = 0.4$

$\lambda_2 : X \to [0,1]$ is defined as $\lambda_2(a) = 0.5$; $\lambda_2(b) = 0.4$; $\lambda_2(c) = 0.7$

$\lambda_3 : X \to [0,1]$ is defined as $\lambda_3(a) = 0.4$; $\lambda_3(b) = 0$; $\lambda_3(c) = 0$

$\mu_1 : X \to [0,1]$ is defined as $\mu_1(a) = 0.4$; $\mu_1(b) = 0$; $\mu_1(c) = 0$

$\mu_2 : X \to [0,1]$ is defined as $\mu_2(a) = 0.7$; $\mu_2(b) = 0.2$; $\mu_2(c) = 0$

$\mu_3 : X \to [0,1]$ is defined as $\mu_3(a) = 0.5$; $\mu_3(b) = 0$; $\mu_3(c) = 0.3$.

Clearly, $T_1 = \{0, \lambda_1, \lambda_2, 1\}$ and $T_2 = \{0, \mu_1, \mu_2, 1\}$ are fuzzy topologies on X. On computation, one can see that $cl_{T_1}(\lambda_2) = 1$, $cl_{T_1}(1-\mu_1) = 1$, $cl_{T_1}(1-\mu_2) = 1$, $cl_{T_1}(1-\mu_3) = 1$, $cl_{T_2}(\mu_2) = 1$ and $cl_{T_2}(1-\lambda_1) = 1$. Hence (X, T_1) and (X, T_2) are pairwise fuzzy irresolvable spaces. But, $cl_{T_1}cl_{T_2}(\lambda_1) = cl_{T_1}(1-\mu_2) = 1$ and $cl_{T_2}cl_{T_1}(\lambda_1) = cl_{T_2}(1-\lambda_1) = 1$, implies that λ_1 is a pairwise fuzzy dense set in (X, T_1, T_2). Also, $cl_{T_1}cl_{T_2}(1-\lambda_1) = cl_{T_1}(1) = 1$ and $cl_{T_2}cl_{T_1}(1-\lambda_1) = cl_{T_2}(1-\lambda_1) = 1$ in (X, T_1, T_2), implies that $1-\lambda_1$ is a pairwise fuzzy dense set in (X, T_1, T_2). Hence, for the pairwise fuzzy dense set λ_1 in (X, T_1, T_2), $cl_{T_i}cl_{T_j}(1-\lambda_1) = 1$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy resolvable space. Eventhough (X, T_1) and (X, T_2) are fuzzy irresolvable spaces, the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space.

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4. Characterizations of pairwise fuzzy resolvable spaces

Proposition 4.1 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ defined on X such that

(i) $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ in (X, T_1, T_2).

(ii) $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(1-\lambda) = 0 = int_{T_2}int_{T_1}(1-\lambda)$ in (X, T_1, T_2).

Proof: Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then there exists a fuzzy set λ defined on X such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}cl_{T_2}(1-\lambda) = 1 = cl_{T_2}cl_{T_1}(1-\lambda)$ in (X, T_1, T_2).

(i) Now, $cl_{T_1}cl_{T_2}(1-\lambda) = 1$, implies that $cl_{T_1}[cl_{T_2}(1-\lambda)] = cl_{T_1}[1 - int_{T_2}(\lambda)] = 1 - int_{T_1}int_{T_2}(\lambda) = 1$ in (X, T_1, T_2) and $cl_{T_2}[cl_{T_1}(1-\lambda)] = cl_{T_2}[1 - int_{T_1}(\lambda)] = 1 - int_{T_2}int_{T_1}(\lambda) = 1$, in (X, T_1, T_2). Hence $cl_{T_1}cl_{T_2}(1-\lambda) = 1 = cl_{T_2}cl_{T_1}(1-\lambda)$ in (X, T_1, T_2), implies that $1 - int_{T_1}int_{T_2}(\lambda) = 1 = 1 - int_{T_2}int_{T_1}(\lambda)$ in (X, T_1, T_2). Then, $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$, in (X, T_1, T_2). Thus, $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ in (X, T_1, T_2).

