On Fuzzy Regular Volterra Spaces

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Abstract

The aim of this paper is to introduce the concepts of regular $G_\delta$-sets, regular $F_\sigma$-sets and regular Volterra spaces in fuzzy setting are introduced and studied. Several characterizations of fuzzy regular Volterra spaces in terms of fuzzy regular $F_\sigma$-sets, fuzzy first category sets, fuzzy residual sets and fuzzy $\sigma$-nowhere dense sets are also established in this paper.

Key words: Fuzzy open set, fuzzy dense set, fuzzy nowhere dense set, fuzzy $\sigma$-nowhere dense set, fuzzy $G_\delta$-set, fuzzy $F_\sigma$-set, fuzzy first category set, fuzzy residual set, fuzzy $\beta$-open set and fuzzy Volterra spaces.

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1. Introduction

In 1970, J. Mack [6] introduced the concepts of regular $G_\delta$-sets and regular $F_\sigma$-sets in classical topology. K.K. Azad [1] introduced fuzzy regular open and fuzzy regular closed sets in 1981. The concepts of regular $G_\delta$-sets and regular $F_\sigma$-sets in fuzzy setting are introduced and studied in this paper. By using fuzzy regular $G_\delta$-sets, the concept of fuzzy regular Volterra spaces is introduced in this paper. Several characterizations of fuzzy regular Volterra spaces in terms of fuzzy regular $F_\sigma$-sets, fuzzy first category sets, fuzzy residual sets and fuzzy $\sigma$-nowhere dense sets are also established in this paper.

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2. Preliminaries

In 1965, L.A. Zadeh [10] introduced the concept of fuzzy set $\lambda$ on a base set $X$ as a function from $X$ into the unit interval $I = [0, 1]$. This function is also called a membership function. A membership function is a generalization of a characteristic function.

**Definition 2.1** [5] Let $\lambda$ and $\mu$ be fuzzy sets in $X$. Then for all $x \in X$,

1. $\lambda = \mu \iff \lambda(x) = \mu(x)$,
2. $\lambda \leq \mu \iff \lambda(x) \leq \mu(x)$,
3. $\psi = \lambda \lor \mu \iff \psi(x) = \max\{\lambda(x), \mu(x)\}$,
4. $\delta = \lambda \land \mu \iff \delta(x) = \min\{\lambda(x), \mu(x)\}$,
5. $\eta = \lambda^c \iff \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i/i \in I\}$ of fuzzy sets in $X$, the union $\psi = \lor_i \lambda_i$ and intersection $\delta = \land_i \lambda_i$ are defined by $\psi(x) = \sup_i \{\lambda_i(x), x \in X\}$, and $\delta(x) = \inf_i \{\lambda_i(x), x \in X\}$.

The fuzzy set $0_X$ is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set $1_X$ defined as $1_X(x) = 1$, for all $x \in X$.

**Definition 2.2** [5] A fuzzy topology is a family $\mathcal{T}$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $\Phi, X \in \mathcal{T}$,
2. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
3. If $A_i \in \mathcal{T}$, for each $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$.

$\mathcal{T}$ is called a fuzzy topology for $X$ and the pair $(X, \mathcal{T})$ is a fuzzy topological space or fts in short. Every member of $\mathcal{T}$ is called a $\mathcal{T}$-open fuzzy set. A fuzzy set is $\mathcal{T}$-closed if and only if its complement is $\mathcal{T}$-open. When no confusion is likely to arise, we shall call a $\mathcal{T}$-open ($\mathcal{T}$-closed) fuzzy set simply an open (closed) fuzzy set.

**Lemma 2.3** [1] For a family $\mathcal{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space $X$. Then, $\lor cl \lambda_\alpha \leq cl(\lor \lambda_\alpha)$. In case $\mathcal{A}$ is a finite set, $\lor cl \lambda_\alpha = cl(\lor \lambda_\alpha)$. Also $\lor int \lambda_\alpha \leq int(\lor \lambda_\alpha)$.

