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Ulam Stability of new type additive functional equation in Multi-Banach Spaces

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Abstract

In this Paper, we establish the General solution and stability of a new type additive functional equation is of the form $f(2x+5y) - f(x+4y) = f(x) + f(y)$ in Multi-Banach Space in the sense of Hyers-Ulam.

Key words: Hyers-Ulam stability, Multi-Banach Spaces, Additive Functional Equation, Fixed Point Method.

AMS classification: 39B52, 32B72,32B82

1. Introduction

The stability problem of functional equation was raised by S.M. Ulam [17] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. Note that the affirmtive solution to this question was given in the next year by D.H. Hyers [14] in 1941. In the year 1950, T. Aoki [1] generalized Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M.Rassias [13] for linear mappings by considering an unbounded Cauchy difference. In 1994, a further generalization of Th.M. Rassias theorem was obtained by P.Gavruta [4].
Then, the stability problem of several functional equations has been extensively investigated by a many number of authors, and there are many interesting results concerning this problem ([3, 5, 6, 8, 16, 9, 10, 11, 12, 15]). Motivated by the above discussions, we prove the general solution of a new type additive functional equation

$$f(2x+5y) - f(x+4y) = f(x) + f(y). \tag{1}$$

Also, we investigate the Hyers-Ulam Stability of the above functional equation in Multi-Banach Spaces by using fixed point method.

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$$Df(x,y) = f(2x+5y) - f(x+4y) - f(x) - f(y).$$

Now, Let us recall the definitions of Multi-Banach Space.

Definition 1.1 [2] A Multi- norm on $\left\{\mathcal{A}^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\mathcal{A}^k$ for each $k \in \mathbb{N}, \|x\|_1 = \|x\|$ for each $x \in \mathcal{A}$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

1. $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right)\right\|_k = \|(x_1 ... x_k)\|_k$, for $\sigma \in \Psi_k, x_1, ..., x_k \in \mathcal{A}$;
2. $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1 ... x_k)\|_k$
   for $\alpha_1 ... \alpha_k \in \mathbb{C}, x_1, ..., x_k \in \mathcal{A}$;
3. $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$, for $x_1, ..., x_{k-1} \in \mathcal{A}$;
4. $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$ for $x_1, ..., x_{k-1} \in \mathcal{A}$.

2. General Solution of Additive Fuctional Equation (1)

In this section, we prove the general solution of the additive functional equation (1). Throughout this section, we consider $\mathcal{A}$ and $\mathcal{B}$ are real vector spaces.

Theorem 2.1 A mapping $f : \mathcal{A} \to \mathcal{B}$ is additive if and only if f satisfies the functional equation (1) for all $x, y \in \mathcal{A}$.

Proof: Suppose that f is additive. Let us consider the additive functional equation

$$f(x+y) = f(x) + f(y) \tag{2}$$

for all $x, y \in \mathcal{A}$. Letting $x = y = 0$ in (2), we get $f(0) = 0$. Replacing $x = x, y = -x$ in (2), we get $f(-x) = -f(x)$. Hence, f is an odd mapping. Replacing $x = x, y = 2x$ in (2), we get $f(2x) = 2f(x)$ and also, replacing $x = x, y = nx$ in (2), we arrive at $f(nx) = nf(x)$ respectively. Setting $x = x + 4y, y = x + y$ in (2), we obtain that

$$f(2x+5y) = f(x+4y) + f(x+y) \tag{3}$$

for all $x, y \in \mathcal{A}$. Using (2), we get the functional equation (1). Conversely, assume that f satisfies the functional equation (1). Letting $x = y = 0$ in (1), we get $f(0) = 0$. Replacing $(x, y) = (x, 0)$ and $(0, x)$ in (1), we obtain $f(2x) = 2f(x)$ and $f(3x) = 3f(x)$ respectively. Setting $(x, y) = (5x, 2y)$ in (1), we get