(ii) Now, $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ in (X, T_1, T_2), implies that $1 - cl_{T_1}cl_{T_2}(\lambda) = 0 = 1 - cl_{T_2}cl_{T_1}(\lambda)$ and hence by lemma 2.3, $int_{T_1}int_{T_2}(1-\lambda) = 0 = int_{T_2}int_{T_1}(1-\lambda)$, in (X, T_1, T_2). Thus, from (i) $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(1-\lambda) = 0 = int_{T_2}int_{T_1}(1-\lambda)$ in (X, T_1, T_2).

Proposition 4.2 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ defined on X such that $int_{T_1}(\lambda) \neq 0$; $int_{T_2}(\lambda) \neq 0$ and $cl_{T_1}(\lambda) \neq 1$; $cl_{T_2}(\lambda) \neq 1$, in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then there exists a fuzzy set λ defined on X such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}cl_{T_2}(1-\lambda) = 1 = cl_{T_2}cl_{T_1}(1-\lambda)$ in (X, T_1, T_2). Now $int_{T_1}cl_{T_2}(1-\lambda) < cl_{T_1}cl_{T_2}(1-\lambda)$ and $int_{T_2}cl_{T_1}(1-\lambda) < cl_{T_2}cl_{T_1}(1-\lambda)$ in (X, T_1, T_2). This implies that $int_{T_1}cl_{T_2}(1-\lambda) < 1$ and $int_{T_2}cl_{T_1}(1-\lambda) < 1$, in (X, T_1, T_2). Then, by lemma 2.3, $1 - cl_{T_1}int_{T_2}(\lambda) < 1$ and $1 - cl_{T_2}int_{T_1}(\lambda) < 1$, in (X, T_1, T_2) and hence $1 - 1 < cl_{T_1}int_{T_2}(\lambda)$ and $1 - 1 < cl_{T_2}int_{T_1}(\lambda)$, implies that $cl_{T_1}int_{T_2}(\lambda) \neq 0$ and $cl_{T_2}int_{T_1}(\lambda) \neq 0$ and hence $int_{T_2}(\lambda) \neq 0$ and $int_{T_1}(\lambda) \neq 0$, in (X, T_1, T_2). (For otherwise, if $int_{T_2}(\lambda) = 0$ and $int_{T_1}(\lambda) = 0$ in (X, T_1, T_2), then $cl_{T_1}int_{T_2}(\lambda) = cl_{T_1}(0) = 0$ and $cl_{T_2}int_{T_1}(\lambda) = cl_{T_2}(0) = 0$ in (X, T_1, T_2), a contradiction.)

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Now $int_{T_1}cl_{T_2}(\lambda) < cl_{T_1}cl_{T_2}(\lambda)$ and $int_{T_2}cl_{T_1}(\lambda) < cl_{T_2}cl_{T_1}(\lambda)$, implies that $int_{T_1}cl_{T_2}(\lambda) < 1$ and $int_{T_2}cl_{T_1}(\lambda) < 1$, in (X, T_1, T_2) and hence $1 - int_{T_1}cl_{T_2}(\lambda) > 0$ and $1 - int_{T_2}cl_{T_1}(\lambda) > 0$, in (X, T_1, T_2). Therefore $cl_{T_1}int_{T_2}(1 - \lambda) > 0$ and $cl_{T_2}int_{T_1}(1 - \lambda) > 0$, in (X, T_1, T_2). This implies that $int_{T_2}(1 - \lambda) \neq 0$ and $int_{T_1}(1 - \lambda) \neq 0$ and hence $1 - cl_{T_2}(\lambda) \neq 0$ and $1 - cl_{T_1}(\lambda) \neq 0$. That is, $cl_{T_2}(\lambda) \neq 1$ and $cl_{T_1}(\lambda) \neq 1$, in (X, T_1, T_2).

Proposition 4.3 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists atleast one pairwise fuzzy dense set λ in (X, T_1, T_2) such that λ is not a pairwise fuzzy open set in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Assume that each pairwise fuzzy dense set is a pairwise fuzzy open set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy resolvable space, by proposition 4.1, there exists a fuzzy set λ defined on X such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$, in (X, T_1, T_2). Hence, for the pairwise fuzzy dense set λ, $int_{T_1}int_{T_2}(\lambda) = 0 \neq \lambda$ and $int_{T_2}int_{T_1}(\lambda) = 0 \neq \lambda$, a contradiction to the assumption that λ is a pairwise fuzzy open set in (X, T_1, T_2) for which $int_{T_1}(\lambda) = \lambda$ and $int_{T_2}(\lambda) = \lambda$ and hence $int_{T_1}int_{T_2}(\lambda) = \lambda = int_{T_2}int_{T_1}(\lambda)$, in (X, T_1, T_2). Hence the assumption that each pairwise fuzzy dense set is a pairwise fuzzy open set, does not hold. Therefore, there must be atleast one pairwise fuzzy dense set λ in (X, T_1, T_2) such that λ is not a pairwise fuzzy open set in (X, T_1, T_2).