**Definition 2.4** [2] A fuzzy set $\lambda$ in a fuzzy topological space $(X, \mathcal{T})$ is called a fuzzy $F_\alpha$-set in $(X, \mathcal{T})$ if $\lambda = \lor_{i=1}^\infty (\lambda_i)$, where $1 - \lambda_i \in \mathcal{T}$ for $i \in I$. 
Definition 2.5 [2] A fuzzy set \( \lambda \) in a fuzzy topological space \((X, T)\) is called a fuzzy \( G_\delta \)-set in \((X, T)\) if \( \lambda = \bigwedge_{i=1}^{\infty}(\lambda_i) \), where \( \lambda_i \in T \) for \( i \in I \).

Definition 2.6 [7] A fuzzy set \( \lambda \) in a fuzzy topological space \((X, T)\) is called a fuzzy dense set if there exists no fuzzy closed set \( \mu \) in \((X, T)\) such that \( \lambda < \mu < 1 \).

Definition 2.7 [7] Let \((X, T)\) be a fuzzy topological space. A fuzzy set \( \lambda \) in \((X, T)\) is called a fuzzy nowhere dense set if there exists no non-zero fuzzy open set \( \mu \) in \((X, T)\) such that \( \mu < \text{cl}(\lambda) \). That is, \( \text{int} \text{cl}(\lambda) = 0 \).

Definition 2.8 [7] Let \((X, T)\) be a fuzzy topological space. A fuzzy set \( \lambda \) in \((X, T)\) is called a fuzzy first category set if \( \lambda = \bigvee_{i=1}^{\infty}(\lambda_i) \), where \((\lambda_i)\)'s are fuzzy nowhere dense sets in \((X, T)\). Any other fuzzy set in \((X, T)\) is said to be of fuzzy second category.

Definition 2.9 [7] Let \( \lambda \) be a fuzzy first category set in a fuzzy topological space \((X, T)\). Then \( 1 - \lambda \) is called a fuzzy residual set in \((X, T)\).

Definition 2.10 [8] Let \((X, T)\) be a fuzzy topological space. A fuzzy set \( \lambda \) in \((X, T)\) is called a fuzzy \( \sigma \)-nowhere dense set if \( \lambda \) is a fuzzy \( F_\sigma \)-set in \((X, T)\) such that \( \text{int} \text{cl}(\lambda) = 0 \).

Definition 2.11 A fuzzy set \( \lambda \) in a fuzzy topological space \( X \) is called

1. fuzzy pre-open if \( \lambda \leq \text{int cl}(\lambda) \) and fuzzy pre-closed if \( \text{cl int}(\lambda) \leq \lambda \).
2. fuzzy semi-open if \( \lambda \leq \text{cl int}(\lambda) \) and fuzzy semi-closed if \( \text{int cl}(\lambda) \leq \lambda \).
3. fuzzy \( \beta \)-open if \( \lambda \leq \text{cl int cl}(\lambda) \) and fuzzy \( \beta \)-closed if \( \text{int cl int}(\lambda) \leq \lambda \).
4. fuzzy regular open if \( \text{int cl}(\lambda) = \lambda \) and fuzzy regular closed if \( \text{cl int}(\lambda) = \lambda \).

Definition 2.12 [9] A fuzzy topological space \((X, T)\) is called a fuzzy Volterra space if \( \text{cl}(\bigwedge_{i=1}^{\infty}(\lambda_i)) = 1 \), where \((\lambda_i)\)'s are fuzzy dense and fuzzy \( G_\delta \)-sets in \((X, T)\).

Theorem 2.13 [1] In a fuzzy topological space \((X, T)\),

(a). The closure of a fuzzy open set is a fuzzy regular closed set
(b). The interior of a fuzzy closed set is a fuzzy regular open set.

3. Fuzzy regular \( G_\delta \)-sets

Definition 3.1 A fuzzy set \( \lambda \) in a fuzzy topological space \((X, T)\) is called a fuzzy regular \( G_\delta \)-set if \( \lambda = \bigwedge_{i=1}^{\infty}(\text{int}(\lambda_i)) \), where \( 1 - \lambda_i \in T \).
Definition 3.2 A fuzzy set \( \mu \) in a fuzzy topological space \((X, T)\) is called a fuzzy regular \( F_\sigma \)-set if \( \mu = \vee_{i=1}^{\infty} (cl(\mu_i)) \), where \( \mu_i \in T \).