$$10f(x+y) - f(5x+8y) = 5f(x) + 2f(y) \tag{4}$$

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for all $x, y \in \mathcal{A}$. Setting $(x, y) = (4x, y)$ in (1), we get

$$f(8x + 5y) - 4f(x + y) = 4f(x) + f(y) \tag{5}$$

for all $x, y \in \mathcal{A}$. Interchanging x and y in (5), we arrive at

$$f(5x + 8y) - 4f(x + y) = 4f(y) + f(x) \tag{6}$$

for all $x, y \in \mathcal{A}$. Adding (4) and (6), we arrive at (2). This completes the proof.

3. Hyers-Ulam Stability of Additive Functional Equation (1) in Multi-Banach Spaces

Theorem 3.1 Let $\mathcal{A}$ be a linear space and let $\left((\mathcal{B}^k, \|.\|_k) : k \in \mathbb{N}\right)$ be a Multi-Banach space. Suppose that η is a non-negative real number and $f : \mathcal{A} \to \mathcal{B}$ be a function satisfying the inequality

$$\sup_{k\in\mathbb{N}} \|(\mathcal{D}f(x_1, y_1), ..., \mathcal{D}f(x_k, y_k))\|_k \leq \eta \tag{7}$$

$\forall x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$. Then there exists a unique Additive mapping $\mathbb{A} : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathbb{A}(x_1), ..., f(x_k) - \mathbb{A}(x_k))\|_k \leq \eta. \tag{8}$$

Proof: Letting $y_i = 0$ where $i = 1, 2...k$ in (7), we arrive

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 2f(x_1), ..., f(2x_k) - 2f(x_k))\| \leq \eta \tag{9}$$

Dividing on both sides by 2 in (9), we arrive at

$$\sup_{k\in\mathbb{N}} \left\| \left(f(x_1) - \frac{1}{2}f(2x_1), .., f(x_k) - \frac{1}{2}f(2x_k) \right) \right\| \leq \frac{\eta}{2} \tag{10}$$

Let $\Psi = \{l : \mathcal{A} \to \mathcal{B} | l(0) = 0\}$ and introduce the generalized metric d defined on Ψ by

$$d(l, m) = \inf \left\{ \Psi \in [0, \infty] | \sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \Psi \quad \forall \quad x_1, ..., x_k \in \mathcal{A} \right\}$$

Then it is easy to show that (Ψ, d) is a generalized complete metric space, See [7].

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We define an operator $\mathcal{J} : \Psi \to \Psi$ by

$$\mathcal{J}l(x) = \frac{1}{2}l(2x), \qquad \forall x \in \mathcal{A}$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l, m \in \Psi$, let $\Psi \in [0, \infty]$ be an arbitary constant with $d(l, m) \leq \Psi$. From the definition if follows that

$$\sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \Psi \quad \forall x_1, ..., x_k \in \mathcal{A}.$$

Therefore, $\sup_{k\in\mathbb{N}} \|(\mathcal{J}l(x_1) - \mathcal{J}m(x_1), ..., \mathcal{J}l(x_k) - \mathcal{J}m(x_k))\|_k \leq \frac{1}{2}\Psi$
$\forall x_1, ..., x_k \in \mathcal{A}$. Hence,it holds that

$$d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{2}\Psi d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{2}d(l, m)$$