Proposition 4.4 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ defined on X such that $cl_{T_i}cl_{T_j}(\lambda) + int_{T_i}int_{T_j}(\lambda) = 1$ $(i \neq j$ and $i, j = 1, 2)$, in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then, by proposition 4.1, there exists a fuzzy set λ defined on X such that $cl_{T_i}cl_{T_j}(\lambda) = 1$ and $int_{T_i}int_{T_j}(\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$, in (X, T_1, T_2). Now $cl_{T_i}cl_{T_j}(\lambda) = 1 = 1 - 0 = 1 - int_{T_i}int_{T_j}(\lambda)$ in (X, T_1, T_2). Then $cl_{T_i}cl_{T_j}(\lambda) = 1 - int_{T_i}int_{T_j}(\lambda)$ in (X, T_1, T_2) and hence $cl_{T_i}cl_{T_j}(\lambda) + int_{T_i}int_{T_j}(\lambda) = 1$, in (X, T_1, T_2).

Proposition 4.5 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ in (X, T_1, T_2) such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}int_{T_2}(\lambda) \geq 0$ and $cl_{T_2}int_{T_1}(\lambda) \geq 0$ in (X, T_1, T_2).

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Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then, there exists a pairwise fuzzy dense set λ in (X, T_1, T_2) such that $cl_{T_1}cl_{T_2}(1-\lambda) = 1 = cl_{T_2}cl_{T_1}(1-\lambda)$. Now

$$\begin{aligned} cl_{T_1}int_{T_2}(\lambda) &= 1 - [1 - cl_{T_1}int_{T_2}(\lambda)] \\ &= 1 - int_{T_1}cl_{T_2}(1-\lambda) \\ &\geq 1 - cl_{T_1}cl_{T_2}(1-\lambda) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

and hence $cl_{T_1}int_{T_2}(\lambda) \geq 0$ in (X, T_1, T_2). Also,

$$\begin{aligned} cl_{T_2}int_{T_1}(\lambda) &= 1 - [1 - cl_{T_2}int_{T_1}(\lambda)] \\ &= 1 - int_{T_2}cl_{T_1}(1-\lambda) \\ &\geq 1 - cl_{T_2}cl_{T_1}(1-\lambda) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

and hence $cl_{T_2}int_{T_1}(\lambda) \geq 0$ in (X, T_1, T_2). Therefore, there exists a pairwise fuzzy dense set λ in (X, T_1, T_2) such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}int_{T_2}(\lambda) \geq 0$ and $cl_{T_2}int_{T_1}(\lambda) \geq 0$ in (X, T_1, T_2).

Proposition 4.6 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ defined on X such that $int_{T_i}int_{T_j}(\lambda \wedge (1-\lambda)) = 0$ ($i \neq j$ and $i, j = 1, 2$) in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then, there exists a pairwise fuzzy dense set λ in (X, T_1, T_2) such that $cl_{T_i}cl_{T_j}(1-\lambda) = 1$, in (X, T_1, T_2). That is, $cl_{T_i}cl_{T_j}(\lambda) = 1$ and $cl_{T_i}cl_{T_j}(1-\lambda) = 1$, in (X, T_1, T_2). Now

$$int_{T_i}int_{T_j}(1-\lambda) = 1 - cl_{T_i}cl_{T_j}(\lambda) = 1 - 1 = 0 \text{ and}$$
$$int_{T_i}int_{T_j}(\lambda) = 1 - \left(1 - int_{T_i}int_{T_j}(\lambda)\right) = 1 - \left(cl_{T_i}cl_{T_j}(1-\lambda)\right)$$
$$= 1 - 1 = 0 in (X, T_1, T_2).$$

Now $int_{T_i}int_{T_j}(\lambda \wedge (1-\lambda)) = int_{T_i}[int_{T_j}(\lambda \wedge (1-\lambda))]$
$$= int_{T_i}[int_{T_j}(\lambda) \wedge int_{T_j}(1-\lambda)]$$
$$= int_{T_i}int_{T_j}(\lambda) \wedge int_{T_i}int_{T_j}(1-\lambda) = 0 \wedge 0 = 0$$

and hence $int_{T_i}int_{T_j}(\lambda \wedge (1-\lambda)) = 0$, in (X, T_1, T_2).