Proposition 3.3 If \( \lambda \) is a fuzzy regular \( G_\delta \)-set in a fuzzy topological space \((X, T)\) if and only if \( 1 - \lambda \) is a fuzzy regular \( F_\sigma \)-set in \((X, T)\).

Proof: Let \( \lambda \) be a fuzzy regular \( G_\delta \)-set in \((X, T)\). Then \( \lambda = \bigwedge_{i=1}^{\infty} (int(\lambda_i)) \), where \( 1 - \lambda_i \in T \). Now \( 1 - \lambda = 1 - \bigwedge_{i=1}^{\infty} (int(\lambda_i)) = \bigvee_{i=1}^{\infty} (1 - int(\lambda_i)) = \bigvee_{i=1}^{\infty} (cl(1 - \lambda_i)). \) Let \( \mu_i = 1 - \lambda_i \). Then \( \mu_i \in T \). Hence \( 1 - \lambda = \bigvee_{i=1}^{\infty} (cl(\mu_i)), \mu_i \in T \). Therefore \( 1 - \lambda \) is a fuzzy regular \( F_\sigma \)-set in \((X, T)\).

Conversely, let \( \lambda \) be a fuzzy regular \( F_\sigma \)-set in \((X, T)\). Then \( \lambda = \bigvee_{i=1}^{\infty} (cl(\mu_i)), \mu_i \in T \). Now \( 1 - \lambda = 1 - \bigvee_{i=1}^{\infty} (cl(\mu_i)) = \bigwedge_{i=1}^{\infty} (1 - cl(\mu_i)) = \bigwedge_{i=1}^{\infty} (int(1 - \mu_i)). \) Let \( 1 - \mu_i = \lambda_i \). Then implies that \( \mu_i = 1 - \lambda_i \) and \( 1 - \lambda_i \in T \). Hence \( 1 - \lambda = \bigwedge_{i=1}^{\infty} (int(\lambda_i)) \), where \( 1 - \lambda_i \in T \). Therefore \( 1 - \lambda \) is a fuzzy regular \( G_\delta \)-set in \((X, T)\).

Proposition 3.4 Let \((X, T)\) be a fuzzy topological space.

(1). If \( \lambda \) is a fuzzy regular \( G_\delta \)-set in \((X, T)\), then \( \lambda = \bigwedge_{i=1}^{\infty} (\delta_i) \), where \( (\delta_i)'s \) are fuzzy regular open sets in \((X, T)\).

(2). If \( \lambda \) is a fuzzy regular \( F_\sigma \)-set in \((X, T)\), then \( \lambda = \bigvee_{i=1}^{\infty} (\mu_i) \), where \( (\mu_i)'s \) are fuzzy regular closed sets in \((X, T)\).

Proof: (1). Let \( \lambda \) be a fuzzy regular \( G_\delta \)-set in \((X, T)\). Then \( \lambda = \bigwedge_{i=1}^{\infty} (int(\lambda_i)) \), where \( 1 - \lambda_i \in T \). Now \( 1 - \lambda_i \in T \) implies that \( \lambda_i \) is a fuzzy closed set in \((X, T)\). By theorem 2.13, \( int(\lambda_i) \) is a fuzzy regular open set in \((X, T)\). Let \( \delta_i = int(\lambda_i) \). Then \( \lambda = \bigwedge_{i=1}^{\infty} (\delta_i) \), where \( (\delta_i)'s \) are fuzzy regular open sets in \((X, T)\).

(2). Let \( \lambda \) be a fuzzy regular \( F_\sigma \)-set in \((X, T)\). Then \( \lambda = \bigvee_{i=1}^{\infty} (cl(\mu_i)) \), where \( \mu_i \in T \). Now \( \mu_i \in T \) implies that \( cl(\mu_i) \) is a fuzzy regular closed set in \((X, T)\). Let \( \eta_i = cl(\mu_i) \). Then \( \lambda = \bigvee_{i=1}^{\infty} (\eta_i) \), where \( (\eta_i)'s \) are fuzzy regular closed sets in \((X, T)\).