$\forall l, m \in \Psi$. This Means that $\mathcal{J}$ is strictly contractive operator on Ψ with the Lipschitz constant $L = \frac{1}{2}$.
By (10), we have $d(\mathcal{J}f, f) \leq \frac{\eta}{2}$. Applying the Theorem 2.2 in [18], we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $\mathbb{A} : \mathcal{A} \to \mathcal{B}$ such that

$$\mathbb{A}(2x) = 2\mathbb{A}(x) \qquad \forall x \in \mathcal{A}.$$

Moreover, we have $d(\mathcal{J}^n f, \mathbb{A}) \to 0$, gives that

$$\mathbb{A}(a) = \lim_{n\to\infty} \mathcal{J}^n f(a) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{A}$. Also, $d(f, \mathbb{A}) \leq \frac{1}{1 - \mathcal{L}} d(\mathcal{J}f, f)$ gives the inequality

$$d(f, \mathbb{A}) \leq \eta.$$

Doing $x_1 =, ..., = x_k = 2^n x$ and $y_1 =, ..., = y_k = 2^n y$ in (7) and dividing by 2^n. Now, applying the property (a) of multi-norms, we have
$\|\mathcal{D}f(2^n x, 2^n y)\| \leq \lim_{n\to\infty} \frac{1}{2^n} = 0$ for all $x, y \in \mathcal{A}$. The uniqueness of $\mathbb{A}$ follows from the fact that $\mathbb{A}$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathbb{A}(x_1), ..., f(x_k) - \mathbb{A}(x_k))\|_k \leq \ell$$

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for all $x_1, ..., x_k \in \mathcal{A}$. Hence the proof.

Corollary 3.2 Let $\mathcal{A}$ be a linear space, and let $(\mathcal{B}^n, \|.\|_n)$ be a Multi-Banach Spaces. Let $\theta > 0$, $0 < r < 1$ and $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $f(0) = 0$

$$\sup_{k\in\mathbb{N}} \|Df(x_1, y_1), ..., Df(x_k, y_k)\|_k \leq \theta \left(\|x_1\|^r + \|y_1\|^r, ..., \|x_k\|^r + \|y_k\|^r\right) \quad (11)$$

for all $x_1, ..., x_k, y_1 ..., y_k \in \mathcal{A}$. Then there exists a unique mapping $\mathbb{A} : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathbb{A}(x_1), ..., f(x_k) - \mathbb{A}(x_k))\| \leq \frac{1}{2 - 2^r}(\|x_1\|^r, ..., \|x_k\|^r) \quad (12)$$

Proof: Proof is similar to that of Theorem 3.1 replacing η by $\theta \left(\|x_1\|^r + \|y_1\|^r, .., \|x_k\|^r + \|y_k\|^r\right)$, we arrive the result.

Corollary 3.3 Let $\mathcal{A}$ be a linear space, and let $(\mathcal{B}^n, \|.\|_n)$ be a multi-Banach space. Let $\theta > 0, 0 < p + q = r < 1$ and $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $f(0) = 0$

$$\sup_{k\in\mathbb{N}} \|\mathcal{D}f(x_1, y_1), ..., \mathcal{D}f(x_k, y_k)\|_k \leq \theta \left(\|x_1\|^p . \|y_1\|^q, .., \|x_k\|^p . \|y_k\|^q\right) \quad (13)$$

for all $x_1, ..., x_k, y_1 ..., y_k \in \mathcal{A}$. Then there exists a unique mapping $\mathbb{A} : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathbb{A}(x_1), ..., f(x_k) - \mathbb{A}(x_k))\| \leq \frac{1}{2 - 2^r}(\|x_1\|^{p+q}, ..., \|x_k\|^{p+q}) \quad (14)$$

Proof: Proof is similar to that of Theorem 3.1 replacing η by $\theta \left(\|x_1\|^p . \|y_1\|^q, .., \|x_k\|^p . \|y_k\|^q\right)$ we arrive the result.