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The following proposition gives a condition, in terms of pairwise fuzzy dense set, for a fuzzy bitopological space to be a pairwise fuzzy resolvable space.

Proposition 4.7 If there exists a fuzzy set λ defined on X such that $\lambda \wedge (1-\lambda)$ is a pairwise fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Proof. Let λ be a fuzzy set defined on X such that $cl_{T_i} cl_{T_j} (\lambda \wedge (1-\lambda)) = 1$ $(i \neq j$ and $i, j = 1, 2)$, in (X, T_1, T_2). Now

$$\begin{aligned} cl_{T_i} cl_{T_j} (\lambda \wedge (1-\lambda)) &= cl_{T_i} \left[cl_{T_j} (\lambda \wedge (1-\lambda)) \right] \\ &\leq cl_{T_i} \left[cl_{T_j}(\lambda) \wedge cl_{T_j}(1-\lambda) \right] \\ &\leq cl_{T_i} cl_{T_j}(\lambda) \wedge cl_{T_i} cl_{T_j}(1-\lambda). \end{aligned}$$

That is, $cl_{T_i} cl_{T_j} (\lambda \wedge (1-\lambda)) \leq cl_{T_i} cl_{T_j}(\lambda) \wedge cl_{T_i} cl_{T_j}(1-\lambda)$, in (X, T_1, T_2) and hence $1 \leq cl_{T_i} cl_{T_j}(\lambda) \wedge cl_{T_i} cl_{T_j}(1-\lambda)$, in (X, T_1, T_2). That is, $cl_{T_i} cl_{T_j}(\lambda) \wedge cl_{T_i} cl_{T_j}(1-\lambda) = 1$, in (X, T_1, T_2). This implies that $cl_{T_i} cl_{T_j}(\lambda) = 1$ and $cl_{T_i} cl_{T_j}(1-\lambda) = 1$, in (X, T_1, T_2). Thus, for a pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_i} cl_{T_j}(1-\lambda) = 1$, in (X, T_1, T_2) implies that (X, T_1, T_2) is a pairwise fuzzy resolvable space.

The following proposition establishes a condition for a fuzzy bitopological space to be a pairwise fuzzy resolvable space.

Proposition 4.8 If $\wedge_{k=1}^{N}(\mu_k) = 0$, where (μ_k)'s are pairwise fuzzy dense sets in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Proof. Suppose that $\wedge_{k=1}^{N}(\mu_k) = 0$, where $cl_{T_i} cl_{T_j}(\mu_k) = 1$ $(i \neq j$ and $i, j = 1, 2)$, in (X, T_1, T_2). Then $1 - \wedge_{k=1}^{N}(\mu_k) = 1 - 0 = 1$, in (X, T_1, T_2). This implies that $\vee_{k=1}^{N}(1-\mu_k) = 1$, in (X, T_1, T_2). Now $1 - cl_{T_i} cl_{T_j}(\mu_k) = 1 - 1 = 0$, in (X, T_1, T_2). This implies that $int_{T_i} int_{T_j}(1-\mu_k) = 0$, in (X, T_1, T_2). Let $\lambda_k = 1 - \mu_k$. Then, $\vee_{k=1}^{N}(\lambda_k) = 1$, where $int_{T_i} int_{T_j}(\lambda_k) = 0$, in (X, T_1, T_2). Hence, by theorem 2.9, (X, T_1, T_2) is a pairwise fuzzy resolvable space.

The following proposition gives a condition, in terms of pairwise fuzzy dense sets, for a fuzzy bitopological space to be a pairwise fuzzy resolvable space.