Proposition 3.5 If \( \lambda \) is a fuzzy regular \( G_\delta \)-set in a fuzzy topological space \((X, T)\), then \( \lambda \) is a fuzzy \( G_\delta \)-set in \((X, T)\).

Proof: Let \( \lambda \) be a fuzzy regular \( G_\delta \)-set in \((X, T)\). Then by proposition 3.4, \( \lambda = \bigwedge_{i=1}^{\infty} (\delta_i) \), where \( (\delta_i)'s \) are fuzzy regular open sets in \((X, T)\). Since every fuzzy regular
open set is a fuzzy open set in \((X, T)\), \((\delta_i)\)'s are fuzzy open sets in \((X, T)\). Hence \(\lambda = \bigwedge_{i=1}^{\infty}(\delta_i)\), where \(\delta_i \in T\). Therefore \(\lambda\) is a fuzzy \(G_\delta\)-set in \((X, T)\).

**Proposition 3.6** If \(\lambda\) is a fuzzy regular \(F_\sigma\)-set in a fuzzy topological space \((X, T)\), then \(\lambda\) is a fuzzy \(F_\sigma\)-set in \((X, T)\).

Proof: Let \(\lambda\) be a fuzzy regular \(F_\sigma\)-set in \((X, T)\). Then by proposition 3.4 \(\lambda = \bigvee_{i=1}^{\infty}(\eta_i)\), where \((\eta_i)\)'s are fuzzy regular closed sets in \((X, T)\). Since every fuzzy regular closed set is a fuzzy closed set in \((X, T)\), \((\eta_i)\)'s are fuzzy closed sets in \((X, T)\). Hence \(\lambda = \bigvee_{i=1}^{\infty}(\eta_i)\), where \(1 - \eta_i \in T\). Therefore \(\lambda\) is a fuzzy \(F_\sigma\)-set in \((X, T)\).

**Proposition 3.7** If \(cl(\bigwedge_{i=1}^{\infty}int(\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy closed sets in a fuzzy topological space \((X, T)\), then \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X, T)\).

Proof: Suppose that \(cl(\bigwedge_{i=1}^{\infty}int(\lambda_i)) = 1\), where \((\lambda_i)\)'s are fuzzy closed sets in \((X, T)\). But \(cl(\bigwedge_{i=1}^{\infty}int(\lambda_i)) \leq \bigwedge_{i=1}^{\infty}clint(\lambda_i)\). Then, \(1 \leq \bigwedge_{i=1}^{\infty}clint(\lambda_i)\). That is, \(\bigwedge_{i=1}^{\infty}clint(\lambda_i) = 1\). This implies that \(clint(\lambda_i) = 1\), \(\ldots\), (1). Since \((\lambda_i)\)'s are fuzzy closed sets in \((X, T)\), \(cl(\lambda_i) = \lambda_i\). Then \(clintcl(\lambda_i) = 1\). From (1), \(\lambda_i \leq clintcl(\lambda_i)\). Therefore, \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X, T)\).

**Proposition 3.8** If a fuzzy regular \(G_\delta\)-set \(\lambda\) is a fuzzy dense set in a fuzzy topological space \((X, T)\), then \(\lambda = \bigwedge_{i=1}^{\infty}int(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X, T)\).

Proof: Let \(\lambda\) be a fuzzy regular \(G_\delta\)-set in \((X, T)\) such that \(cl(\lambda) = 1\). Then \(\lambda = \bigwedge_{i=1}^{\infty}int(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy closed sets in \((X, T)\) and \(cl[\bigwedge_{i=1}^{\infty}int(\lambda_i)] = cl(\lambda) = 1\). Then, by proposition 3.7 \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X, T)\). Therefore, \(\lambda = \bigwedge_{i=1}^{\infty}int(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-open sets in \((X, T)\).