Theorem 3.4 Let $\phi : X^{2k} \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\phi(x_1, y_1, ..., x_k, y_k) \leq 2L\phi(\frac{x_1}{2}, \frac{y_1}{2}, ..., \frac{x_k}{2}, \frac{y_k}{2}) \quad (15)$$

for all $x_i, y_i \in \mathcal{A}$, where $i = 1, 2, .., k$. If $f : \mathcal{A} \to \mathcal{B}$ is a mapping with $f(0) = 0$ satisfies inequality

$$\|Df(x_1, y_1), ..., Df(x_k, y_k)\| \leq \phi(x_1, y_1, ..., x_k, y_k) \quad (16)$$

for all $x_i, y_i \in X$, where $i = 1, 2, .., k$. Then there exists a unique additive mapping

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$\mathbb{A}: \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x_1) - \mathbb{A}(x_1), ..., f(x_k) - \mathbb{A}(x_k)\| \leq \frac{L}{1-L}\phi(x_1, 0, .., x_k, 0) \tag{17}$$

for all $x_i, y_i \in X$, where $i = 1, 2, .., k$.

Proof: Letting $y_i = 0$, where $i = 1, 2, .., k$ in (16), we arrive at

$$\|f(2x_1) - 2f(x_1), ..., f(2x_k) - 2f(x_k)\| \leq \phi(x_1, 0, ..., x_k, 0) \tag{18}$$

for all $x_1, x_2, ..., x_k \in X$. And also diving by 2 in above equation, we get

$$\left\|f(x_1) - \frac{1}{2}f(2x_1), ..., f(x_k) - \frac{1}{2}f(2x_k)\right\| \leq \frac{1}{2}\phi(x_1, 0, .., x_k, 0) \tag{19}$$

for all $x_1, x_2, ..., x_k \in X$. Consider the set $S := \{g : \mathcal{A} \to \mathcal{B} : g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f, g) = inf\left\{\mu \in \mathbb{R} : \|g(x_1) - h(x_1), ..., g(x_k) - h(x_k)\| \leq \mu\phi(x_1, 0, ..., x_k, 0)\right\}$$

where $\inf \varphi = \infty$. It is easy to show that (S, d) is complete. Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) = \frac{1}{2}f(2x) \quad \forall x \in X.$$

Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then we have

$$\begin{aligned}\|Jg(x_1) - Jh(x_1), ..., Jg(x_k) - Jh(x_k)\| &\leq \left\|\frac{1}{2}g(2x_1) - \frac{1}{2}h(2x_1), .., \frac{1}{2}g(2x_k) - \frac{1}{2}h(2x_k)\right\| \\ &\leq L\phi\left(x_1, 0, .., x_k, 0\right)\end{aligned}$$

for all $x_1, x_2, ..., x_k \in X$. Thus $d(g, h) = \epsilon$ gives that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (19) that $m(g, Jg) \leq L$.
By Theorem, there exists a mapping $\mathbb{A} : \mathcal{A} \to \mathcal{B}$ satisfying the following:
(1) $\mathbb{A}$ is a fixed point of J, i.e.

$$\mathbb{A}(2x) = 2\mathbb{A}(x) \tag{20}$$

for all $x \in X$. The mapping $\mathbb{A}$ is a unique fixed point of J in the set $M = \{g \in S : d(g, h) < \infty\}$. This gives that $\mathbb{A}$ is a unique mapping satisfying (20)

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such that there exists a $\mu \in (0, \infty)$ satisfying the inequality

$$\|f(x_1) - \mathbb{A}(x_1), .., f(x_k) - \mathbb{A}(x_k)\| \leq \mu\phi(x_1, 0, .., x_k, 0)$$

for all $x_1, x_2, ..., x_k \in X$.
(2). $m(J^n f, \mathbb{A}) \to 0$ as $n \to \infty$, which gives that

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \mathbb{A}(x) \quad \forall x \in X.$$

(3). $m(f, \mathbb{A}) \leq \frac{1}{1-L} m(f, Jf)$ which gives the inequality $m(f, \mathbb{A}) \leq \frac{L}{1-L}$. Thus that the inequality (17) holds. It follows from (15) and (16) that $\|D\mathbb{A}(x, y)\| = 0$. Hence $\mathbb{A} : \mathcal{A} \to \mathcal{B}$ is a additive mapping.

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