Proposition 4.9 If λ_1 and λ_2 are any two pairwise fuzzy dense sets in a fuzzy

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bitopological space (X, T_1, T_2) such that $\lambda_1 \leq (1 - \lambda_2)$, then (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Proof. Let λ_1 and λ_2 be any two pairwise fuzzy dense sets in (X, T_1, T_2) such that $\lambda_1 \leq (1 - \lambda_2)$. Now $\lambda_1 \leq (1 - \lambda_2)$ in (X, T_1, T_2), implies that $cl_{T_i} cl_{T_j}(\lambda_1) \leq cl_{T_i} cl_{T_j}(1 - \lambda_2)$ $(i \neq j$ and $i, j = 1, 2)$.........(A). Since λ_1 is a pairwise fuzzy dense set, $cl_{T_i} cl_{T_j}(\lambda_1) = 1$ in (X, T_1, T_2). Then, from (A), $1 \leq cl_{T_i} cl_{T_j}(1 - \lambda_2)$. That is, $cl_{T_i} cl_{T_j}(1 - \lambda_2) = 1$ in (X, T_1, T_2). Hence, there exists a pairwise fuzzy dense set λ_2 in (X, T_1, T_2) such that $cl_{T_i} cl_{T_j}(1 - \lambda_2) = 1$, in (X, T_1, T_2) and therefore (X, T_1, T_2) is a pairwise fuzzy resolvable space.

The following proposition gives a condition, in terms of pairwise fuzzy nowhere dense set, for a fuzzy bitopological space to be a pairwise fuzzy resolvable space.

Proposition 4.10 If there exists a pairwise fuzzy nowhere dense set λ in a fuzzy bitopological space (X, T_1, T_2) such that $int_{T_i} cl_{T_j}(1 - \lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$, then (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Proof. Suppose that there exists a pairwise fuzzy nowhere dense set λ in (X, T_1, T_2) such that $int_{T_i} cl_{T_j}(1-\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$. Now $int_{T_i} cl_{T_j}(1-\lambda) = 0$, implies that $1 - cl_{T_i} int_{T_j}(\lambda) = 0$, in (X, T_1, T_2). Then $cl_{T_i} int_{T_j}(\lambda) = 1$, in (X, T_1, T_2). But $cl_{T_i} int_{T_j}(\lambda) \leq cl_{T_i} cl_{T_j}(\lambda)$, in (X, T_1, T_2). Then, $1 \leq cl_{T_i} cl_{T_j}(\lambda)$ in (X, T_1, T_2). That is, $cl_{T_i} cl_{T_j}(\lambda) = 1$, in (X, T_1, T_2). Hence, λ is a pairwise fuzzy dense set in (X, T_1, T_2).

Since λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2), $int_{T_i} cl_{T_j}(\lambda) = 0$ in (X, T_1, T_2). But $int_{T_i} int_{T_j}(\lambda) \leq int_{T_i} cl_{T_j}(\lambda)$ in (X, T_1, T_2), implies that $int_{T_i} int_{T_j}(\lambda) \leq 0$, in (X, T_1, T_2). That is, $int_{T_i} int_{T_j}(\lambda) = 0$ in (X, T_1, T_2). Then, $1 - int_{T_i} int_{T_j}(\lambda) = 1 - 0 = 1$, in (X, T_1, T_2) and hence $cl_{T_i} cl_{T_j}(1-\lambda) = 1$ in (X, T_1, T_2). Therefore, $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Thus, there exists a pairwise fuzzy dense set λ in (X, T_1, T_2) such that $cl_{T_i} cl_{T_j}(1 - \lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$, in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Definition 4.11 Let λ be a fuzzy set defined on a non-empty set X. The $(1, 2)$-fuzzy boundary of λ and $(2, 1)$-fuzzy boundary of λ are defined as follows:

$$Bd_{12}(\lambda) = cl_{T_1} cl_{T_2}(\lambda) \wedge cl_{T_1} cl_{T_2}(1 - \lambda)$$

$$Bd_{21}(\lambda) = cl_{T_2} cl_{T_1}(\lambda) \wedge cl_{T_2} cl_{T_1}(1 - \lambda)$$

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If $Bd_{12}(\lambda) = Bd_{21}(\lambda) = \mu$, then μ is called the pairwise fuzzy boundary of λ in (X, T_1, T_2).