**Proposition 3.9** If \(int(\mu) = 0\), where \(\mu\) is a fuzzy regular \(F_\sigma\)-set in a fuzzy topological space \((X, T)\), then \(\mu = \bigvee_{i=1}^{\infty}cl(\lambda_i)\), where \((\lambda_i)\)'s are fuzzy \(\beta\)-closed sets in \((X, T)\).

Proof: Let \(\mu\) be a fuzzy regular \(F_\sigma\)-set in \((X, T)\) such that \(int(\mu) = 0\). Then, by proposition 3.3, \(1 - \mu\) is a fuzzy regular \(G_\delta\)-set in \((X, T)\) and \(cl(1 - \mu) = 1 - int(\mu) = 1 - 0 = 1\). Now, by proposition 3.8 \(1 - \mu = \bigwedge_{i=1}^{\infty}int(\mu_i)\), where \((\mu_i)\)'s are fuzzy \(\beta\)-open sets in \((X, T)\). Hence \(\mu = 1 - \bigwedge_{i=1}^{\infty}int(\mu_i) = \bigvee_{i=1}^{\infty}(1 - int(\mu_i)) = \)

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\text{Since } (\mu_i)\text{'s are fuzzy } \beta\text{-open sets, } (1 - \mu_i)\text{'s are fuzzy } \beta\text{-closed sets in } (X, T). \text{ Let } \lambda_i = 1 - \mu_i. \text{ Therefore, } \mu = \bigvee_{i=1}^{\infty} cl(\lambda_i), \text{ where } (\lambda_i)\text{'s are fuzzy } \beta\text{-closed sets in } (X, T).$

\section{Fuzzy regular Volterra spaces}

\textbf{Definition 4.1} A fuzzy topological space \((X, T)\) is called a fuzzy regular Volterra space if \(\bigwedge_{i=1}^{N} (\lambda_i) = 1\), where \((\lambda_i)\)’s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\).

\textbf{Proposition 4.2} If \(int\left( \bigvee_{i=1}^{N} (\mu_i) \right) = 0\) where \((\mu_i)\)’s are fuzzy regular \(F_\sigma\)-sets with \(int(\mu_i) = 0\) in a fuzzy topological space \((X, T)\), then \((X, T)\) is a fuzzy regular Volterra space.

\textbf{Proof:} Suppose that \(int\left( \bigvee_{i=1}^{N} (\mu_i) \right) = 0\), where \((\mu_i)\)’s are fuzzy regular \(F_\sigma\)-sets with \(int(\mu_i) = 0\). Now \(1 - int\left( \bigvee_{i=1}^{N} (\mu_i) \right) = 1\). Then, \(cl\left(1 - \bigvee_{i=1}^{N} (\mu_i)\right) = 1\). This implies that \(cl\left( \bigwedge_{i=1}^{N} (1 - \mu_i) \right) = 1\). Since \((\mu_i)\)’s are fuzzy regular \(F_\sigma\)-sets in \((X, T)\), by proposition 3.3, \((1 - \mu_i)\)’s are fuzzy regular \(G_\delta\)-sets in \((X, T)\). Also, \(int(\mu_i) = 0\) implies that \(1 - int(\mu_i) = 1\). Then, \(cl(1 - \mu_i) = 1\). Let \(\lambda_i = 1 - \mu_i\). Then \((\lambda_i)\)’s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). Hence, \(cl\left( \bigwedge_{i=1}^{N} (\lambda_i) \right) = 1\), where \((\lambda_i)\)’s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). Therefore \((X, T)\) is a fuzzy regular Volterra space.

\textbf{Remark:} In view of the propositions 3.9 and 4.2 one will have the following result: “If \(int\left( \bigvee_{i=1}^{N} (\eta_i) \right) = 0\), where \((\eta_i)\)’s are fuzzy \(\beta\)-closed sets in a fuzzy topological space \((X, T)\), then \((X, T)\) is a fuzzy regular Volterra space”.

\textbf{Proposition 4.3} If a fuzzy topological space \((X, T)\) is a fuzzy Volterra space, then \((X, T)\) is a fuzzy regular Volterra space.