Proposition 4.12 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set μ in (X, T_1, T_2) such that $Bd_{12}(\mu) = 1$ and $Bd_{21}(\mu) = 1$ in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then, there exists a pairwise fuzzy dense set μ in (X, T_1, T_2) such that $cl_{T_1}cl_{T_2}(1-\mu) = 1 = cl_{T_2}cl_{T_1}(1-\mu)$. Now for the fuzzy set μ, $Bd_{12}(\mu) = cl_{T_1}cl_{T_2}(\mu) \wedge cl_{T_1}cl_{T_2}(1-\mu) = 1 \wedge 1 = 1$ and $Bd_{21}(\mu) = cl_{T_2}cl_{T_1}(\mu) \wedge cl_{T_2}cl_{T_1}(1-\mu) = 1 \wedge 1 = 1$ and hence $Bd_{12}(\mu) = 1$ and $Bd_{21}(\mu) = 1$ in (X, T_1, T_2).

Proposition 4.13 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ defined on X such that $Bd_{12}(1-\lambda) = 1 = Bd_{21}(1-\lambda)$ in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy resolvable space. Then, by proposition 4.1, there exists a fuzzy set λ defined on X such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ in (X, T_1, T_2). Now

$$\begin{aligned} Bd_{12}(1-\lambda) &= cl_{T_1}cl_{T_2}(1-\lambda) \wedge cl_{T_1}cl_{T_2}\,(1-(1-\lambda)) \\ &= cl_{T_1}cl_{T_2}(1-\lambda) \wedge cl_{T_1}cl_{T_2}(\lambda) \\ &= cl_{T_1}cl_{T_2}(1-\lambda) \wedge 1 \\ &= cl_{T_1}cl_{T_2}(1-\lambda) \\ &= 1 - int_{T_1}int_{T_2}(\lambda) \\ &= 1-0 \\ &= 1 \end{aligned}$$

and hence $Bd_{12}(1-\lambda) = 1$ in (X, T_1, T_2).

$$\begin{aligned} Bd_{21}(1-\lambda) &= cl_{T_2}cl_{T_1}(1-\lambda) \wedge cl_{T_2}cl_{T_1}\,(1-(1-\lambda)) \\ &= cl_{T_2}cl_{T_1}(1-\lambda) \wedge cl_{T_2}cl_{T_1}(\lambda) \\ &= cl_{T_2}cl_{T_1}(1-\lambda) \wedge 1 \end{aligned}$$

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$$
\begin{aligned}
&= cl_{T_2}cl_{T_1}(1-\lambda)\\
&= 1-int_{T_2}int_{T_1}(\lambda)\\
&= 1-0\\
&= 1
\end{aligned}
$$

and hence $Bd_{21}(1-\lambda) = 1$ in (X, T_1, T_2). Thus, $Bd_{12}(1-\lambda) = 1 = Bd_{21}(1-\lambda)$ in (X, T_1, T_2).

The following proposition gives a condition, in terms of pairwise fuzzy boundary of fuzzy set, for a fuzzy bitopological space to be a pairwise fuzzy resolvable space.

Proposition 4.14 If the fuzzy set 1_X is the pairwise fuzzy boundary of some fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy resolvable space.

Proof. Suppose that the fuzzy set 1_X is the pairwise fuzzy boundary of some fuzzy set λ in (X, T_1, T_2). Then $Bd_{12}(\lambda) = 1$ and $Bd_{21}(\lambda) = 1$, in (X, T_1, T_2). That is, $cl_{T_1}cl_{T_2}(\lambda) \wedge cl_{T_1}cl_{T_2}(1-\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) \wedge cl_{T_2}cl_{T_1}(1-\lambda) = 1$, in (X, T_1, T_2). These are possible only if $cl_{T_1}cl_{T_2}(\lambda) = 1$, $cl_{T_1}cl_{T_2}(1-\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$, $cl_{T_2}cl_{T_1}(1-\lambda) = 1$, in (X, T_1, T_2). That is, $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}cl_{T_2}(1-\lambda) = 1 = cl_{T_2}cl_{T_1}(1-\lambda)$, in (X, T_1, T_2) and hence λ and $1-\lambda$ are pairwise fuzzy dense sets in (X, T_1, T_2). Hence there exists a pairwise fuzzy dense λ in (X, T_1, T_2) such that $cl_{T_i}cl_{T_j}(1-\lambda) = 1$ ($i \neq j$ and $i, j = 1, 2$) in (X, T_1, T_2). Therefore, (X, T_1, T_2) is a pairwise fuzzy resolvable space.

5. Conclusion

The concept of pairwise fuzzy resovable spaces have studied in detail.

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