\textbf{Proof:} Let \((X, T)\) be a fuzzy Volterra space. Let \(\lambda = cl\left( \bigwedge_{i=1}^{N} (\lambda_i) \right) \ldots\ldots(1)\), where \((\lambda_i)\)’s are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). By proposition 3.5, the fuzzy regular \(G_\delta\)-sets \((\lambda_i)\)’s are fuzzy \(G_\delta\)-sets in \((X, T)\). Since \((X, T)\) is a fuzzy Volterra space, \(cl\left( \bigwedge_{i=1}^{N} (\lambda_i) \right) = 1\). \ldots\ldots(2)\), where \((\lambda_i)\)’s are fuzzy dense and fuzzy \(G_\delta\)-sets in \((X, T)\). Hence, from (1) and (2), \(\lambda = 1\). Therefore \((X, T)\) is a fuzzy regular Volterra space.

\textbf{Proposition 4.4} If a fuzzy topological space \((X, T)\) is a fuzzy regular Volterra space, then \(int\left( \bigvee_{i=1}^{N} (\mu_i) \right) = 0\), where \((\mu_i)\)’s are fuzzy \(\sigma\)-nowhere dense sets in \((X, T)\).
Proof: Let \((X, T)\) be a fuzzy regular Volterra space. Then \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\), where \((\lambda_i)'s\) are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). Now \(1 - cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 0\) implies that \(int(\bigvee_{i=1}^{N}(1 - \lambda_i)) = 0\). Since \((\lambda_i)'s\) are fuzzy regular \(G_\delta\)-sets, by proposition 3.3, \((1 - \lambda_i)'s\) are fuzzy regular \(F_\sigma\)-sets in \((X, T)\). By proposition 3.6, \((1 - \lambda_i)'s\) are fuzzy \(F_\sigma\)-sets in \((X, T)\). Also, \(cl(\lambda_i) = 1\) implies that \(1 - cl(\lambda_i) = 0\) and hence \(int(1 - \lambda_i) = 0\). Let \(\mu_i = 1 - \lambda_i\). Then \((\mu_i)'s\) are fuzzy \(F_\sigma\)-sets with \(int(\mu_i) = 0\). Then, by the definition of fuzzy \(\sigma\)-nowhere dense sets, \((\mu_i)'s\) are fuzzy \(\sigma\)-nowhere dense sets in \((X, T)\). Hence \(int(\bigvee_{i=1}^{N}(\mu_i)) = 0\), where \((\mu_i)'s\) are fuzzy \(\sigma\)-nowhere dense sets in \((X, T)\).

**Proposition 4.5** If \(int(\lambda) = 0\) for a fuzzy regular \(F_\sigma\)-set \(\lambda\) in a fuzzy topological space \((X, T)\), then \(\lambda\) is a fuzzy first category set in \((X, T)\).

Proof: Let \(\lambda\) be a fuzzy regular \(F_\sigma\)-set in \((X, T)\). Then \(\lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))\), where \(\mu_i \in T\). Now \(int(\lambda) = 0\) implies that \(int(\bigvee_{i=1}^{\infty}(cl(\mu_i))) = 0\). But \(\bigvee_{i=1}^{\infty}(int(cl(\mu_i))) \leq int(\bigvee_{i=1}^{\infty}(cl(\mu_i))) = 0\). Then \(\bigvee_{i=1}^{\infty}(int(cl(\mu_i))) = 0\). This implies that \(int(cl(\mu_i)) = 0\). Hence \(\mu_i\) is a fuzzy nowhere dense set in \((X, T)\). Also \(int(cl(\mu_i)) = int(cl(\mu_i)) = 0\) implies that \(cl(\mu_i)\) is a fuzzy nowhere dense set in \((X, T)\). Hence \(\lambda = \bigvee_{i=1}^{\infty}(cl(\mu_i))\), where \((cl(\mu_i))'s\) are fuzzy nowhere dense sets in \((X, T)\). Therefore \(\lambda\) is a fuzzy first category set in \((X, T)\).

Remark: In view of the propositions 3.9 and 4.5, one will have the following result: “If \(int(\lambda) = 0\), for a fuzzy regular \(F_\sigma\)-set in a fuzzy topological space \((X, T)\), then \(\lambda = \bigvee_{i=1}^{\infty}(cl(\lambda_i))\), where \((\lambda_i)'s\) are fuzzy \(\beta\)-closed sets in \((X, T)\), is a fuzzy first category set in \((X, T)\)”.

**Proposition 4.6** If a fuzzy regular \(G_\delta\)-set \(\lambda\) is a fuzzy dense set in a fuzzy topological space \((X, T)\), then \(\lambda\) is a fuzzy residual set in \((X, T)\).

Proof: Let \(\lambda\) be a fuzzy regular \(G_\delta\)-set with \(cl(\lambda) = 1\). Then \(1 - \lambda\) is a fuzzy regular \(F_\sigma\)-set with \(1 - cl(\lambda) = 0\). That is, \(1 - \lambda\) is a fuzzy regular \(F_\sigma\)-set with \(int(1 - \lambda) = 0\). Then by proposition 4.5, \(1 - \lambda\) is a fuzzy first category set in \((X, T)\). Therefore \(\lambda\) is a fuzzy residual set in \((X, T)\).

**Proposition 4.7** If a fuzzy topological space \((X, T)\) is a fuzzy regular Volterra space, then \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\) where \((\lambda_i)'s\) are fuzzy residual sets in \((X, T)\).

Proof: Let \((X, T)\) be a fuzzy regular Volterra space. Then \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\), where \((\lambda_i)'s\) are fuzzy dense and fuzzy regular \(G_\delta\)-sets in \((X, T)\). By proposition 4.6
(λᵢ)’s are fuzzy residual sets in (X, T). Hence \( \text{cl} \left( \bigwedge_{i=1}^{N} (\lambda_i) \right) = 1 \), where (λᵢ)’s are fuzzy residual sets in (X, T).

**Proposition 4.8** If a fuzzy topological space (X, T) is a fuzzy regular Volterra space, then \( \text{int} \left( \bigvee_{i=1}^{N} (\mu_i) \right) = 0 \), where (μᵢ)’s are fuzzy first category sets in (X, T).

Proof: Let (X, T) be a fuzzy regular Volterra space. Then \( \text{cl} \left( \bigwedge_{i=1}^{N} (\lambda_i) \right) = 1 \), where (λᵢ)’s are fuzzy dense and fuzzy regular Gδ-sets in (X, T). Now \( 1 - \text{cl} \left( \bigwedge_{i=1}^{N} (\lambda_i) \right) = 0 \) implies that \( \text{int} \left( 1 - \bigwedge_{i=1}^{N} (\lambda_i) \right) = 0 \). Then, \( \text{int} \left( \bigvee_{i=1}^{N} (1 - \lambda_i) \right) = 0 \).

Now (λᵢ)’s are fuzzy regular Gδ-sets in (X, T) implies that \( (1 - \lambda_i)’ \)’s are fuzzy regular Fσ-sets in (X, T). Also \( \text{cl}(\lambda_i) = 1 \) implies that \( 1 - \text{cl}(\lambda_i) = 0 \). Then \( \text{int}(1 - \lambda_i) = 0 \). Hence, \( (1 - \lambda_i)’ \)’s are fuzzy regular Fσ-sets with \( \text{int}(1 - \lambda_i) = 0 \). Therefore by proposition 4.5, \( (1 - \lambda_i)’ \)’s are fuzzy first category sets in (X, T). Let \( \mu_i = 1 - \lambda_i \). Hence if (X, T) is a fuzzy regular Volterra space, then \( \text{int} \left( \bigvee_{i=1}^{N} (\mu_i) \right) = 0 \), where (μᵢ)’s are fuzzy first category sets in (X, T).

**5.Conclusion**

In this paper, the concepts of fuzzy regular Gδ-sets, fuzzy regular Fσ-sets and fuzzy regular Volterra spaces have introduced and studied. Several characterizations of fuzzy regular Volterra spaces have established in this paper.

**References**


